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Differential Geometry II - Smooth Manifolds

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CHAPTER 1

TOPOLOGICAL AND SMOOTH MANIFOLDS

1.1 Topological Manifolds

Definition 1.1. A *topological manifold of dimension* $n \geq 0$ (or topological n-manifold) is a topological space M with the following properties:

- M is a *Hausdorff space*: for each pair of distinct points $p, q \in M$ there are disjoint open sets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.
- M is *second-countable*: there is a countable basis for the topology of M.
- M is *locally Euclidean* of dimension n: each point of M has a neighborhood which is homeomorphic to an open set of $\mathbb{R}^n$; that is, for each $p \in M$ we can find
    - an open subset $U \subseteq M$ containing p,
    - an open subset $\widehat{U} \subseteq \mathbb{R}^n$, and
    - a homeomorphism $\varphi \colon U \to \widehat{U}$.

Recall: Let X and Y be topological spaces. A continuous bijective map $F \colon X \to Y$ with continuous inverse is called a *homeomorphism*. If there exists a homeomorphism from X to Y, then we say that X and Y are *homeomorphic*.

Comments:

(1) Every topological manifold has, by definition, a specific, well-defined dimension. Thus, we do not consider spaces of mixed dimension, such as the disjoint union of a plane and a line, to be manifolds at all. It can be shown (using de Rham cohomology or singular homology, see [Lee13, Theorem 17.26] or [Lee11, Chapter 13], respectively) that the dimension of a (non-empty) topological manifold is in fact a topological invariant:

A non-empty topological n-manifold cannot be homeomorphic to a topological m-manifold unless $n = m$.

(2) The three conditions in Definition 1.1 ensure that topological manifolds behave in the ways we expect from our experience with Euclidean spaces. For example, in a Hausdorff space, finite subsets are closed and limits of convergent sequences are unique. The motivation for second-countability is less evident, but stems from the existence of so-called partitions of unity; see Section 2.2.

(3) There are also examples of topological spaces which are not topological manifolds; see *Exercise Sheet* 1. For example:

- The line with two origins is locally Euclidean and second-countable, but not Hausdorff.
- A disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

(4) The empty set $\emptyset$ satisfies the definition of a topological manifold for every $n \in \mathbb{N}$. For the most part we will ignore this special case. But because it is useful in certain contexts to allow the empty manifold, we choose not to exclude it from the definition.

Definition 1.2. Let M be a topological n-manifold. A *coordinate chart* on M is a pair (U, φ), where U is an open subset of M and $\varphi\colon U \to \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$. The set U is called a *coordinate domain*, or a *coordinate neighborhood* of each of its points. If, in addition, $\varphi(U)$ is an open ball in $\mathbb{R}^n$, then U is called a *coordinate ball*; if $\varphi(U)$ is an open cube in $\mathbb{R}^n$, then U is called a *coordinate cube*. The map φ called a *(local) coordinate map* and its component functions $(x^1, x^2, \dots, x^n)$, defined by $\varphi(p) = \big(x^1(p), x^2(p), \dots, x^n(p)\big)$, are called *local coordinates* on U.

Figure 1.1: A coordinate chart

By definition of a topological manifold, each point $p \in M$ is contained in the domain of some chart (U, φ). If $\varphi(p) = 0$, then we say that the chart is *centered at p*. If (U, φ) is any chart whose domain contains p, it is easy to obtain a new chart centered at p by subtracting the constant vector $\varphi(p)$.

Example 1.3 (Topological manifolds)**.**

(0) The basic example of a topological n-manifold is $\mathbb{R}^n$ itself. It is Hausdorff, because it is a metric space, and it is second-countable, because the collection of all open balls with rational centers and rational radii is a countable basis for the topology.

Moreover, every *open* subset of a topological n-manifold is itself a topological n-manifold (with the subspace topology), because the Hausdorff and second-countability properties are inherited by (arbitrary) subspaces.

(1) *Graphs of continuous functions*: Let $U \subseteq \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}^k$ be a continuous function. *The graph of f* is the subset

$$\Gamma(f) \coloneqq \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k \mid x \in U,\ y = f(x)\big\} \subseteq \mathbb{R}^n \times \mathbb{R}^k$$

endowed with the subspace topology. Let $\pi_1 \colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be the projection onto the first factor and let $\varphi \colon \Gamma(f) \to U$ be the restriction of π_1 to $\Gamma(f)$:

$$\varphi(x, y) = x, \ (x, y) \in \Gamma(f).$$

Since φ is the restriction of a continuous map, it is continuous; and it is homeomorphism, because it has a continuous inverse given by $\varphi^{-1}(x) = \big(x, f(x)\big)$. Thus, $\Gamma(f)$ is a topological manifold of dimension n. In fact, $\Gamma(f)$ is homeomorphic to U itself, and $\big(\Gamma(f), \varphi\big)$ is a *global* coordinate chart, called *graph coordinates*.

The same observation applies to any subset of $\mathbb{R}^{n+k}$ defined by setting any k of the coordinates (not necessarily the last k) equal to some continuous function of the other n, which are restricted to lie in an open subset of $\mathbb{R}^n$. (This observation will be used in (2) below for $k = 1$.)

(2) *Spheres*: For each $n \in \mathbb{N}$, *the unit n-sphere* is the subset

$$\mathbb{S}^n := \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, |x| = 1 \right\} \subseteq \mathbb{R}^{n+1} .$$

It is Hausdorff and second-countable, because it is a subspace of $\mathbb{R}^{n+1}$. To show that it is locally Euclidean, for each $i \in \{1, \ldots, n+1\}$ consider the sets

$$U_i^+ := \left\{ (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} \mid x^i > 0 \right\}$$

and

$$U_i^- := \left\{ (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} \mid x^i < 0 \right\}.$$

Consider also the *(open) unit ball* of dimension n

$$\mathbb{B}^n := \left\{ x \in \mathbb{R}^n \,\middle|\, |x| < 1 \right\}$$

and the continuous function

$$f \colon \mathbb{B}^n \to \mathbb{R}, \ u \mapsto \sqrt{1 - |u|^2} .$$

Then for each $i \in \{1, \ldots, n+1\}$ it is easy to check that $U_i^+ \cap \mathbb{S}^n$ is the graph of the function

$$x^i = f(x^1, \ldots, \underset{\hookrightarrow \text{omitted}}{\widehat{x^i}}, \ldots, x^{n+1})$$

and that $U_i^- \cap \mathbb{S}^n$ is the graph of the function

$$x^i = -f(x^1, \ldots, \underset{\hookrightarrow \text{omitted}}{\widehat{x^i}}, \ldots, x^{n+1}) .$$

Thus, each subset $U_i^\pm \cap \mathbb{S}^n$ is locally Euclidean of dimension n, see (1) above, and the maps

$$\varphi_i^\pm \colon U_i^\pm \cap \mathbb{S}^n \to \mathbb{B}^n$$

$$(x^1, \ldots, x^{n+1}) \mapsto (x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1})$$

are graph coordinates for $\mathbb{S}^n$. Since each point of $\mathbb{S}^n$ is in the domain of at least one of these $2n+2$ charts (see Figure 1.2), we conclude that $\mathbb{S}^n$ is a topological n-manifold.

(3) *Projective spaces*: see Appendix A.

⇝ We will encounter many more examples of topological manifolds later in the course and in the exercise sheets as well.

Figure 1.2: Charts for $\mathbb{S}^n$

1.2 Smooth Manifolds

$\leadsto$ Topological manifolds are:

- suitable for the study of topological properties (e.g. compactness, connectedness, simple connectedness, etc)
- not suitable for doing calculus: being "differentiable" is *not* an invariant under homeomorphisms; in other words, it is *not* a topological property. For instance, the map

  $$\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \varphi(u,v) \mapsto \left(\sqrt[3]{u}, \sqrt[3]{v}\right)$$

  is a homeomorphism, the map

  $$f\colon \mathbb{R}^2 \to \mathbb{R},\ f(x,y) \mapsto x$$

  is differentiable, but the composite map

  $$(f \circ \varphi)(u,v) = \sqrt[3]{u}$$

  is not differentiable at $(0,0)$.

$\leadsto$ To make sense of derivatives of maps between manifolds, we need to introduce a new kind of manifold. It will be a topological manifold with some extra structure (in addition to its topology), which will allow us to decide which maps are "smooth".

$\leadsto$ Plausible definition of "smoothness" of a function on M:
$f\colon M \to \mathbb{R}$ is smooth if $f \circ \varphi^{-1}\colon \widehat{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ is smooth (in the usual sense), which makes sense if it does not depend on the choice of coordinate chart (U,φ). To guarantee this independence, we will restrict our attention to "smooth charts".

Definition 1.4. Let M be a topological manifold. If (U,φ) and (V,ψ) are two charts such that $U \cap V \neq \varnothing$, then the composite map $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ is called the *transition map from φ to ψ* (see Figure 1.3) and is clearly a homeomorphism. Two charts (U,φ) and (V,ψ) are said to be *smoothly compatible* if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is diffeomorphism (i.e., smooth and bijective with smooth inverse). Since $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$, smoothness of this map is to be interpreted in the ordinary sense of having continuous partial derivatives of all orders (C^∞).

An *atlas* for M is a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible. Finally, a smooth atlas $\mathcal{A}$ on M is called *maximal* (or *complete*) if it is not properly contained in any larger smooth atlas. This just means that any chart which is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$.

Remark 1.5. To show that an atlas is smooth, we need only verify that each transition map $\psi \circ \varphi^{-1}$ is smooth whenever (U,φ) and (V,ψ) are charts in $\mathcal{A}$; once we have proved this, it follows that $\psi \circ \varphi^{-1}$ is a diffeomorphism because its inverse $\varphi \circ \psi^{-1} = (\psi \circ \varphi^{-1})^{-1}$ is one of the transition maps we have already shown to be smooth. Alternatively, given two particular charts (U, φ) and (V, ψ), it is often easier to show that they are smoothly compatible by verifying that $\psi \circ \varphi^{-1}$ is smooth with non-singular Jacobian at each point of its domain, since then the *Inverse Function Theorem* = [Lee13, Theorem C.34] implies that $\psi \circ \varphi^{-1}$ is a diffeomorphism, see [Lee13, Corollary C.36].

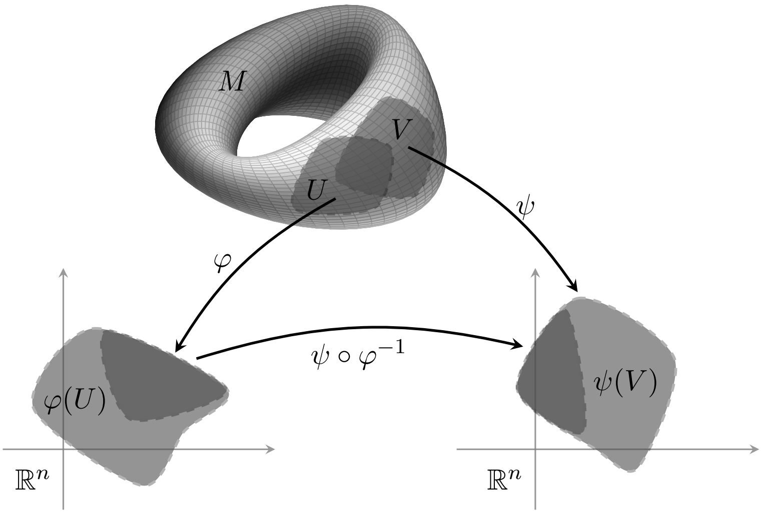


Figure 1.3: A transition map

Comment: Our plan is to define a "smooth structure" on M by giving it a smooth atlas, and to define a function $f\colon M \to \mathbb{R}$ to be "smooth" if and only if $f \circ \varphi^{-1}$ is smooth in the sense of ordinary calculus for each coordinate chart (U, φ) in this atlas. There is one minor technical problem with this approach (which led to the definition of a maximal smooth atlas): in general, there will be many possible atlantes that give the "same" smooth structure to M, in that they all determine the same collection of smooth functions on M. For example, consider the following pair of smooth atlantes on $\mathbb{R}^n$:

$$\begin{aligned} \mathcal{A}_1 &:= \big\{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\big\}, \\ \mathcal{A}_2 &:= \big\{(B_1(x), \mathrm{Id}_{B_1(x)}) \mid x \in \mathbb{R}^n\big\}. \end{aligned}$$

Although these are different smooth atlantes, clearly a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is smooth with respect to either atlas if and only if it is smooth in the sense of ordinary calculus.

We can now define the main concept of this chapter.

Definition 1.6. Let M be a topological manifold. A *smooth structure* on M is a maximal smooth atlas. A *smooth manifold* is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

A smooth structure is an additional piece of data that must be added to a topological manifold before we are entitled to talk about a "smooth manifold". Note that a given positive-dimensional topological manifold may have many smooth structures (in fact, if it has one, then it has uncountably many distinct ones; see [Lee13, Problem 1.6] and the short discussion that follows), but it may also admit no smooth structures at all.

Exercise 1.7: Consider the topological manifold $\mathbb{R}$ together with the two atlantes

$$\mathcal{A} = \big\{(\mathbb{R}, \mathrm{Id}_{\mathbb{R}})\big\} \quad \text{and} \quad \mathcal{A}' = \big\{(\mathbb{R}, \psi)\big\}$$

where $\psi\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$. Show that the corresponding smooth structures on $\mathbb{R}$ are different, but they are diffeomorphic to each other, i.e., there is a diffeomorphism $(\mathbb{R}, \mathrm{Id}_{\mathbb{R}}) \to (\mathbb{R}, \psi)$.

Solution: The union of the atlases $\mathcal{A} := \{(\mathbb{R}, \mathrm{Id}_{\mathbb{R}})\}$ and $\mathcal{A}' := \{(\mathbb{R}, \psi)\}$ is not a smooth atlas, because the transition map $\mathrm{Id}_{\mathbb{R}} \circ \psi^{-1}\colon y \mapsto y^{1/3}$ is not differentiable at the origin. Hence, these atlases determine different smooth structures on $\mathbb{R}$.

Consider now the map

$$\begin{aligned} F\colon (\mathbb{R}, \mathcal{A}) &\to (\mathbb{R}, \mathcal{A}') \\ x &\mapsto x^{1/3}. \end{aligned}$$

The coordinate representation of this map is

$$\widehat{F}(t) = \big(\psi \circ F \circ \mathrm{Id}_{\mathbb{R}}^{-1}\big)(t) = t,$$

which is smooth. The coordinate representation of its inverse is

$$\widehat{F^{-1}}(s) = \big(\mathrm{Id}_{\mathbb{R}} \circ F^{-1} \circ \psi^{-1}\big)(s) = s,$$

which is smooth as well. Hence, F is a diffeomorphism.

We exhibited in Exercise 1.7 two distinct smooth structures on $\mathbb{R}$, but then proved that they are diffeomorphic. It is in fact true that any two smooth structures on $\mathbb{R}$ are diffeomorphic to each other; in other words, $\mathbb{R}$ admits a unique smooth structure up to diffeomorphism. However, there are topological manifolds admitting several smooth structures which are not diffeomorphic (google, for example, "exotic spheres").

It is generally not convenient to define a smooth structure by explicitly describing a maximal smooth atlas, because such an atlas contains very many charts. The next result shows that we need only specify some smooth atlas.

Proposition 1.8. *Let M be a topological manifold.*

(a) *Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called* the smooth structure determined by $\mathcal{A}$*.*

(b) *Two smooth atlantes for M determine the same smooth structure if and only if their union is a smooth atlas.*

Proof.

(a) Given a smooth atlas $\mathcal{A}$ on M, set

$$\overline{\mathcal{A}} := \big\{(U, \varphi) \text{ chart for } M \mid \forall (V, \psi) \in \mathcal{A} : (U, \varphi) \text{ and } (V, \psi) \text{ are smoothly compatible}\big\}.$$

By definition of a smooth atlas we have $\mathcal{A} \subseteq \overline{\mathcal{A}}$. Now, let $\mathcal{A}'$ be a smooth atlas on M such that $\overline{\mathcal{A}} \subseteq \mathcal{A}'$ and take $(U', \varphi') \in \mathcal{A}'$. Since it holds that $\mathcal{A} \subseteq \mathcal{A}'$, we infer that (U', φ') is smoothly compatible with every chart $(U, \varphi) \in \mathcal{A}$ (by virtue of $\mathcal{A}'$ being a smooth atlas). Hence, $(U', \varphi') \in \overline{\mathcal{A}}$, which implies that $\overline{\mathcal{A}} = \mathcal{A}'$; in particular, $\overline{\mathcal{A}}$ is a smooth atlas on M. As $\mathcal{A}'$ was arbitrary, we also conclude that $\overline{\mathcal{A}}$ is maximal.

It remains to show that $\overline{\mathcal{A}}$ the unique maximal smooth atlas containing $\mathcal{A}$. So let $\mathcal{A}'$ be a maximal smooth atlas containing $\mathcal{A}$. In particular, any chart in $\mathcal{A}'$ is smoothly compatible with any chart in $\mathcal{A}$, and thus $\mathcal{A}' \subseteq \overline{\mathcal{A}}$. By maximality, we conclude that $\mathcal{A}' = \overline{\mathcal{A}}$, and thus we obtain the uniqueness.

(b) Assume first that two smooth atlantes $\mathcal{A}_1$ and $\mathcal{A}_2$ for M determine the same smooth structure, that is, both $\mathcal{A}_1$ and $\mathcal{A}_2$ are contained in the same (unique) maximal smooth atlas for M. Then every chart $(U, \varphi) \in \mathcal{A}_1$ is smoothly compatible with every chart $(V, \psi) \in \mathcal{A}_2$, so the union $\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas for M.

Conversely, assume that the union $\mathcal{A}_1 \cup \mathcal{A}_2$ of two smooth atlantes $\mathcal{A}_1$ and $\mathcal{A}_2$ for M is also a smooth atlas for M. Then every chart $(U, \varphi) \in \mathcal{A}_1$ is smoothly compatible with every chart $(V, \psi) \in \mathcal{A}_2$. If $\bar{\mathcal{A}}_1$ (resp. $\bar{\mathcal{A}}_2$) is the smooth structure on M determined by $\mathcal{A}_1$ (resp. $\mathcal{A}_2$), then by the construction in (a) we infer that $\mathcal{A}_1 \subseteq \bar{\mathcal{A}}_2$ and $\mathcal{A}_2 \subseteq \bar{\mathcal{A}}_1$, and hence $\bar{\mathcal{A}}_1 = \bar{\mathcal{A}}_2$ due to the uniqueness in (a). ☐

For example, if a topological manifold M can be covered by a single chart, then the smooth compatibility condition is trivially satisfied, so any such chart determines automatically a smooth structure on M; see Example 1.3(1) and Example 1.10(1).

Definition 1.9. Let $(M, \mathcal{A})$ be a smooth manifold. Any chart (U, φ) contained in the maximal smooth atlas $\mathcal{A}$ is called a *smooth coordinate chart*. The corresponding coordinate map φ is called a *smooth coordinate map*, and its domain U is called a *smooth coordinate domain*, or *smooth coordinate neighborhood* of each of its points. A *smooth coordinate ball* is a smooth coordinate domain whose image under a smooth coordinate map is a ball in Euclidean space. A *smooth coordinate cube* is defined similarly.

Here is how one usually thinks about (smooth) coordinate charts on a smooth manifold. Once we choose a (smooth) coordinate chart (U, φ) on M^n, the (smooth) coordinate map $\varphi\colon U \to \widehat{U} \subseteq \mathbb{R}^n$ can be thought of as giving a temporary *identification* between U and $\widehat{U}$. Using this identification, while we work in this chart, we can think of U simultaneously as an open subset of M and as an open subset of $\mathbb{R}^n$. Under this identification, we can represent a point $p \in M$ by its coordinates $(x^1, \ldots, x^n) = \varphi(p)$, and think of this n-tuple as *being* the point p. We typically express this by saying "$(x^1, \ldots, x^n)$ is the (local) coordinate representation for p" or "$p = (x^1, \ldots, x^n)$ in local coordinates".

Example 1.10 (Smooth manifolds)**.**

(0) For each $n \in \mathbb{N}$ the Euclidean space $\mathbb{R}^n$ is a smooth n-manifold with smooth structure determined by the atlas $\big\{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\big\}$. We call this the *standard smooth structure on* $\mathbb{R}^n$ and the resulting coordinate map the *standard coordinates* on $\mathbb{R}^n$. (Unless we explicitly say otherwise, we always use this smooth structure on $\mathbb{R}^n$.) With respect to this smooth structure, the smooth coordinate charts for $\mathbb{R}^n$ are exactly those charts (U, φ) such that φ is diffeomorphism (in the usual sense) from $U \subseteq \mathbb{R}^n$ to another open set $\widehat{U} \subseteq \mathbb{R}^n$.

(1) *Graphs of smooth functions*: If $U \subseteq \mathbb{R}^n$ is an open set and if $f\colon U \to \mathbb{R}^k$ is a smooth function, then by Example 1.3(1) the graph $\Gamma(f)$ of f is a topological n-manifold

in the subspace topology. Since $\Gamma(f)$ is covered by the single graph coordinate chart $\varphi\colon \Gamma(f) \to U$, we can put a canonical smooth structure on $\Gamma(f)$ by declaring the graph coordinate chart $\big(\Gamma(f), \varphi\big)$ to be a smooth chart.

(2) *Spheres*: The unit n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is a topological n-manifold according to Example 1.3(2). We put a smooth structure on $\mathbb{S}^n$ as follows. For each $i \in \{1, \dots, n+1\}$ consider the graph coordinate charts $(U_i^\pm \cap \mathbb{S}^n, \varphi_i^\pm)$. For any $i \neq j$ and any choice of ± signs, the transition maps $\varphi_i^\pm \circ (\varphi_j^\pm)^{-1}$ and $\varphi_i^\pm \circ (\varphi_j^\mp)^{-1}$ are easily computed. For example, when $i < j$, we get:

$$\begin{aligned}\big(\varphi_i^+ \circ (\varphi_j^+)^{-1}\big)(u^1, \dots, u^n) &= \varphi_i^+\big(u^1, \dots, \underset{\underset{j\text{-th}}{\downarrow}}{\sqrt{1-|u|^2}}, \dots, u^n\big) \\ &= \big(u^1, \dots, \underset{\underset{i\text{-th}}{\downarrow}}{\widehat{u^i}}, \dots, \underset{\underset{j\text{-th}}{\downarrow}}{\sqrt{1-|u|^2}}, \dots, u^n\big)\,,\end{aligned}$$

and similar formulas hold in the other cases. When $i = j$, the domains of φ_i^+ and φ_i^- are disjoint, so there is nothing to check. Thus, the collection of charts

$$\big\{(U_i^\pm \cap \mathbb{S}^n, \varphi_i^\pm)\big\}_{i=1}^{n+1}$$

is a smooth atlas, so it determines a smooth structure on $\mathbb{S}^n$, which we call its *standard smooth structure*.

(3) *Projective spaces*: see Appendix A.

(4) *Open submanifolds*: If U is any open subset of $\mathbb{R}^n$, then U is a topological n-manifold, and the single chart (U, Id_U) determines a smooth structure on U.

More generally, let M be a smooth n-manifold and let $U \subseteq M$ be an open subset. Define an atlas on U by

$$\mathcal{A}_U = \big\{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subseteq U\big\}\,.$$

Every point $p \in U$ is contained in the domain of some chart (W, φ) for M. If we set $V = W \cap U$, then $(V, \varphi|_V)$ is a chart in $\mathcal{A}_U$ whose domain contains p. Therefore, U is covered by the domains of the charts in $\mathcal{A}_U$, and it is easy to verify that $\mathcal{A}_U$ is a smooth atlas for U. In conclusion, any open subset of M is itself a smooth n-manifold in a natural way. Endowed with this smooth structure, we call any open subset an *open submanifold of* M.

⇝ We will encounter many more examples of smooth manifolds later in the course and in the exercise sheets as well.

In the examples we have seen so far, we constructed a smooth manifold structure in two stages: we started with a topological space and checked that it was a topological manifold, and then we specified a smooth structure (by means of a smooth atlas due to Proposition 1.8(a)). The following lemma shows how, given a set with suitable "charts" that overlap smoothly, we can use these charts to define both a topology and a smooth structure on the set.

Lemma 1.11 (Smooth manifold chart lemma)**.** *Let M be a set. Suppose that we are given a collection $\{U_\alpha\}$ of subsets of M together with maps $\varphi_\alpha\colon U_\alpha \to \mathbb{R}^n$ such that the following properties are satisfied:*

(i) *For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.*

(ii) *For each α and β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.*

(iii) *Whenever $U_\alpha \cap U_\beta \neq \varnothing$, the map*

$$\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth.

(iv) *Countably many of the sets U_α cover M.*

(v) *Whenever $p, q \in M$ with $p \neq q$, either there exists some U_α containing both p and q or there exist disjoint sets U_α and U_β with $p \in U_\alpha$ and $q \in U_\beta$.*

Then M has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

Proof. For the details of the proof we refer to [Lee13, Lemma 1.35]. The basic idea is to define a topology on M by taking all sets of the form $\varphi_\alpha^{-1}(V)$, where $V \subseteq \mathbb{R}^n$ is open, as a basis. $\square$

CHAPTER 2

SMOOTH MAPS

The main reason for introducing smooth structures was to enable us to define smooth functions on manifolds and smooth maps between manifolds. In Section 2.1 we carry out this project. In Section 2.2 we introduce a powerful tool for blending together locally defined smooth objects, called *partitions of unity*. They are used throughout smooth manifold theory for building global smooth objects out of local ones. At the end of this chapter we give the first applications of partitions of unity.

2.1 Smooth Maps

Definition 2.1. Let M be a smooth n-manifold and let $f\colon M \to \mathbb{R}^k$ be a function, where $k, n \in \mathbb{N}$. We say that f is *smooth* if for every point $p \in M$ there exists a smooth chart (U, φ) for M such that $p \in U$ and the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$.

Figure 2.1: Definition of smooth functions

Remark 2.2. Let M be a smooth n-manifold. The set $C^\infty(M)$ of all smooth real-valued functions on M is an $\mathbb{R}$-vector space: sums and constant multiples of smooth functions are smooth. According to Exercise 2.21, $C^\infty(M)$ is infinite-dimensional when $n \geq 1$. Moreover, pointwise multiplication turns $C^\infty(M)$ into a commutative ring and an associative and commutative $\mathbb{R}$-algebra.

Definition 2.3. Let M be a topological n-manifold. Given a function $f\colon M \to \mathbb{R}^k$ and a chart (U, φ) for M, the function

$$\widehat{f} = f \circ \varphi^{-1} \colon \varphi(U) \to \mathbb{R}^k$$

is called *the coordinate representation of* f.

Let M be a smooth manifold and let $f\colon M \to \mathbb{R}^k$ be a function on M. By definition, f is smooth if and only if its coordinate representation is smooth in *some* smooth chart around each point. According to Remark 2.7, smooth functions have smooth coordinate representations in *every* smooth chart; that is, $f \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{R}^k$ is smooth for *every* smooth chart (U, φ) for M.

Definition 2.4. Let $F\colon M \to N$ be a map between smooth manifolds. We say that F is a *smooth map* if for every $p \in M$ there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is smooth.

Figure 2.2: Definition of smooth maps

Observe that Definition 2.1 is a special case of Definition 2.4 by taking $N = V = \mathbb{R}^k$ and $\psi = \mathrm{Id}_{\mathbb{R}^k}$.

The first important observation about our definition of smooth maps is that, as one might expect, smoothness implies continuity.

Proposition 2.5. *Every smooth map is continuous.*

Proof. Let $F\colon M \to N$ be a map between smooth manifolds. Fix $p \in M$. Since F is smooth, there are smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is smooth, and hence continuous. Since $F(U) \subseteq V$ and the maps $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ are homeomorphisms, the map

$$F\big|_U = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi\colon U \to V$$

is continuous as a composition of continuous maps. Therefore, F is continuous in a neighborhood of each point, and thus continuous on M. $\square$

Comment: The requirement that

$$\forall p \in M \quad \exists\, (U, \varphi) \ni p \quad \exists\, (V, \psi) \ni F(p) \ \text{ such that } \ F(U) \subseteq V$$

in the definition of smoothness is included precisely so that smoothness automatically implies continuity. [Lee13, Problem 2.1] illustrates what can go wrong if this requirement is omitted.

Definition 2.6. Let $F\colon M \to N$ be a map between topological manifolds. If (U, φ) and (V, ψ) are charts for M and N, respectively, then we call

$$\widehat{F} = \psi \circ F \circ \varphi^{-1}$$

the coordinate representation of F with respect to the given coordinates. It maps the set $\varphi\big(U \cap F^{-1}(V)\big)$ to $\psi(V)$.

Remark 2.7. If $F\colon M \to N$ is a smooth map between smooth manifolds, then the coordinate representation of F with respect to *every* pair of smooth charts for M and N is smooth.

Proof: Fix $p \in M$. Since F is smooth, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is smooth. Pick smooth charts (U', φ') containing p and (V', ψ') containing $F(p)$. Then $V \cap V'$ is an open neighborhood of $F(p)$ in N, and since F is continuous by Proposition 2.5, $F^{-1}(V \cap V')$ is an open neighborhood of p in M, and thus so is $U'' := U \cap U' \cap F^{-1}(V \cap V')$. Consider now the coordinate representation of F with respect to the smooth charts (U', φ') and (V', ψ') with domain of definition $\varphi'(U'')$ and observe that

$$\begin{aligned}\psi' \circ F \circ (\varphi')^{-1} &= \psi' \circ \left(\psi^{-1} \circ \psi\right) \circ F \circ \left(\varphi^{-1} \circ \varphi\right) \circ (\varphi')^{-1} \\ &= \left(\psi' \circ \psi^{-1}\right) \circ \left(\psi \circ F \circ \varphi^{-1}\right) \circ \left(\varphi \circ (\varphi')^{-1}\right).\end{aligned}$$

Thus, $\psi' \circ F \circ (\varphi')^{-1}$ is smooth on its domain of definition as a composition of smooth maps between open subsets of Euclidean spaces; indeed, $\psi \circ F \circ \varphi^{-1}$ is smooth and both $\psi' \circ \psi^{-1}$ and $\varphi \circ (\varphi')^{-1}$ are diffeomorphisms. This proves the claim.

Proposition 2.8 (Equivalent characterizations of smoothness). *Let M and N be smooth manifolds and let $F\colon M \to N$ be a map. Then F is smooth if and only if either of the following conditions is satisfied:*

(a) *For every $p \in M$ there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi\big(U \cap F^{-1}(V)\big)$ to $\psi(V)$.*

(b) *F is continuous and there exist smooth atlases $\big\{(U_\alpha, \varphi_\alpha)\big\}$ and $\big\{(V_\beta, \psi_\beta)\big\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha\big(U_\alpha \cap F^{-1}(V_\beta)\big)$ to $\psi_\beta(V_\beta)$.*

Proof. See [*Exercise Sheet* 3, *Exercise* 1]. ☐

Proposition 2.9 (Smoothness is a local property). *Let M and N be smooth manifolds and let $F\colon M \to N$ be a map. The following statements hold:*

(a) *If every point $p \in M$ has a neighborhood U such that $F|_U$ is smooth, then F is smooth.*

(b) *If F is smooth, then its restriction to every open subset of M is smooth.*

Proof. See [*Exercise Sheet* 3, *Exercise* 2]. ☐

The next result is essentially just a restatement of the previous proposition, but it gives a highly useful way of constructing smooth maps.

Lemma 2.10 (Gluing lemma for smooth maps). *Let M and N be smooth manifolds and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Suppose that for each $\alpha \in A$ we are given a smooth map $F_\alpha\colon U_\alpha \to N$ such that the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta \in A$. Then there exists a unique smooth map $F\colon M \to N$ such that $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.*

Proposition 2.11. *Let M, N and P be a smooth manifolds.*

(a) *Every constant map $c\colon M \to N$ is smooth.*

(b) *The identity map Id_M of M is smooth.*

(c) *If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota\colon U \hookrightarrow M$ is smooth.*

(d) *If $F\colon M \to N$ and $G\colon N \to P$ are smooth, then so is $G \circ F\colon M \to P$.*

Proof. See [*Exercise Sheet* 3*, Exercise* 3]. □

Comment: We now have enough information in order to produce a number of interesting examples of smooth maps. In spite of the apparent complexity of the definition, it is usually not hard to prove that a particular map is smooth. There are basically only three common ways to do so:

(1) Write the map in smooth local coordinates and recognize its component functions as compositions of smooth elementary functions.

(2) Exhibit the map as a composition of maps that are known to be smooth.

(3) Use some special-purpose theorem that applies to the particular case under consideration.

We give below an example of a smooth map utilizing the first method, while another such example is given in *Exercise A.6*.

Example 2.12. Consider the unit n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ with its standard smooth structure, see Example 1.3(2) and Example 1.10(2). The inclusion map $\iota\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is continuous (inclusion map of topological spaces). In fact, it is a smooth map, because its coordinate representation with respect to any of the graph coordinates is

$$\begin{aligned}\widehat{\iota}(u^1,\dots,u^n) &= \big(\iota \circ (\varphi_i^{\pm})^{-1}\big)(u^1,\dots,u^n)\\ &= \Big(u^1,\dots,u^{i-1},\underset{i\text{-th}}{\underset{\downarrow}{\pm\sqrt{1-|u|^2}}},u^i,\dots,u^n\Big)\,,\end{aligned}$$

which is smooth on its domain (the set where $|u|^2 < 1$).

⇝ We will encounter many more examples of smooth maps later in the course and in the exercise sheets as well.

2.1.1 Diffeomorphisms

Definition 2.13. Let M and N be smooth manifolds. A *diffeomorphism* from M to N is a smooth bijective map $M \to N$ that has smooth inverse. We say that M and N are *diffeomorphic* if there exists a diffeomorphism between them.

Example 2.14 (Diffeomorphisms)**.**

(1) Consider the maps

$$F\colon \mathbb{B}^n \to \mathbb{R}^n\,, \quad x \mapsto \frac{x}{\sqrt{1-|x|^2}}$$

and

$$G\colon \mathbb{R}^n \to \mathbb{B}^n\,, \quad y \mapsto \frac{y}{\sqrt{1+|y|^2}}\,.$$

These maps are smooth, and it is straightforward to check that they are inverses to each other. Thus, they are both diffeomorphisms, so $\mathbb{B}^n$ is diffeomorphic to $\mathbb{R}^n$.

(2) If M is any smooth manifold and if (U,φ) is any smooth coordinate chart on M, then the coordinate map $\varphi\colon U \to \varphi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism. Indeed, it has an identity map as a coordinate representation.

Proposition 2.15 (Properties of diffeomorphisms)**.**

(a) *Every composition of diffeomorphisms is a diffeomorphism.*

(b) *Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.*

(c) *Every diffeomorphism is a homeomorphism and an open map.*

(d) *The restriction of a diffeomorphism to an open submanifold is a diffeomorphism onto its image.*

(e) *"Diffeomorphic" is an equivalence relation on the class of all smooth manifolds.*

Proof. Exercise! (See also Proposition 4.9.) ☐

Just as two topological spaces are considered to be "the same" if they are homeomorphic, two smooth manifolds are essentially indistinguishable if they are diffeomorphic. The central concern of smooth manifold theory is the study of properties of smooth manifolds that are preserved by diffeomorphisms. The next result shows that the dimension is one such property (cf. p. 1):

Theorem 2.16 (Diffeomorphism invariance of the dimension)**.** *A non-empty smooth m-manifold cannot be diffeomorphic to a non-empty smooth n-manifold unless $m = n$.*

Proof. Let M be a non-empty smooth m-manifold, let N be a non-empty smooth n-manifold, and assume that there exists a diffeomorphism $F\colon M \to N$. Choose any point $p \in M$ and consider smooth charts (U,φ) for M containing p and (V,ψ) for N containing $F(p)$. Then (the restriction of the coordinate representation) $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is a diffeomorphism from an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$. It is now a consequence of the *chain rule* that $m = n$, see [Lee13, Proposition C.4]. ☐

2.2 Partitions of Unity

We briefly discuss here partitions of unity, which are tools for "blending together" local smooth objects into global ones without necessarily assuming that they agree on overlaps (cf. Lemma 2.10). They are indispensable in smooth manifold theory, and we will soon see some first applications of partitions of unity. For further information we refer to [Lee13, Chapter 2, Partitions of Unity].

Definition 2.17. Let M be a topological space and let $f\colon M \to \mathbb{R}^k$ be a function. The *support* of f is defined as

$$\operatorname{supp} f = \overline{\{p \in M \mid f(p) \neq 0\}}.$$

Moreover,

- if $\operatorname{supp} f$ is contained in some open subset $U \subseteq M$, then we say that f is *supported in* U;
- if $\operatorname{supp} f$ is a compact set (e.g., if M is a compact space), then we say that f is *compactly supported*.

Definition 2.18. Let M be a topological space and let $\mathfrak{X} = (X_\alpha)_{\alpha\in A}$ be an open cover of M, indexed by a set A. A *partition of unity subordinate to* $\mathfrak{X}$ is an indexed family $(\psi_\alpha)_{\alpha\in A}$ of continuous functions $\psi_\alpha\colon M \to \mathbb{R}$ with the following properties:

(i) $0 \leq \psi_\alpha(x) \leq 1, \ \forall\, \alpha \in A, \ \forall\, x \in M$.

(ii) $\operatorname{supp} \psi_\alpha \subseteq X_\alpha, \ \forall\, \alpha \in A$.

(iii) The family of supports $\big\{\operatorname{supp} \psi_\alpha\big\}_{\alpha\in A}$ is *locally finite*, i.e., every point $p \in M$ has a neighborhood W_p such that $W_p \cap \operatorname{supp} \psi_\alpha = \emptyset$ for all but a finite number of $\alpha \in A$.

(iv) $\displaystyle\sum_{\alpha\in A} \psi_\alpha(x) = 1, \ \forall\, x \in M$.

If now M is a smooth manifold, then a *smooth partition of unity* is one for which each of the functions ψ_α is smooth.

Observe that, due to the local finiteness condition (iii), the sum in (iv) has only finitely many non-zero terms in a neighborhood of each point, so there is no issue of convergence.

Theorem 2.19 (Existence of smooth partitions of unity)**.** *Let M be a smooth manifold and let $\mathfrak{X} = (X_\alpha)_{\alpha\in A}$ be an open cover of M. Then there exists a smooth partition of unity subordinate to $\mathfrak{X}$.*

Proof. For a detailed proof of the statement we refer to [Lee13, Theorem 2.23], see also [Lee09, Theorem 1.73]. □

Comment: The hypothesis that M is second-countable is used implicitly in the proof of Theorem 2.19 via the following characterization: *If X is a locally Euclidean Hausdorff space, then X is second-countable if and only if it is paracompact and has countably many connected components*, see [Lee13, Problem 1.5]. In particular, *every topological manifold is paracompact*, see [Lee13, Theorem 1.15].

2.2.1 Applications

In this subsection we present three interesting applications of partitions of unity.

① **Existence of smooth bump functions:**

If M is a topological space, $A \subseteq M$ is a closed subset and $U \subseteq M$ is an open subset such that $A \subseteq U$, a continuous function $\psi \colon M \to \mathbb{R}$ is called a *bump function for A supported in U* if

- $0 \leq \psi(x) \leq 1, \ \forall\, x \in M$,
- $\psi \equiv 1$ on A, and
- $\operatorname{supp} \psi \subseteq U$.

In other words, a bump function is a continuous real-valued function that is equal to 1 on a specified set and zero outside a specified neighborhood of that set.

Proposition 2.20. *Let M be a smooth manifold. For every closed subset $A \subseteq M$ and any open subset $U \subseteq M$ containing A, there exists a smooth bump function for A supported in U.*

Proof. Set $U_0 := U$ and $U_1 := M \setminus A$, and let $\{\psi_0, \psi_1\}$ be a smooth partition of unity subordinate to the open cover $\{U_0, U_1\}$ of M. Since $\psi_1 \equiv 0$ on A by construction, and therefore $\psi_0 = \sum_{i=0}^{1} \psi_i \equiv 1$ on A, the function ψ_0 has the required properties. $\square$

Exercise 2.21: If M is a smooth manifold of dimension $n \geq 1$, then the vector space $C^\infty(M)$ is infinite-dimensional.

[Hint: Show that if $f_1, \dots, f_k$ are elements of $C^\infty(M)$ with non-empty disjoint supports, then they are linearly independent.]

② **Extension lemma for smooth functions:**

In view of Lemma 2.10 it is often possible to construct smooth maps by "gluing together" maps defined on *open* subsets. However, one cannot expect to "glue together" smooth maps defined on *closed* subsets and obtain a smooth result. For example, the two functions

$$f_+ \colon [0, +\infty) \to \mathbb{R}, \ \ x \mapsto +x,$$
$$f_- \colon (-\infty, 0] \to \mathbb{R}, \ \ x \mapsto -x,$$

are both smooth and agree at the point 0 where they overlap, but the continuous function $f \colon \mathbb{R} \to \mathbb{R}, \ x \mapsto |x|$ that they define is clearly not smooth at the origin. Our second application of partitions of unity is an important result concerning the possibility of extending smooth functions from closed sets.

Let M and N be smooth manifolds and let $A \subseteq M$ be an arbitrary subset. We say that a map $F \colon A \to N$ is *smooth on A* if it admits a smooth extension in a neighborhood of each point; namely, if for every $p \in A$ there exists an open subset $W \subseteq M$ containing p and a smooth map $\widetilde{F} \colon W \to N$ whose restriction to $W \cap A$ agrees with F.

Lemma 2.22 (Extension lemma for smooth functions)**.** *Let M be a smooth manifold, let $A \subseteq M$ be a closed subset, and let $f\colon A \to \mathbb{R}^k$ be a smooth function. For any open subset $U \subseteq M$ containing A, there exists a smooth function $\widetilde{f}\colon M \to \mathbb{R}^k$ such that $\widetilde{f}|_A = f$ and* $\operatorname{supp} \widetilde{f} \subseteq U$*.*

Proof. For each $p \in A$ choose an open neighborhood W_p of p and a smooth function $\widetilde{f}_p\colon W_p \to \mathbb{R}^k$ such that

$$\widetilde{f}_p|_{W_p \cap A} = f. \tag{2.1}$$

Replacing W_p by $W_p \cap U$, we may assume that $W_p \subseteq U$. Observe that the family of sets $\{W_p\}_{p\in A} \cup (M \setminus A)$ is an open cover of M. Let $\{\psi_p\}_{p\in A} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp} \psi_p \subseteq W_p$ and $\operatorname{supp} \psi_0 \subseteq M \setminus A$.

For each $p \in A$, the product $\psi_p \widetilde{f}_p$ is smooth on W_p, and has a smooth extension to all of M if we interpret it to be zero on $M \setminus \operatorname{supp} \psi_p$. (The extended function is smooth because the two definitions agree on the open subset $W_p \setminus \operatorname{supp} \psi_p$ where they overlap.) Thus, we can define the function

$$\widetilde{f}\colon M \to \mathbb{R}^k, \quad x \mapsto \sum_{p\in A} \psi_p(x)\, \widetilde{f}_p(x).$$

Since the collection of supports $\{\operatorname{supp} \psi_p\}_{p\in A}$ is locally finite by construction, the sum actually has only finitely many non-zero terms in a neighborhood of any point of M, and therefore defines a smooth function. If $x \in A$, then $\psi_0(x) = 0$ by construction and $\widetilde{f}_p(x) = f(x)$ for each p such that $\psi_p(x) \neq 0$ by (2.1), so

$$\widetilde{f}(x) = \sum_{p\in A} \psi_p(x)\, \widetilde{f}_p(x) = \Bigl(\psi_0(x) + \sum_{p\in A} \psi_p(x)\Bigr) f(x) = f(x).$$

Thus, $\widetilde{f}$ is indeed an extension of f. Finally, we have

$$\operatorname{supp} \widetilde{f} \subseteq \overline{\bigcup_{p\in A} \operatorname{supp} \psi_p} = \bigcup_{p\in A} \operatorname{supp} \psi_\alpha \subseteq U,$$

where the equality in the middle is a property of locally finite collections, see [Lee13, Lemma 1.13]. □

Comments:

(1) The conclusion of the extension lemma can be false if A is not closed, see [Lee13, Exercise 2.27].

(2) The assumption in the extension lemma that the codomain is $\mathbb{R}^k$, and not some other manifold, is necessary: for other codomains, extensions can fail to exist for topological reasons. For example, the identity $\mathrm{Id}_{\mathbb{S}^1}\colon \mathbb{S}^1 \to \mathbb{S}^1$ is smooth, but does not even have a *continuous* extension to a map from $\mathbb{R}^2$ to $\mathbb{S}^1$.

(3) [Lee13, Corollary 6.27] asserts that a smooth map from a closed subset of a smooth manifold into a smooth manifold has a smooth extension if and only if it has a continuous one.

③ **Closed subsets as level sets:** The next result tells us that every closed subset of a smooth manifold can be expressed as a level set of some smooth real-valued function. This remarkable fact will not be used anywhere in these notes, so we omit its proof and we refer to [Lee13, Theorem 2.29] for the details.

Theorem 2.23. *Let M be a smooth manifold. If K is a closed subset of M, then there exists a smooth non-negative function $f\colon M \to \mathbb{R}$ such that $f^{-1}(0) = K$.*

Exercise 2.24: Let A and B be disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Solution: By Theorem 2.23 there exist non-negative smooth functions f_A and f_B on M such that

$$f_A^{-1}(0) = A \quad \text{and} \quad f_B^{-1}(0) = B. \tag{2.2}$$

Consider now the function

$$f\colon M \to \mathbb{R},\ x \mapsto \frac{f_A(x)}{f_A(x) + f_B(x)}$$

and observe that it is well-defined (that is, $f_A(x) + f_B(x) \neq 0$ for all $x \in M$) due to (2.2) and since $A \cap B = \emptyset$. Moreover, f is smooth as a quotient of smooth functions, and it satisfies

$$0 \leq f(x) \leq 1 \quad \text{for all } x \in M,$$

since f_A and f_B are non-negative. Finally, it follows from (2.2) that

$$f^{-1}(0) = A \quad \text{and} \quad f^{-1}(1) = B.$$

Hence, $f \in C^\infty(M)$ has the desired properties.

CHAPTER 3

THE TANGENT BUNDLE

The central idea of calculus is *linear approximation*. For example, a real-valued function of one variable can be approximated by its tangent line, a surface in $\mathbb{R}^3$ by its tangent plane, or a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ by its total derivative. In order to make sense of calculus on manifolds, we need to introduce the *tangent space to a manifold at a point*, which can be thought of as a sort of "linear model" for the manifold near the point. We carry out this task in Section 3.1, while in Section 3.2 we show how a smooth map between manifolds yields a linear map between tangent spaces, called the *differential* of the map, which generalizes the total derivative of a map between Euclidean spaces. The rest of the chapter is devoted to more concrete computations involving the differential and to the construction of the *tangent bundle* of a given smooth manifold.

3.1 Tangent Vectors

Given a point $a \in \mathbb{R}^n$, we define *the geometric tangent space to* $\mathbb{R}^n$ *at* a to be the set

$$\mathbb{R}^n_a := \{a\} \times \mathbb{R}^n = \big\{(a, v) \,\big|\, v \in \mathbb{R}^n\big\}.$$

A *geometric tangent vector* in $\mathbb{R}^n$ is an element of $\mathbb{R}^n_a$ for some $a \in \mathbb{R}^n$. We abbreviate (a, v) as v_a or $v|_a$, and we think of v_a as the vector v with initial point at a.

Figure 3.1: Geometric tangent space

The set $\mathbb{R}^n_a$ is an $\mathbb{R}$-vector space under the natural operations

$$\begin{aligned} v_a + w_a &:= (v + w)_a, \\ \lambda\, v_a &:= (\lambda\, v)_a. \end{aligned}$$

The vectors $e_i|_a$, $1 \le i \le n$, (where e_i denotes the i-th standard basis vector of $\mathbb{R}^n$) are a basis of $\mathbb{R}^n_a$. In fact, $\mathbb{R}^n_a$ is essentially the same as $\mathbb{R}^n$ itself; the only reason why we add the index a is so that the geometric tangent spaces $\mathbb{R}^n_a$ and $\mathbb{R}^n_b$ at distinct points a and b are disjoint sets.

Geometric tangent vectors provide a means of taking directional derivatives of functions. For example, any geometric tangent vector $v \in \mathbb{R}^n_a$ yields a map

$$\begin{aligned} \mathrm{D}_v\big|_a \colon C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \mathrm{D}_v\big|_a f = \mathrm{D}_v f(a) = \frac{d}{dt}\bigg|_{t=0} f(a+tv), \end{aligned}$$

the directional derivative of f in the direction v at a. This operation is $\mathbb{R}$-linear and satisfies the product rule:

$$\mathrm{D}_v\big|_a(fg) = f(a)\,\mathrm{D}_v\big|_a g + g(a)\,\mathrm{D}_v\big|_a f.$$

If $v_a = v^i e_i|_a$ in terms of the standard basis, then by the chain rule $\mathrm{D}_v\big|_a f$ can be written more concretely as

$$\mathrm{D}_v\big|_a f = v^i\,\frac{\partial f}{\partial x^i}(a). \tag{3.1}$$

In particular, if $v_a = e_j|_a$, then

$$\mathrm{D}_{e_j}\big|_a f = \frac{\partial f}{\partial x^j}(a). \tag{3.2}$$

With this construction in mind, we make the following definition.

Definition 3.1. Given $a \in \mathbb{R}^n$, a map $w\colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a *derivation at a* if it is $\mathbb{R}$-linear and satisfies the product rule:

$$w(fg) = f(a)\,w(g) + g(a)\,w(f).$$

We denote by $\mathrm{T}_a\mathbb{R}^n$ the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a. Clearly, $\mathrm{T}_a\mathbb{R}^n$ is an $\mathbb{R}$-vector space under the natural operations

$$\begin{aligned} (w_1 + w_2)(f) &\coloneqq w_1 f + w_2 f, \\ (\lambda w) f &\coloneqq \lambda\, w f. \end{aligned}$$

The most important fact about $\mathrm{T}_a\mathbb{R}^n$ is that it is finite-dimensional; in fact, it is naturally isomorphic to the geometric tangent space $\mathbb{R}^n_a$ that we defined above. The proof relies on the following lemma.

Lemma 3.2 (Properties of Derivations)**.** *Let $a \in \mathbb{R}^n$, $w \in \mathrm{T}_a\mathbb{R}^n$ and $f, g \in C^\infty(\mathbb{R}^n)$.*

(a) *If f is constant, then $wf = 0$.*

(b) *If $f(a) = g(a) = 0$, then $w(fg) = 0$.*

Proof.

(a) Consider the constant function $f_1 \equiv 1 \in C^\infty(\mathbb{R}^n)$. By the product rule we obtain

$$wf_1 = w(f_1 \cdot f_1) = \overset{1}{\cancel{f_1(a)}}\,wf_1 + \overset{1}{\cancel{f_1(a)}}\,wf_1 = 2wf_1,$$

which implies that $wf_1 = 0$. Now, since $f \equiv c$ is constant, by linearity we obtain

$$wf = w(cf_1) = c\,wf_1 = 0.$$

(b) Follows immediately from the product rule. □

Proposition 3.3. *Let $a \in \mathbb{R}^n$.*

(a) *For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map*

$$\begin{aligned} \mathrm{D}_v\big|_a \colon C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \mathrm{D}_v\big|_a f = \mathrm{D}_v f(a) = \frac{d}{dt}\bigg|_{t=0} f(a+tv) \end{aligned}$$

(directional derivative of f in the direction v at a) is a derivation of $C^\infty(\mathbb{R}^n)$ at a.

(b) *The map*

$$\begin{aligned} \Phi \colon \mathbb{R}^n_a &\to \mathrm{T}_a\mathbb{R}^n \\ v &\mapsto \mathrm{D}_v\big|_a \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism.

(c) *The n derivations*

$$\frac{\partial}{\partial x^1}\bigg|_a, \dots, \frac{\partial}{\partial x^n}\bigg|_a$$

defined by

$$\frac{\partial}{\partial x^i}\bigg|_a f := \frac{\partial f}{\partial x^i}(a)\,, \quad 1 \le i \le n,$$

form a basis of $\mathrm{T}_a\mathbb{R}^n$, and thus

$$\dim_{\mathbb{R}} \mathrm{T}_a\mathbb{R}^n = n.$$

Proof.

(a) Easy to check using calculus.

(b) *Linearity:* For every $f \in C^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} \Phi(\lambda_1 v_1 + \lambda_2 v_2)(f) &= \mathrm{D}_{\lambda_1 v_1 + \lambda_2 v_2}\big|_a(f) \\ &= \frac{d}{dt}\bigg|_{t=0} f\big(a + t(\lambda_1 v_1 + \lambda_2 v_2)\big) \\ &= \mathrm{D}f(a) \cdot (\lambda_1 v_1 + \lambda_2 v_2) \\ &= \lambda_1 \frac{d}{dt}\bigg|_{t=0} f(a+tv_1) + \lambda_2 \frac{d}{dt}\bigg|_{t=0} f(a+tv_2) \\ &= \lambda_1 \Phi(v_1)(f) + \lambda_2 \Phi(v_2)(f) \\ &= \big(\lambda_1\Phi(v_1) + \lambda_2\Phi(v_2)\big)(f), \end{aligned}$$

which shows the $\mathbb{R}$-linearity of Φ.

Injectivity: Suppose that $\Phi(v_a) = \mathrm{D}_v|_a = \mathbf{0}$ is the zero derivation. Writing $v_a = v^i\, e_i|_a$ in terms of the standard basis and considering the j-th coordinate function $x^j \colon \mathbb{R}^n \to \mathbb{R}$, thought of as a smooth function on $\mathbb{R}^n$, we obtain:

$$0 = \mathrm{D}_v\big|_a\, x^j \overset{(3.1)}{=\!=} v^i \frac{\partial}{\partial x^i}(x^j)\bigg|_{x=a} = v^j,$$

where the last equality follows because

$$\frac{\partial x^j}{\partial x^i} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Hence, $v_a = 0 \in \mathbb{R}^n_a$.

Surjectivity: Let $w \in \mathrm{T}_a\mathbb{R}^n$. Set $v := v^i\, e_i|_a \in \mathbb{R}^n_a$, where $v^i := w(x^i) \in \mathbb{R}$. We will show that $w = \Phi(v) = \mathrm{D}_v|_a$. To this end, let $f \in C^\infty(\mathbb{R}^n)$. By Taylor's theorem [Lee13, Theorem C.15] we can write

$$\begin{aligned} f(x) = f(a) &+ \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)\,(x^i - a^i) + \\ &+ \sum_{i,j=1}^n (x^i - a^i)\,\underbrace{(x^j - a^j) \int_0^1 (1-t)\, \frac{\partial^2 f}{\partial x^i \partial x^j}\big(a + t(x-a)\big)\, dt}\,. \end{aligned}$$

Note that each term in the last sum above is a product of two smooth functions of x that vanish at $x = a$: one is $(x^i - a^i)$ and the other is $(x^j - a^j) \cdot$ (integral). (The integral is a smooth function of x by iterative application of [Lee13, Theorem C.14].) By Lemma 3.2(b) the derivation w annihilates this entire sum. Thus, thanks again to the $\mathbb{R}$-linearity of w and Lemma 3.2(a), we obtain

$$\begin{aligned} wf &= \cancelto{0}{w(f(a))} + \sum_{i=1}^n w\left(\frac{\partial f}{\partial x^i}(a)\,(x^i - a^i)\right) \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)\,\big(\cancelto{v^i}{w(x^i)} - \cancelto{0}{w(a^i)}\big) \\ &= \sum_{i=1}^n v^i\, \frac{\partial f}{\partial x^i}(a) = \mathrm{D}_v\big|_a\, f. \end{aligned}$$

(c) Follows immediately from (b) and (3.2). ☐

Definition 3.4. Let M be a smooth manifold and let $p \in M$. A map $v \colon C^\infty(M) \to \mathbb{R}$ is called a *derivation at* p if it is $\mathbb{R}$-linear and satisfies the product rule:

$$v(fg) = f(p)\, v(g) + g(p)\, v(f), \quad \forall\, f, g \in C^\infty(M).$$

We denote by T_pM the set of all derivations of $C^\infty(M)$ at p. Clearly, T_pM is an $\mathbb{R}$-vector space, called *the tangent space to* M *at* $p \in M$. An element of T_pM is called a *tangent vector at* p.

Lemma 3.5 (Properties of Derivations)**.** *Let M be a smooth manifold, $p \in M$, $v \in \mathrm{T}_pM$ and $f, g \in C^\infty(M)$.*

(a) *If f is constant, then $vf = 0$.*

(b) *If $f(p) = g(p) = 0$, then $v(fg) = 0$.*

Proof. Exercise! (Essentially identical to the proof of Lemma 3.2.) ☐

With the motivation of geometric tangent vectors in $\mathbb{R}^n$ in mind, we visualize tangent vectors to M as "arrows" that are tangent to M and whose base points are attached to M at the given point. For other descriptions of tangent vectors to M, see Section 3.5 and [Lee13, Chapter 3, Alternative Descriptions of the Tangent Space].

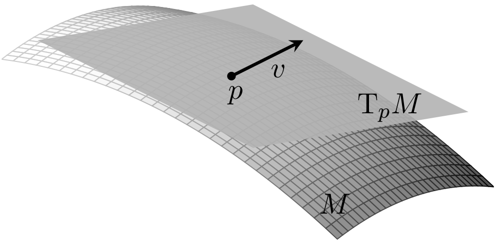


Figure 3.2: Tangent vector to a manifold at a point

3.2 The Differential of a Smooth Map

Definition 3.6. If $F\colon M \to N$ is a smooth map, then for each $p \in M$ we define a map

$$\mathrm{d}F_p\colon \mathrm{T}_pM \to \mathrm{T}_{F(p)}N,$$

called *the differential* (or *tangent map*) *of F at p*, as follows. Given $v \in \mathrm{T}_pM$, we let $\mathrm{d}F_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by

$$\mathrm{d}F_p(v)(f) = v(f \circ F)\,.$$

The operator $\mathrm{d}F_p(v)\colon C^\infty(N) \to \mathbb{R}$ is a derivation at $F(p)$. Indeed, it is $\mathbb{R}$-linear, since v is so, and it satisfies the product rule:

$$\begin{aligned}\mathrm{d}F_p(v)(fg) &= v\big((fg)\circ F\big) = v\big((f\circ F)(g\circ F)\big)\\ &= (f\circ F)(p)\,v(g\circ F) + (g\circ F)(p)\,v(f\circ F)\\ &= f\big(F(p)\big)\,\mathrm{d}F_p(v)(g) + g\big(F(p)\big)\,\mathrm{d}F_p(v)(f).\end{aligned}$$

Proposition 3.7 (Properties of differentials)**.** *Let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps and let $p \in M$.*

(a) *$\mathrm{d}F_p\colon \mathrm{T}_pM \to \mathrm{T}_{F(p)}N$ is an $\mathbb{R}$-linear map.*

(b) *$\mathrm{d}(G\circ F) = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p\colon \mathrm{T}_pM \to \mathrm{T}_{(G\circ F)(p)}P$.*

(c) $\mathrm{d}(\mathrm{Id}_M)_p = \mathrm{Id}_{\mathrm{T}_pM} \colon \mathrm{T}_pM \to \mathrm{T}_pM$.

(d) *If* F *is a diffeomorphism, then* $\mathrm{d}F_p \colon \mathrm{T}_pM \to \mathrm{T}_{F(p)}N$ *is an isomorphism, and it holds that* $(\mathrm{d}F_p)^{-1} = \mathrm{d}(F^{-1})_{F(p)}$.

Proof. See [*Exercise Sheet* 4, *Exercise* 1]. ☐

Our first important application of the differential will be to use coordinate charts to relate the tangent space to a point on a smooth manifold with the Euclidean tangent space. But there is an important technical issue that we must address first. While the tangent space is defined in terms of smooth functions on the whole manifold, coordinate charts are in general defined only on open subsets. The key point, expressed in the next proposition, is that tangent vectors act locally.

Proposition 3.8. *Let* M *be a smooth manifold,* $p \in M$ *and* $v \in \mathrm{T}_pM$. *If* $f, g \in C^\infty(M)$ *agree on some neighborhood of* p*, then* $vf = vg$.

Proof. Set $h := f - g$ and observe that h is a smooth function on M that vanishes in a neighborhood U of p. By Proposition 2.20 there exists a smooth bump function ψ for $\operatorname{supp} h$ supported in $M \setminus \{p\}$ (this is an open subset of M which contains $\operatorname{supp} h$, since $h(x) = 0$ for all $x \in U$). Since $\psi \equiv 1$ where h is non-zero, the product ψh is identically equal to h. Since $h(p) = \psi(p) = 0$, by Lemma 3.5(b) we obtain $v(h) = v(\psi h) = 0$, so $v(f) = v(g)$ by linearity. ☐

Using Proposition 3.8, we can identify the tangent space to an open submanifold with the tangent space to the whole manifold.

Proposition 3.9 (The tangent space to an open submanifold)**.** *Let* M *be a smooth manifold, let* $U \subseteq M$ *be an open subset and let* $\iota \colon U \hookrightarrow M$ *be the inclusion map. For every* $p \in U$*, the differential* $\mathrm{d}\iota_p \colon \mathrm{T}_pU \to \mathrm{T}_pM$ *is an isomorphism.*

Proof. Recall that U is a smooth manifold by Example 1.10(4) and that ι is a smooth map by Proposition 2.11(c). Fix now $p \in U$, consider the differential $\mathrm{d}\iota_p \colon \mathrm{T}_pU \to \mathrm{T}_pM$, and let V be a neighborhood of p such that $\overline{V} \subseteq U$.

Injectivity: Let $v \in \mathrm{T}_pU$ such that $\mathrm{d}\iota_p(v) = 0 \in \mathrm{T}_pM$. If $f \in C^\infty(U)$ is arbitrary, then by Lemma 2.22 there exists $\widetilde{f} \in C^\infty(M)$ such that $\widetilde{f}|_{\overline{V}} = f$. Since then f and $\widetilde{f}|_U$ are smooth functions on U that agree in a neighborhood of p, Proposition 3.8 implies

$$vf = v\big(\widetilde{f}\big|_U\big) = v\big(\widetilde{f} \circ \iota\big) = \mathrm{d}\iota_p(v)\left(\widetilde{f}\right) = 0.$$

Hence, $v = 0 \in \mathrm{T}_pU$, and thus $\mathrm{d}\iota_p$ is injective.

Surjectivity: Let $w \in \mathrm{T}_pM$. Define

$$v \colon C^\infty(U) \to \mathbb{R}, \ f \mapsto w\widetilde{f},$$

where $\widetilde{f}$ is any smooth function on M that agrees with f on $\overline{V}$, see Lemma 2.22. By Proposition 3.8, vf is independent of the choice of $\widetilde{f}$, so v is well-defined, and it is easy to check that it is a derivation of $C^\infty(U)$ at p. For any $g \in C^\infty(M)$ we have

$$\mathrm{d}\iota_p(v)(g) = v(g \circ \iota) = w(\widetilde{g \circ \iota}) = wg,$$

where the last equality follows from the fact that $g \circ \iota$, $\widetilde{g \circ \iota}$ and g all agree on V. Hence, $\mathrm{d}\iota_p(v) = w$, and thus $\mathrm{d}\iota_p$ is surjective. ☐

Given an open subset $U \subseteq M$, the isomorphism $\mathrm{d}\iota_p$ from T_pU to T_pM is canonically defined, independent of any choices. From now on we *identify* T_pU with T_pM for any $p \in U$. This identification just amounts to the observation that $\mathrm{d}\iota_p(v)$ is the same derivation as v, though of as acting on functions on the bigger manifold M instead of on functions on U. Since the action of a derivation on a function depends only on the values of the function in an arbitrarily small neighborhood (see Proposition 3.8), this is a harmless identification. In particular, this means that any tangent vector $v \in \mathrm{T}_pM$ can be unambiguously applied to the functions defined only in a neighborhood of p, not necessarily on all of M.

Proposition 3.10. *If M is a smooth n-manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional $\mathbb{R}$-vector space.*

Proof. Fix $p \in M$ and let (U, φ) be a smooth coordinate chart containing p. Since $\varphi\colon U \to \widehat{U} \subseteq \mathbb{R}^n$ is a diffeomorphism by Example 2.14(2), $\mathrm{d}\varphi_p\colon \mathrm{T}_pU \to \mathrm{T}_{\varphi(p)}\widehat{U}$ is an isomorphism by Proposition 3.7(d). Since Proposition 3.9 guarantees that $\mathrm{T}_pU \cong \mathrm{T}_pM$ and $\mathrm{T}_{\varphi(p)}\widehat{U} \cong \mathrm{T}_{\varphi(p)}\mathbb{R}^n$, it follows from Proposition 3.3(c) that

$$\dim \mathrm{T}_pM = \dim \mathrm{T}_{\varphi(p)}\mathbb{R}^n = n. \qquad \square$$

3.3 Computations in Local Coordinates

In this section we indicate how to do computations with tangent vectors and differentials in local coordinates.

Let M be a smooth manifold and let (U, φ) be a smooth coordinate chart on M. As already explained in the proof of Proposition 3.10, $\mathrm{d}\varphi_p\colon \mathrm{T}_pM \to \mathrm{T}_{\varphi(p)}\mathbb{R}^n$ is an $\mathbb{R}$-linear isomorphism (for each $p \in U$). By Proposition 3.3(c) the derivations

$$\left.\frac{\partial}{\partial x^1}\right|_{\varphi(p)}, \ldots, \left.\frac{\partial}{\partial x^n}\right|_{\varphi(p)}$$

form a basis of $\mathrm{T}_{\varphi(p)}\mathbb{R}^n$. Therefore, the preimages of these vectors under the isomorphism $\mathrm{d}\varphi_p$ form a basis of T_pM. These vectors are denoted by $\left.\frac{\partial}{\partial x^i}\right|_p$ and are characterized by

$$\left.\frac{\partial}{\partial x^i}\right|_p := (\mathrm{d}\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) \overset{\text{Prop.}}{\underset{3.7(d)}{=\!=}} \mathrm{d}(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right). \tag{3.3}$$

Unwinding the definitions, we see that $\left.\frac{\partial}{\partial x^i}\right|_p$ acts on a function $f \in C^\infty(U)$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} (f \circ \varphi^{-1}) = \frac{\partial \widehat{f}}{\partial x^i}(\widehat{p}),$$

where $\widehat{f} := f \circ \varphi^{-1}$ is the coordinate representation of f and $\widehat{p} = (p^1, \ldots, p^n) = \varphi(p)$ is the coordinate representation of p. In other words, $\left.\frac{\partial}{\partial x^i}\right|_p$ is the derivation at p that takes the i-th partial derivative of (the coordinate representation of) f at (the coordinate representation of) p. The vectors $\left.\frac{\partial}{\partial x^i}\right|_p$ are called *the coordinate vectors at p* associated with the given coordinate system. In the special case of standard coordinates on $\mathbb{R}^n$, the vectors $\frac{\partial}{\partial x^i}\big|_p$ are literally the partial derivative operators.

To summarize, if M is a smooth n-manifold and if $p \in M$, then T_pM is an n-dimensional $\mathbb{R}$-vector space, and for any smooth coordinate chart $\big(U, (x^i)\big)$ containing p, the coordinate vectors $\big\{\frac{\partial}{\partial x^i}\big|_p\big\}_{i=1}^n$ form a basis for T_pM. Thus, a tangent vector $v \in \mathrm{T}_pM$ can be written uniquely as a linear combination

$$v = v^i \frac{\partial}{\partial x^i}\bigg|_p .$$

The ordered basis $\big(\frac{\partial}{\partial x^i}\big|_p\big)$ is called a *coordinate basis for* T_pM and the numbers (v^i) are called *the components of* v with respect to the coordinate basis. If v is known, then its components can be easily computed from its action on the coordinate functions. Indeed, for each $j \in \{1, \dots, n\}$, the components of v are given by $v^j = v(x^j)$ (where we think of x^j as a smooth real-valued function on U), because

$$v(x^j) = \left(v^i \frac{\partial}{\partial x^i}\bigg|_p\right)(x^j) = v^i \frac{\partial x^j}{\partial x^i}(p) = v^j .$$

Next, we explore how differentials look in local coordinates. We begin by considering the case of a smooth map $F\colon U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ between open subsets of Euclidean spaces. For any $p \in U$ we will determine the matrix of $\mathrm{d}F_p\colon \mathrm{T}_p\mathbb{R}^n \to \mathrm{T}_{F(p)}\mathbb{R}^m$ in terms of the standard coordinate bases. Denoting by $(x^1, \dots, x^n)$ (respectively $(y^1, \dots, y^m)$) the coordinates in the domain (respectively codomain), we use the chain rule to compute the action of $\mathrm{d}F_p$ on a typical basis vector as follows:

$$\begin{aligned} \mathrm{d}F_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) f &= \frac{\partial}{\partial x^i}\bigg|_p (f \circ F) = \frac{\partial f}{\partial y^j}\big(F(p)\big)\,\frac{\partial F^j}{\partial x^i}(p) \\ &= \left(\frac{\partial F^j}{\partial x^i}(p)\,\frac{\partial}{\partial y^j}\bigg|_{F(p)}\right) f. \end{aligned}$$

Thus,

$$\mathrm{d}F_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial F^j}{\partial x^i}(p)\,\frac{\partial}{\partial y^j}\bigg|_{F(p)} . \tag{3.4}$$

In other words, the matrix of $\mathrm{d}F_p$ in terms of the coordinate bases is

$$\begin{pmatrix} \dfrac{\partial F^1}{\partial x^1}(p) & \cdots & \dfrac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F^m}{\partial x^1}(p) & \cdots & \dfrac{\partial F^m}{\partial x^n}(p) \end{pmatrix},$$

that is, the *Jacobian matrix of* F *at* p, which is the matrix representation of the total derivative $\mathrm{D}F(p)\colon \mathbb{R}^n \to \mathbb{R}^m$. Therefore, in this case, $\mathrm{d}F_p\colon \mathrm{T}_p\mathbb{R}^n \to \mathrm{T}_{F(p)}\mathbb{R}^m$ corresponds to the total derivative $\mathrm{D}F(p)\colon \mathbb{R}^n \to \mathbb{R}^m$, under the usual identification of Euclidean spaces with their tangent spaces (see Proposition 3.3).

We now consider the more general case of a smooth map $F\colon M \to N$ between two smooth manifolds. Choosing smooth coordinate charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$, we obtain the coordinate representation

$$\widehat{F} = \psi \circ F \circ \varphi^{-1}\colon \varphi\big(U \cap F^{-1}(V)\big) \to \psi(V)$$

of F, see Figure 3.3, and we also denote by $\widehat{p} = \varphi(p)$ the coordinate representation of p. By the computation above, $\mathrm{d}\widehat{F}_{\widehat{p}}$ is represented with respect to the standard coordinates bases by the Jacobian matrix of $\widehat{F}$ at $\widehat{p}$. Using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ \widehat{F}$, we compute

$$\begin{aligned}
\mathrm{d}F_p\bigg(\frac{\partial}{\partial x^i}\bigg|_p\bigg) &\overset{\text{dfn}}{=\!=} \mathrm{d}F_p\bigg(\mathrm{d}(\varphi^{-1})_{\widehat{p}}\bigg(\frac{\partial}{\partial x^i}\bigg|_{\widehat{p}}\bigg)\bigg) \\
&\underset{3.7(b)}{\overset{\text{Prop.}}{=\!=}} \mathrm{d}(\underbrace{F \circ \varphi^{-1}}_{\psi^{-1}\circ\widehat{F}})_{\widehat{p}}\bigg(\frac{\partial}{\partial x^i}\bigg|_{\widehat{p}}\bigg) \\
&\underset{3.7(b)}{\overset{\text{Prop.}}{=\!=}} \mathrm{d}(\psi^{-1})_{\widehat{F}(\widehat{p})}\bigg(\mathrm{d}\widehat{F}_{\widehat{p}}\bigg(\frac{\partial}{\partial x^i}\bigg|_{\widehat{p}}\bigg)\bigg) \\
&\overset{(3.4)}{=\!=} \mathrm{d}(\psi^{-1})_{\widehat{F}(\widehat{p})}\bigg(\frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\,\frac{\partial}{\partial y^j}\bigg|_{\widehat{F}(\widehat{p})}\bigg) \\
&\underset{\text{linearity}}{\overset{\text{dfn and}}{=\!=}} \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\,\frac{\partial}{\partial y^j}\bigg|_{F(p)}. \qquad (3.5)
\end{aligned}$$

Thus, $\mathrm{d}F_p$ is represented in coordinate bases by the Jacobian matrix of (the coordinate representation of) F. In fact, the definition of the differential was cooked up precisely in order to give a coordinate-independent meaning to the Jacobian matrix.

Figure 3.3: The differential in coordinates

Finally, suppose that $\big(U, \varphi = (x^i)\big)$ and $\big(V, \psi = (\widetilde{x}^i)\big)$ are two smooth charts on M and that $p \in U \cap V$. Any tangent vector at p can be represented with respect to either coordinates basis $\big(\frac{\partial}{\partial x^i}\big|_p\big)$ or $\big(\frac{\partial}{\partial \widetilde{x}^i}\big|_p\big)$. How are the two representations related?

Figure 3.4: Change of coordinates

In this situation it is customary to write the transition map $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ in the following notation:

$$(\psi \circ \varphi^{-1})(x) = \big(\widetilde{x}^1(x), \dots, \widetilde{x}^n(x)\big)\,.$$

Here we are indulging in a typical abuse of notation: in the expression $\widetilde{x}^i(x)$ we think of $\widetilde{x}^i$ as a coordinate *function* (whose domain is an open subset of M, identified with an open subset of $\mathbb{R}^n$), but we think of x as representing a *point* (in this case, in $\varphi(U \cap V)$). By (3.4) we have

$$\mathrm{d}(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} \right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(\varphi(p)) \, \frac{\partial}{\partial \widetilde{x}^j}\bigg|_{\psi(p)} .$$

Using the definition of coordinate vectors, we obtain

$$\begin{aligned}
\frac{\partial}{\partial x^i}\bigg|_p &\overset{(3.3)}{=\!=\!=} \mathrm{d}(\varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} \right) \\
&\underset{3.7(b)}{\overset{\text{Prop.}}{=\!=\!=}} \mathrm{d}(\psi^{-1})_{\psi(p)} \cdot \mathrm{d}(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} \right) \\
&=\!=\!= \mathrm{d}(\psi^{-1})_{\psi(p)} \left(\frac{\partial \widetilde{x}^j}{\partial x^i}(\varphi(p)) \, \frac{\partial}{\partial \widetilde{x}^j}\bigg|_{\psi(p)} \right) \\
&\underset{\text{linearity}}{\overset{(3.3)}{=\!=\!=}} \frac{\partial \widetilde{x}^j}{\partial x^i} \underbrace{(\varphi(p))}_{=\widehat{p}} \frac{\partial}{\partial \widetilde{x}^j}\bigg|_p .
\end{aligned} \tag{3.6}$$

(This formula looks exactly the same as the chain rule for partial derivatives in $\mathbb{R}^n$.) Applying this to the components of a vector

$$v = v^i \, \frac{\partial}{\partial x^i}\bigg|_p = \widetilde{v}^j \, \frac{\partial}{\partial \widetilde{x}^j}\bigg|_p ,$$

we find that the components of v transform by the rule

$$\widetilde{v}^j = \frac{\partial \widetilde{x}^j}{\partial x^i}(\widehat{p}) \, v^i . \tag{3.7}$$

3.4 The Tangent Bundle

Definition 3.11. Let M be a smooth manifold. The *tangent bundle* of M is denoted by $\mathrm{T}M$ and is defined as the disjoint union of the tangent spaces at all points of M:

$$\mathrm{T}M = \bigsqcup_{p \in M} \mathrm{T}_p M .$$

We usually write an element of this disjoint union as an ordered pair (p, v) with $p \in M$ and $v \in \mathrm{T}_p M$; we sometimes also write v_p for (p, v). The tangent bundle comes equipped with a natural projection map $\pi \colon \mathrm{T}M \to M$, which sends each vector in $\mathrm{T}_p M$ to the point p at which it is tangent: $(p, v) \mapsto p$.

For example, when $M = \mathbb{R}^n$, using Proposition 3.3, we see that the tangent bundle of $\mathbb{R}^n$ can be canonically identified with the disjoint union of its geometric tangent spaces, which in turn is just the Cartesian product of $\mathbb{R}^n$ with itself:

$$\mathrm{T}(\mathbb{R}^n) = \bigsqcup_{p \in \mathbb{R}^n} \mathrm{T}_p \mathbb{R}^n \cong \bigsqcup_{p \in \mathbb{R}^n} \mathbb{R}^n_p = \bigsqcup_{p \in \mathbb{R}^n} \{p\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n .$$

An element of this Cartesian product can be thought of as representing either the geometric tangent vector v_p or the derivation $\mathrm{D}_v|_p$ defined in Proposition 3.3(a). In general, however, the tangent bundle of a smooth manifold cannot be identified in a natural way with a Cartesian product, because there is no canonical way to identify tangent spaces at distinct points with each other.

The tangent bundle of a smooth manifold can be thought of simply as a disjoint union of vector spaces, but it is much more than that. The next proposition shows that the tangent bundle of a smooth manifold can be considered as a smooth manifold in its own right. For its proof we will use Lemma 1.11.

Proposition 3.12. *For any smooth n-manifold M, the tangent bundle $\mathrm{T}M$ has a natural topology and smooth structure that make it into a smooth $(2n)$-manifold. With respect to this structure, the projection $\pi\colon \mathrm{T}M \to M$ is smooth.*

Proof. We begin by defining the maps that will become our smooth charts. Given any smooth chart (U,φ) for M, observe that $\pi^{-1}(U)$ is the set of all tangent vectors to M at all points of U. Denote by $(x^1,\dots,x^n)$ the coordinate functions of φ, and define a map

$$\begin{gathered}\widetilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^{2n},\\ \widetilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = \big(x^1(p),\dots,x^n(p),v^1,\dots,v^n\big)\,. \end{gathered}\tag{3.8}$$

Its image is the set $\varphi(U)\times\mathbb{R}^n$, which is an open subset of $\mathbb{R}^{2n}$. It is a bijection onto its image, because its inverse can be explicitly written as

$$\widetilde{\varphi}^{-1}\big(x^1,\dots,x^n,v^1,\dots,v^n\big) = v^i \frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(x)}.$$

Now, suppose that we are given two smooth charts (U,φ) and (V,ψ) for M, and consider the corresponding "charts" $\big(\pi^{-1}(U),\widetilde{\varphi}\big)$ and $\big(\pi^{-1}(V),\widetilde{\psi}\big)$ for $\mathrm{T}M$. The sets

$$\widetilde{\varphi}\big(\pi^{-1}(U)\cap\pi^{-1}(V)\big) = \varphi(U\cap V)\times\mathbb{R}^n$$

and

$$\widetilde{\psi}\big(\pi^{-1}(U)\cap\pi^{-1}(V)\big) = \psi(U\cap V)\times\mathbb{R}^n$$

are open in $\mathbb{R}^{2n}$, and the transition map

$$\widetilde{\psi}\circ\widetilde{\varphi}^{-1}\colon \varphi(U\cap V)\times\mathbb{R}^n \to \psi(U\cap V)\times\mathbb{R}^n$$

can be written explicitly as

$$\begin{aligned}\big(\widetilde{\psi}\circ\widetilde{\varphi}^{-1}\big)\big(x^1,\dots,x^n,v^1,\dots,v^n\big) &= \widetilde{\psi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(x)}\right)\\ &\overset{(3.7)}{=} \widetilde{\psi}\left(\left(v^i \frac{\partial \widetilde{x}^j}{\partial x^i}\right)\frac{\partial}{\partial \widetilde{x}^j}\bigg|_{\varphi^{-1}(x)}\right)\\ &= \left(\widetilde{x}^1,\dots,\widetilde{x}^n,\frac{\partial \widetilde{x}^1}{\partial x^i}v^i,\dots,\frac{\partial \widetilde{x}^n}{\partial x^i}v^i\right),\end{aligned}$$

which is clearly smooth.

Choosing a countable cover $\{U_i\}$ of M by smooth coordinate domains, we obtain a countable cover of $\mathrm{T}M$ by coordinate domains $\{\pi^{-1}(U_i)\}$ satisfying conditions (i)-(iv) of Lemma 1.11. To check the Hausdorff condition (v), just note that any two points in the same fiber of π lie in one chart, while if (p, v) and (q, w) lie in different fibers, there exist disjoint smooth coordinate domains U and V for M such that $p \in U$ and $q \in V$, and then $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint coordinate neighborhoods containing (p, v) and (q, w), respectively. This completes the proof of the first part of the statement.

Finally, to check that the projection $\pi\colon \mathrm{T}M \to M$ is smooth, note that with respect to charts (U, φ) for M and $\big(\pi^{-1}(U), \widetilde{\varphi}\big)$ for $\mathrm{T}M$, its coordinate representation $\varphi \circ \pi \circ \widetilde{\varphi}^{-1}$ is $\widehat{\pi}(x, v) = x$, which is clearly smooth. ☐

The coordinates (x^i, v^i) given by (3.8) are called *natural coordinates on* $\mathrm{T}M$.

Proposition 3.13. *If M is a smooth n-manifold which can be covered by a single smooth chart, then its tangent bundle $\mathrm{T}M$ is diffeomorphic to $M \times \mathbb{R}^n$.*

Proof. If (U, φ) is a global smooth chart for M, then φ is, in particular, a diffeomorphism from $U = M$ to an open subset $\widehat{U} \subseteq \mathbb{R}^n$, see Example 2.14(2). The proof of Proposition 3.12 showed that the natural coordinate chart $\widetilde{\varphi}$ is a bijection from $\mathrm{T}M$ to $\widehat{U} \times \mathbb{R}^n$, and the smooth structure on $\mathrm{T}M$ is defined essentially by declaring $\widetilde{\varphi}$ to be diffeomorphism. ☐

Comment: In general, the tangent bundle is not globally diffeomorphic (or ever homeomorphic) to a product of the manifold with $\mathbb{R}^n$.

3.4.1 The Global Differential of a Smooth Map

Let $F\colon M \to N$ be a smooth map. By putting together the differentials of F at all points of M, we obtain a globally defined map between tangent bundles, called *the global differential* and denoted by $\mathrm{d}F\colon \mathrm{T}M \to \mathrm{T}N$. This is just the map whose restriction to each tangent space $\mathrm{T}_pM \subseteq \mathrm{T}M$ is $\mathrm{d}F_p$.

One important feature of the smooth structure we have defined on the tangent bundle is that it makes the differential of a smooth map into a smooth map between tangent bundles.

Exercise 3.14: Show that if $F\colon M \to N$ is a smooth map, then its global differential $\mathrm{d}F\colon \mathrm{T}M \to \mathrm{T}N$ is also a smooth map.

Solution: Using the local expression for dF_p in coordinates, see (3.5), we see that dF has the following coordinate representation in terms of natural coordinates for TM and TN:

$$
\begin{aligned}
\big(\widetilde{\psi} \circ dF \circ \widetilde{\varphi}^{-1}\big)(x^1, \dots, x^n, v^1, \dots, v^n) &= \big(\widetilde{\psi} \circ dF\big)\left(v^i \frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(x)} \right) \\
&= \widetilde{\psi}\left(v^i \frac{\partial \widehat{F}^j}{\partial x^i} \frac{\partial}{\partial y^j}\bigg|_{F \circ \varphi^{-1}(x)} \right) \\
&= \left(\widehat{F}^1(x), \dots, \widehat{F}^n(x), \frac{\partial \widehat{F}^1}{\partial x^i}(x) v^i, \dots, \frac{\partial \widehat{F}^n}{\partial x^i}(x) v^i \right).
\end{aligned}
$$

Since F is smooth, and thus its coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is smooth, the above coordinate representation of dF is smooth, and hence dF is smooth, as claimed.

Proposition 3.15 (Properties of the global differential)**.** *Let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps. The following statements hold:*

(a) $\mathrm{d}(G \circ F) = \mathrm{d}G \circ \mathrm{d}F \colon \mathrm{T}M \to \mathrm{T}P$.

(b) $\mathrm{d}(\mathrm{Id}_M) = \mathrm{Id}_{\mathrm{T}M} \colon \mathrm{T}M \to \mathrm{T}M$.

(c) *If F is a diffeomorphism, then $\mathrm{d}F\colon \mathrm{T}M \to \mathrm{T}N$ is also a diffeomorphism, and it holds that $(\mathrm{d}F)^{-1} = \mathrm{d}(F^{-1})$.*

Proof. All assertions follow immediately from Proposition 3.7. ☐

3.5 Velocity Vectors of Curves

Recall: A *continuous (parametrized) curve* in a topological manifold M is a continuous map $\gamma\colon J \to M$, where $J \subseteq \mathbb{R}$ is an interval.

Our definition of tangent spaces leads to a natural interpretation of velocity vectors.

Definition 3.16. Let M be a smooth manifold.

(a) A *smooth (parametrized) curve* in M is a smooth map $\gamma\colon J \to M$, where $J \subseteq \mathbb{R}$ is an interval.

(b) Given a smooth curve $\gamma\colon J \to M$ in M and an instant $t_0 \in J$, the *velocity of γ at t_0* is defined to be the tangent vector

$$\gamma'(t_0) \coloneqq d\gamma\left(\left.\frac{d}{dt}\right|_{t=t_0}\right) \in T_{\gamma(t_0)}M,$$

where $d/dt|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$. Other common notations for the velocity vector are:

$$\dot{\gamma}(t_0), \quad \frac{d\gamma}{dt}(t_0) \quad \text{and} \quad \left.\frac{d\gamma}{dt}\right|_{t=t_0}.$$

Figure 3.5: Velocity vector of a curve

Assume that M, γ and t_0 are as in Definition 3.16. The tangent vector $\gamma'(t_0)$ acts on functions $f \in C^\infty(M)$ by

$$\gamma'(t_0)f = d\gamma\left(\left.\frac{d}{dt}\right|_{t=t_0}\right)f = \left.\frac{d}{dt}\right|_{t=t_0}(f \circ \gamma) = (f \circ \gamma)'(t_0).$$

In other words, $\gamma'(t_0)$ is the derivation at $\gamma(t_0)$ obtained by taking the derivative of a function along γ. (If t_0 is an endpoint of the interval $J \subseteq \mathbb{R}$, this still holds, provided that we interpret the derivative with respect to t as a one-sided derivative, or equivalently as the derivative of any smooth extension of $f \circ \gamma$ to an open subset of $\mathbb{R}$.)

Now, let (U, φ) be a smooth chart for M with coordinate functions (x^i). If $\gamma(t_0) \in U$, then we can write the coordinate representation of γ as

$$\widehat{\gamma}(t) = \big(\gamma^1(t), \dots, \gamma^n(t)\big),$$

at least for $t \in J$ sufficiently close to $t_0 \in J$, and then the coordinate formula for the differential (3.5) yields

$$\gamma'(t_0) = \frac{d\gamma^i}{dt}(t_0) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)}. \tag{3.9}$$

This means that $\gamma'(t_0)$ is given by essentially the same formula as it would be in Euclidean space: it is the tangent vector whose components in a coordinate basis are the derivatives of the component functions of γ.

The next proposition shows that every tangent vector on a manifold is the velocity vector of some curve. This gives a different and somewhat more geometric way to think about the tangent bundle: it is just the set of all velocity vectors of smooth curves in M.

Proposition 3.17 (Tangent vectors as velocity vectors of smooth curves)**.** *Let M be a smooth manifold. If $p \in M$, then for any $v \in T_pM$ there exists a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.*

Proof. See [*Exercise Sheet* 4, *Exercise* 5(*a*)]. ☐

Proposition 3.18 (The velocity of a composite curve)**.** *If $F\colon M \to N$ is a smooth map and if $\gamma\colon J \to M$ is a smooth curve, then for any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma\colon J \to N$ is given by*

$$(F \circ \gamma)'(t_0) = dF_{\gamma(t_0)}\big(\gamma'(t_0)\big).$$

Proof. See [*Exercise Sheet* 4, *Exercise* 5(*b*)]. ☐

Corollary 3.19 (Computing the differential using a velocity vector)**.** *If $F\colon M \to N$ is a smooth map, $p \in M$ and $v \in T_pM$, then*

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma\colon J \to M$ such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.

Proof. See [*Exercise Sheet* 4, *Exercise* 5(*c*)]. ☐

CHAPTER 4

MAPS OF CONSTANT RANK

Since the differential of a smooth map is supposed to represent the "best linear approximation" to the map near a given point, we can learn a great deal about a map by studying linear-algebraic properties of its differential. The most essential property of the differential is its *rank* (i.e., the dimension of its image). In this chapter we study the ways in which geometric properties of smooth maps can be detected from their differentials. The maps for which differentials give good local models turn out to be the ones whose differentials have constant rank.

4.1 Immersions, Submersions, and Embeddings

Definition 4.1. Given a smooth map $F\colon M \to N$ and a point $p \in M$, *the rank of F at p* is defined to be the rank of the linear map $\mathrm{d}F_p\colon \mathrm{T}_pM \to \mathrm{T}_{F(p)}N$; it is the rank of the Jacobian matrix of F in any smooth chart, or the dimension of the image $\mathrm{Im}(\mathrm{d}F_p) \subseteq \mathrm{T}_{F(p)}N$. If F has the same rank r at every point, then we say that it has *constant rank* and we write $\operatorname{rk} F = r$.

Note that the rank of F at each point is bounded above by $\min\{\dim M, \dim N\}$. If the rank of $\mathrm{d}F_p$ is equal to this upper bound, then we say that F has *full rank at p*. If F has full rank everywhere, then we say that F has *full rank*.

Definition 4.2. A smooth map $F\colon M \to N$ is called

(a) a *smooth immersion* if its differential is injective at each point or, equivalently, if $\operatorname{rk} F = \dim M$;

(b) a *smooth submersion* if its differential is surjective at each point or, equivalently, if $\operatorname{rk} F = \dim N$; and

(c) a *smooth embedding* if it is a smooth immersion that is also *topological embedding*, i.e., a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology.

Comment: A smooth embedding is a map that is both a topological embedding and a smooth immersion, not just a topological embedding that happens to be smooth; see Example 4.5(1).

Smooth immersions and smooth embeddings are essential ingredients in the theory of submanifolds (see Chapter 5), while smooth submersions play a role in smooth manifold theory analogous to the role played by quotient maps in topology (see Subsection 4.3.1 and [Lee13, Chapter 4, Smooth Covering Maps]). We will see that smooth immersions and submersions behave locally like injective and surjective linear maps, respectively (see Theorem 4.12).

Lemma 4.3. *Let $F\colon M \to N$ be a smooth map. If $\mathrm{d}F_p$ is injective (respectively surjective) for some $p \in M$, then p has a neighborhood U such that $F|_U$ is an immersion (respectively submersion).*

Proof. If we choose any smooth coordinates for M near p and for N near $F(p)$, then either hypothesis means that the Jacobian matrix of F in these coordinates has full rank at $p \in M$. By [*Exercise Sheet* 2*, Exercise* 3] we know that the set of $n \times m$ matrices of full rank is an open subset of $\mathrm{M}(n \times m, \mathbb{R})$, where $m = \dim M$ and $n = \dim N$, so by continuity, the Jacobian matrix of F (in coordinates) has full rank in some neighborhood of $p \in M$. ☐

Example 4.4.

(1) If $\gamma\colon J \to M$ is a smooth curve in a smooth manifold M, then γ is an immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$.

(2) If M is a smooth manifold and its tangent bundle $\mathrm{T}M$ is given the smooth manifold structure described in Proposition 3.12, then the projection $\pi\colon \mathrm{T}M \to M$ is a smooth submersion. Indeed, we showed that with respect to any smooth local coordinates (x^i) on an open subset $U \subseteq M$ and the corresponding natural coordinates (x^i, v^i) on $\pi^{-1}(U) \subseteq \mathrm{T}M$, the coordinate representation of π is $\widehat{\pi}(x, v) = x$, and thus

$$J_{\widehat{\pi}} = \begin{pmatrix} \mathrm{Id}_{\dim M} & \mathbb{O} \end{pmatrix},$$

which has rank $\operatorname{rk} J_{\widehat{\pi}} = \dim M$.

(3) If M is a smooth manifold and $U \subseteq M$ is an open subset, then the inclusion map $U \hookrightarrow M$ is a smooth embedding, see Proposition 3.9.

⇝ We will encounter more examples of smooth immersions, smooth embeddings and smooth submersions later in the course and in the exercise sheets as well.

To understand more fully what it means for a map to be a smooth embedding, it is useful to bear in mind some examples of injective smooth maps that are not smooth embeddings. The next three examples illustrate three rather different ways in which this can happen.

Example 4.5.

(1) The map

$$\gamma\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^3, 0)$$

is a smooth map and a topological embedding, but it is *not* a smooth embedding, because $\gamma'(0) = (0, 0)$, see Example 4.4(1).

(2) *The figure-eight curve*: Consider the smooth curve

$$\beta\colon (-\pi,\pi) \to \mathbb{R}^2,\ t \mapsto \big(\sin(2t), \sin t\big).$$

Its image is a set that looks like a figure-eight in the plane, sometimes called a *lemniscate*, see Figure 4.1. It is the locus of points $(x, y) \in \mathbb{R}^2$ such that $x^2 = 4y^2(1 - y^2)$, as one can easily check.

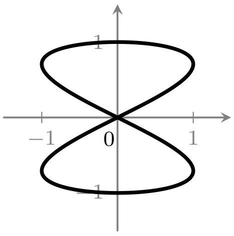


Figure 4.1: Lemniscate

Note that β is injective, since

$$\beta(t_1) = \beta(t_2) \implies t_1 = t_2,$$

and $\beta'(t) \neq 0$ for all $t \in (-\pi, \pi)$, since

$$\|\beta'(t)\|^2 = \big\|\big(2\cos(2t), \cos t\big)\big\|^2 = 4\cos^2(2t) + \cos^2 t \neq 0.$$

Hence, β is an injective smooth immersion, but it is *not* a topological embedding, because its image is compact in the subspace topology, while its domain is not.

(3) *A dense curve on the torus*: Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{C}^2$ denote the torus, and let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. The map

$$\gamma\colon \mathbb{R} \to \mathbb{T}^2,\ t \mapsto \big(\mathrm{e}^{2\pi\mathrm{i}\, t}, \mathrm{e}^{2\pi\mathrm{i}\,\alpha t}\big)$$

is a smooth immersion, because $\gamma'(t)$ never vanishes. It is also injective, because

$$\gamma(t_1) = \gamma(t_2) \implies t_1 - t_2,\ \alpha t_1 - \alpha t_2 \in \mathbb{Z} \implies t_1 = t_2.$$

However, γ is *not* a topological embedding. Indeed, using Dirichlet's approximation theorem [Lee13, Lemma 4.21], it can be shown that $\gamma(0)$ is a limit point of the set $\gamma(\mathbb{Z}) = \{\gamma(n) \mid n \in \mathbb{Z}\}$, while $\mathbb{Z}$ has no limit point in $\mathbb{R}$. It can also be proven that $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$, see [Lee13, Problem 4.4].

Recall: Let $F\colon X \to Y$ be a (not necessarily continuous) map between topological spaces. We say that F is

(a) an *open* map if for every open subset U of X, the image $F(U)$ is an open subset of Y;

(b) a *closed* map if for every closed subset C of X, the image $F(C)$ is a closed subset of Y;

(c) a *proper* map if for every compact subset $K \subseteq Y$, the preimage $F^{-1}(K)$ is a compact subset of X.

The following proposition gives a few simple sufficient criteria for an injective immersion to be an embedding.

Proposition 4.6. *Let* $F\colon M \to N$ *be an injective smooth immersion. If any of the following holds, then* F *is a smooth embedding.*

(a) F *is an open map or a closed map.*

(b) F *is a proper map.*

(c) M *is compact.*

(d) $\dim M = \dim N$.

Proof. We first prove the following three claims of topological nature, which will be subsequently used in the proof of the statement.

– *Claim 1*: Let $F\colon X \to Y$ be a continuous map between topological spaces that is either open or closed. If F is injective, then it is a topological embedding.

– *Proof*: Assume that F is open and injective. (The proof of the assertion is similar when F is closed and injective.) Then $F\colon X \to F(X)$ is bijective, so $F^{-1}\colon F(X) \to X$ exists. If $U \subseteq X$ is open, then $(F^{-1})^{-1}(U) = F(U)$ is open in Y by hypothesis, and therefore also open in $F(X)$ by definition of the subspace topology on $F(X)$. Hence, F^{-1} is continuous, so that F is a topological embedding.

– *Claim 2* (Closed map lemma): Let X be a compact space, let Y be a Hausdorff space, and let $F\colon X \to Y$ be a continuous map. Then F is a closed map.

– *Proof*: Let $K \subseteq X$ be a closed subset. Since X is compact, K is also compact, and since F is continuous, $F(K)$ is also compact. Since Y is Hausdorff, $F(K) \subseteq Y$ is a closed subset. Thus, F is a closed map.

– *Claim 3*: Let X be a topological space and let Y be a locally compact Hausdorff space. Then every proper continuous map $F\colon X \to Y$ is closed.

– *Proof*: Let $K \subseteq X$ be a closed subset. To show that $F(K) \subseteq Y$ is closed, we will show that its complement is open. Let $y \in Y \setminus F(K)$. Since Y is locally compact and Hausdorff, y has an open neighborhood V with compact closure $\overline{V}$, and since F is proper, $F^{-1}(\overline{V}\,)$ is compact. Set $E \coloneqq K \cap F^{-1}(\overline{V}\,)$ and note that E is a compact set. Since F is continuous, $F(E)$ is also compact, and since Y is Hausdorff, $F(E)$ is a closed subset of Y. Set $U \coloneqq V \setminus F(E) = V \cap \big(Y \setminus F(E)\big)$ and observe that U is open neighborhood of y, which is disjoint from $F(K)$. Hence, $Y \setminus F(E)$ is open, which implies that $F(K)$ is closed.

We are now ready to prove the statement.

(a) By assumption and by Claim 1, F is a topological embedding. Since it is also a smooth immersion by assumption, we conclude that F is a smooth embedding.

(b) By assumption and by Claim 3, F is a closed map, so it is a smooth embedding by (a).

(c) By assumption and by Claim 2, F is a closed map, so it is a smooth embedding by (a).

(d) By assumption and by Theorem 4.8 (see also Proposition 4.10(b)), F is a local diffeomorphism (see Definition 4.7), and thus an open map by Proposition 4.9(c). Therefore, F is a smooth embedding by (a). $\square$

Comment: There exist smooth embeddings which are neither open nor closed maps, see [Lee13, Exercise 4.24].

4.2 Local Diffeomorphisms

Recall: Let X and Y be topological spaces. A map $F\colon X \to Y$ is called a *local homeomorphism* if every point $p \in X$ has a neighborhood U such that $F(U)$ is open in Y and $F|_U\colon U \to F(U)$ is a homeomorphism.

Definition 4.7. Let M and N be smooth manifolds. A map $F\colon M \to N$ is called a *local diffeomorphism* if every point $p \in M$ has a neighborhood U such that $F(U)$ is open in N and $F|_U\colon U \to F(U)$ is a diffeomorphism.

The next theorem is the key to the most important properties of local diffeomorphisms.

Theorem 4.8 (Inverse function theorem for manifolds)**.** *Let $F\colon M \to N$ be a smooth map. If $p \in M$ is a point such that the differential $\mathrm{d}F_p$ of F at p is invertible, then there exist connected neighborhoods U_0 of p in M and V_0 of $F(p)$ in N such that $F|_{U_0}\colon U_0 \to V_0$ is a diffeomorphism.*

Proof. The idea is to pass to a coordinate representation of F and to use the *inverse function theorem* for open subsets of Euclidean spaces, which is recalled below.

[Lee13, Theorem C.34]: *Let $W \subseteq \mathbb{R}^n$ be an open set and consider a smooth function $G\colon W \to \mathbb{R}^n$. Suppose that there is a point $a \in W$ such that the Jacobian matrix of G at a is invertible. Then there exist connected open sets U and V such that $a \in U \subseteq W$ and $G(U) \subseteq V \subseteq \mathbb{R}^n$, and moreover the restriction $G|_U\colon U \to V$ admits a smooth inverse; that is, $G|_U$ is a diffeomorphism from U to V.*

Let (U, φ) and (V, ψ) be charts for M and N around p and $F(p)$, respectively, such that $F(U) \subseteq V$, and assume without loss of generality that $\varphi(p) = 0$ and $\psi\big(F(p)\big) = 0$. Set $\widehat{U} := \varphi(U)$ and $\widehat{V} := \psi(V)$, and let

$$\widehat{F} = \psi \circ F \circ \varphi^{-1}\colon \widehat{U} \to \widehat{V}$$

be the coordinate representation of F, which is smooth with $\widehat{F}(0) = 0$. Since $\mathrm{d}F_p$ is invertible, the tangent space to M at p and to N at $F(p)$ must have the same dimension, and thus $\widehat{U}, \widehat{V} \subseteq \mathbb{R}^n$, where $n := \dim M = \dim N$. Observe now that the differential

$$\mathrm{d}\widehat{F}_0 = \mathrm{d}\psi_{F(p)} \circ \mathrm{d}F_p \circ \mathrm{d}(\varphi^{-1})_0$$

is invertible, because $\mathrm{d}F_p$ is invertible by assumption and both $\mathrm{d}(\varphi^{-1})_0$ and $\mathrm{d}\psi_{F(p)}$ are also invertible by Proposition 3.7, since φ and ψ are diffeomorphisms by Example 2.14(2). Note that the matrix representation of $\mathrm{d}\widehat{F}_0$ with respect to the standard coordinates of $\mathbb{R}^n$ is the Jacobian of $\widehat{F}$ at 0. Therefore, by the *inverse function theorem* there are connected open neighborhoods $\widehat{U}_0 \subseteq \widehat{U}$ and $\widehat{V}_0 \subseteq \widehat{V}$ of 0 such that $\widehat{F}|_{\widehat{U}_0} \colon \widehat{U}_0 \to \widehat{V}_0$ is a diffeomorphism. Hence, for $U_0 := \varphi^{-1}(\widehat{U}_0) \ni p$ and $V_0 := \psi^{-1}(\widehat{V}_0) \ni F(p)$, the restriction $F|_{U_0} \colon U_0 \to V_0$ is a diffeomorphism, since we can write it as a composition of diffeomorphisms. ☐

Theorem 4.8 has the following important corollary: a smooth map $F \colon M \to N$ is a local diffeomorphism if and only if its differential dF_p is invertible for all $p \in M$. This also gives a very useful method to prove that some smooth map is a diffeomorphism, without explicitly constructing a smooth inverse: a smooth bijective map $F \colon M \to N$ whose differential dF_p is invertible for all $p \in M$ is a diffeomorphism; see Proposition 4.9(f).

Proposition 4.9 (Elementary properties of local diffeomorphisms)**.**

(a) *Every composition of local diffeomorphisms is a local diffeomorphism.*

(b) *Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.*

(c) *Every local diffeomorphism is a local homeomorphism and an open map.*

(d) *The restriction of a local diffeomorphism to an open submanifold is a local diffeomorphism.*

(e) *Every diffeomorphism is a local diffeomorphism.*

(f) *Every bijective local diffeomorphism is a diffeomorphism.*

(g) *A map between smooth manifolds is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a coordinate representation that is a local diffeomorphism.*

Proof. Exercise! (See also Proposition 2.15.) ☐

Proposition 4.10. *Let M and N be smooth manifolds and let $F \colon M \to N$ be a map. The following statements hold:*

(a) *F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.*

(b) *If $\dim M = \dim N$ and if F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.*

Proof. See [*Exercise Sheet* 5*, Exercise* 3(*c*)]. ☐

Exercise 4.11: Let M, N and P be smooth manifolds, and let $F \colon M \to N$ be a local diffeomorphism. Prove the following assertions:

(a) If $G \colon P \to M$ is continuous, then G is smooth if and only if $F \circ G$ is smooth.

(b) If F is surjective and if $H \colon N \to P$ is any map, then H is smooth if and only if $H \circ F$ is smooth.

4.3 The Rank Theorem

The most important fact about maps of constant rank is the following consequence of the inverse function theorem (see Theorem 4.8), which says that a smooth map of constant rank can be placed locally into a particularly simple canonical form by a change of coordinates. (This is a non-linear version of the canonical form theorem for linear maps; see [Lee13, Theorem B.20].)

Theorem 4.12 (Rank theorem). *Let M and N be smooth manifolds of dimension m and n, respectively, and let $F\colon M \to N$ be a smooth map of constant rank r. For each $p \in M$ there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form*

$$\widehat{F}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0).$$

In particular, if F is a smooth submersion (so that $r = n$), then this becomes

$$\widehat{F}(x^1, \dots, x^n, x^{n+1}, \dots, x^m) = (x^1, \dots, x^n),$$

while if F is a smooth immersion (so that $r = m$), then this becomes

$$\widehat{F}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0).$$

Proof. Since the theorem is local, after choosing smooth coordinates we can replace M and N by open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. The fact that $\mathrm{D}F(p)$ has rank r implies that its matrix has some $r \times r$ submatrix with non-zero determinant. By reordering the coordinates, we may assume that it is the upper left submatrix $\big(\frac{\partial F^i}{\partial x^j}(p)\big)$ for $i, j \in \{1, \dots, r\}$. We relabel the standard coordinates as

$$(x, y) = (x^1, \dots, x^r, y^1, \dots, y^{m-r}) \ \text{ in } \mathbb{R}^m$$

and

$$(v, w) = (v^1, \dots, v^r, w^1, \dots, w^{n-r}) \ \text{ in } \mathbb{R}^n.$$

By initial translation of the coordinates, without loss of generality we may assume that $p = (0, 0)$ and $F(p) = (0, 0)$. If we write

$$F(x, y) = \big(Q(x, y), R(x, y)\big)$$

for some smooth maps $Q\colon U \to \mathbb{R}^r$ and $R\colon U \to \mathbb{R}^{n-r}$, then our hypothesis is that the matrix $\big(\frac{\partial Q^i}{\partial x^j}\big)$ is non-singular at $(0, 0)$.

Define the function

$$\varphi\colon U \to \mathbb{R}^m, \ \varphi(x, y) = \big(Q(x, y), y\big),$$

and observe that its total derivative at $(0, 0)$ is

$$\mathrm{D}\varphi(0, 0) = \begin{pmatrix} \dfrac{\partial Q^i}{\partial x^j}(0, 0) & \dfrac{\partial Q^i}{\partial y^j}(0, 0) \\ \mathbb{O} & \delta^i_j \end{pmatrix},$$

which is non-singular by virtue of the hypothesis. Therefore, by the inverse function theorem [Lee13, Theorem C.34], there are connected neighborhoods U_0 of $(0,0)$ and $\widetilde{U}_0$ of $\varphi(0,0) = (0,0)$ such that $\varphi|_{U_0} \colon U_0 \to \widetilde{U}_0$ is a diffeomorphism. By shrinking U_0 and $\widetilde{U}_0$ if necessary, we may assume that $\widetilde{U}_0 \ni (0,0)$ is an open cube.[1] Writing the inverse map as

$$\varphi^{-1}(x,y) = \big(A(x,y), B(x,y)\big)$$

for some smooth functions $A \colon \widetilde{U}_0 \to \mathbb{R}^r$ and $B \colon \widetilde{U}_0 \to \mathbb{R}^{n-r}$, we see that

$$(x,y) = \big(\varphi \circ \varphi^{-1}\big)(x,y) = \varphi\big(A(x,y), B(x,y)\big) = \Big(Q\big(A(x,y), B(x,y)\big), B(x,y)\Big).$$

Comparing y components shows that $B(x,y) = y$, and therefore φ^{-1} has the form $\varphi^{-1}(x,y) = \big(A(x,y), y\big)$. Comparing now x components and taking this into account also shows that $Q\big(A(x,y), y\big) = x$, and therefore $F \circ \varphi^{-1}$ has the form

$$\big(F \circ \varphi^{-1}\big)(x,y) = \big(x, \widetilde{R}(x,y)\big),$$

where the function $\widetilde{R} \colon \widetilde{U}_0 \to \mathbb{R}^{n-r}$ is defined by $\widetilde{R}(x,y) = R\big(A(x,y), y\big)$. The Jacobian matrix of $F \circ \varphi^{-1}$ at an arbitrary point $(x,y) \in \widetilde{U}_0$ is

$$\mathrm{D}(F \circ \varphi^{-1})(x,y) = \begin{pmatrix} \delta^i_j & \mathbb{O} \\ \dfrac{\partial \widetilde{R}^i}{\partial x^j}(x,y) & \dfrac{\partial \widetilde{R}^i}{\partial y^j}(x,y) \end{pmatrix}.$$

Since composing with a diffeomorphism does not change the rank of a map, the above matrix has rank r everywhere in $\widetilde{U}_0$. The first r columns are obviously linearly independent, so the rank can be r only if the partial derivatives $\frac{\partial \widetilde{R}^i}{\partial y^j}$ vanish identically on $\widetilde{U}_0$, which implies that $\widetilde{R}$ is actually independent of $(y^1, \dots, y^{m-r})$. (This is one reason why we arranged for $\widetilde{U}_0$ to be a cube.) Thus, if we let $S(x) = \widetilde{R}(x,0)$, then we have

$$\big(F \circ \varphi^{-1}\big)(x,y) = \big(x, S(x)\big). \tag{4.1}$$

To complete the proof, we need to define an appropriate smooth chart in some neighborhood of $F(p) = (0,0) \in V$. Consider the open subset

$$V_0 = \big\{(v,w) \in V \mid (v,0) \in \widetilde{U}_0\big\} \subseteq V$$

and note that V_0 is a neighborhood of $(0,0)$. Since $\widetilde{U}_0 \ni (0,0) = \varphi(0,0)$ is a cube and $F \circ \varphi^{-1}$ has the form (4.1), it follows that $\big(F \circ \varphi^{-1}\big)(\widetilde{U}_0) \subseteq V_0$ (because $(v,w) \in \widetilde{U}_0 \Rightarrow \big(F \circ \varphi^{-1}\big)(v,w) = \big(v, S(v)\big) \in V$ by construction and $(v,0) \in \widetilde{U}_0$ by the form of $\widetilde{U}_0$), and hence $F(U_0) \subseteq V_0$. Define the function

$$\psi \colon V_0 \to \mathbb{R}^n, \ \psi(v,w) = \big(v, w - S(v)\big).$$

---

[1] A *closed rectangle* in $\mathbb{R}^k$ is a set of the form $[a^1, b^1] \times \ldots \times [a^k, b^k]$, whereas an *open rectangle* in $\mathbb{R}^k$ is a set of the form $(a^1, b^1) \times \ldots \times (a^k, b^k)$, for real numbers $a^i < b^i$. A (closed or open) rectangle is called a (closed or open) *cube* if all of its side lengths $b^i - a^i$ are equal.

This is an open map and a diffeomorphism onto its image, because its inverse is given explicitly by $\psi^{-1}(s,t) = \big(s, t + S(s)\big)$. Thus, (V_0, ψ) is a smooth chart. It follows now immediately from (4.1) that

$$\big(\psi \circ F \circ \varphi^{-1}\big)(x,y) = \psi\big(x, S(x)\big) = \big(x, S(x) - S(x)\big) = (x, 0)\,,$$

which was to be proved. $\square$

The next corollary can be viewed as a more invariant statement of the rank theorem. It says that maps of constant rank are precisely the ones whose local behavior is the same as that of their differentials.

Corollary 4.13. *Let $F\colon M \to N$ be a smooth map. Assume that M is connected. Then the following are equivalent:*

(a) *For each $p \in M$ there exists smooth charts containing p and $F(p)$ in which the coordinate representation of F is linear.*

(b) *F has constant rank.*

Proof.

(b) $\Rightarrow$ (a): Follows immediately from the rank theorem.

(a) $\Rightarrow$ (b): Since every linear map has constant rank, it follows that the rank of F is constant in a neighborhood of each point, and thus by connectedness it is constant on all of M. $\square$

The rank theorem is a purely local statement. However, it has the following powerful global consequence.

Theorem 4.14 (Global rank theorem)**.** *Let $F\colon M \to N$ be a smooth map of constant rank.*

(a) *If F is surjective, then it is a smooth submersion.*

(b) *If F is injective, then it is a smooth immersion.*

(c) *If F is bijective, then it is a diffeomorphism.*

Proof. Assume that $\dim M = m$, $\dim N = n$ and $\operatorname{rk} F = r$.

(a) See the proof of [Lee13, Theorem 4.14(a)] for the details.

(b) Assume that F is not a smooth immersion, so that $r < m$. By the rank theorem, for each $p \in M$ we can choose smooth charts around p and $F(p)$ in which F has the coordinate representation $\widehat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0)$. Thus, $\widehat{F}(0, \ldots, 0, \varepsilon) = (0, \ldots, 0, 0)$ for any $0 < \varepsilon \ll 1$, which shows that F is not injective, a contradiction.

(c) We have the following implications:

$$\begin{aligned}
F:\text{bijective} \quad &\Longrightarrow \quad F:\text{injective \& surjective}\\
&\xRightarrow[\text{(b)}]{\text{(a)}} \quad F:\text{smooth immersion \& smooth submersion}\\
&\xRightarrow{4.10(a)} \quad F:\text{local diffeomorphism}\\
&\xRightarrow[F\text{:bijective}]{4.9(f)} \quad F:\text{diffeomorphism.}
\end{aligned}$$

$\square$

4.3.1 Applications of the Rank Theorem

① Application to Smooth Immersions:

Theorem 4.15 (Local embedding theorem)**.** *Let $F\colon M \to N$ be a smooth map. Then F is a smooth immersion if and only if every point in M has a neighborhood U such that $F|_U\colon U \to N$ is a smooth embedding.*

Proof. If every point in M has a neighborhood on which F is a smooth embedding, then F has full rank everywhere, so it is a smooth immersion.

Conversely, assume that F is a smooth immersion, and let $p \in M$. We first claim that p has a neighborhood on which F is injective. Indeed, by the rank theorem there is an open neighborhood U_1 of p on which F has a coordinate representation of the form

$$\widehat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0),$$

and thus $F|_{U_1}$ is injective. Now, consider a precompact neighborhood U of p such that $\overline{U} \subseteq U_1$. The restriction of F to $\overline{U}$ is an injective continuous map with compact domain and Hausdorff codomain, so it is a topological embedding according to *Claims 1* and *2* from the proof of Proposition 4.6. Since any restriction of a topological embedding is again a topological embedding, $F|_U$ is both a topological embedding and a smooth immersion, so it is a smooth embedding. ☐

② Application to Smooth Submersions:

Recall: Let $\pi\colon M \to N$ be a continuous map between topological spaces. A *section of* π is a continuous right inverse for π, i.e., a continuous map $\sigma\colon N \to M$ such that $\pi \circ \sigma = \mathrm{Id}_N$. A *local section of* π is a continuous map $\sigma\colon U \to M$ defined on some open subset $U \subseteq N$ and satisfying the analogous relation $\pi \circ \sigma = \mathrm{Id}_U$.

Many of the important properties of smooth submersions follow from the fact that they admit an abundance of smooth local sections, which we prove below.

Theorem 4.16 (Local section theorem)**.** *Let $\pi\colon M \to N$ be a smooth map. Then π is a smooth submersion if and only if every point of M is in the image of a smooth local section of π.*

Proof. Set $m = \dim M$ and $n = \dim N$.

"$\Rightarrow$": Fix $p \in M$ and set $q = \pi(p) \in N$. By the rank theorem we can choose smooth coordinates $(x^1, \ldots, x^m)$ centered at p and $(y^1, \ldots, y^n)$ centered at q in which π has the coordinate representation

$$\widehat{\pi}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n).$$

If $0 < \varepsilon \ll 1$, then the coordinate cube

$$C_\varepsilon := \{x \mid |x^i| < \varepsilon,\ 1 \leq i \leq m\}$$

is a neighborhood of $\widehat{p} = 0 \in \mathbb{R}^m$ whose image under $\widehat{\pi}$ is the coordinate cube

$$C'_\varepsilon := \{y \mid |y^i| < \varepsilon,\ 1 \leq i \leq n\}.$$

The map σ whose coordinate representation is

$$\widehat{\sigma}\colon C'_\varepsilon \to C_\varepsilon,\ \widehat{\sigma}(x^1,\dots,x^n) = (x^1,\dots,x^n,0,\dots,0)$$

is a smooth local section of π satisfying $\sigma(q) = p$, see Figure 4.2.

"$\Leftarrow$": Given $p \in M$, let $\sigma : U \to M$ be a smooth local section of π such that $\sigma(q) = p$, where $q = \pi(\sigma(q)) = \pi(p) \in N$. The equation $\pi \circ \sigma = \mathrm{Id}_U$ implies that $\mathrm{d}\pi_p \circ \mathrm{d}\sigma_q = \mathrm{Id}_{\mathrm{T}_q N}$ by Proposition 3.7(b), which in turn implies that $\mathrm{d}\pi_p$ is surjective. Since $p \in M$ was arbitrary, we conclude that π is a smooth submersion. ☐

Figure 4.2: Local section of a smooth submersion

4.4 Surjective Smooth Submersions

Recall: If X is a topological space, Y is a set, and $\pi\colon X \to Y$ is a surjective map, then *the quotient topology on Y determined by π* is defined by declaring a subset $V \subseteq Y$ to be open if $\pi^{-1}(V)$ is open in X. If X and Y are topological spaces, a map $\pi\colon X \to Y$ is called a *quotient map* if it is surjective and continuous and Y has the quotient topology determined by π.

Proposition 4.17. *Let $\pi\colon M \to N$ be a smooth submersion. Then π is an open map. Moreover, if it is surjective, then it is a quotient map.*

Proof. The second assertion follows from the first one, because a surjective, open and continuous map is a quotient map by [Lee13, Exercise A.29 and Theorem A.38].

It remains to prove that π is an open map. To this end, let W be an open subset of M and let $q \in \pi(W)$. For any $p \in W$ such that $\pi(p) = q$, by Theorem 4.16 there is an open neighborhood U of q on which there exists a smooth local section $\sigma : U \to M$ of π with $\sigma(q) = p$. For each $y \in \sigma^{-1}(W)$, the fact that $\sigma(y) \in W$ implies that $y = \pi\big(\sigma(y)\big) \in \pi(W)$. Thus, $\sigma^{-1}(W)$ is an open neighborhood of q contained in $\pi(W)$, which implies that $\pi(W)$ is open. This completes the proof. ☐

The next three theorems provide important tools that are frequently used when studying submersions and demonstrate that surjective smooth submersions play a role in smooth manifold theory analogous to quotient maps in topology.

Theorem 4.18 (Characteristic property of surjective smooth submersions)**.** *Let $\pi\colon M \to N$ be a surjective smooth submersion. For any smooth manifold P, a map $F\colon N \to P$ is smooth if and only if $F \circ \pi\colon M \to P$ is smooth.*

$$\begin{tikzcd}
M \arrow[d, "\pi"'] \arrow[dr, "F\circ\pi"] & \\
N \arrow[r, "F"'] & P
\end{tikzcd}$$

Proof. See [*Exercise Sheet* 6, *Exercise* 2]. ☐

- ⇝ [*Exercise Sheet* 6, *Exercise* 3] explains the sense in which the above property is "characteristic".
- ⇝ [*Exercise Sheet* 6, *Exercise* 4] shows that the converse of the Theorem 4.18 is false.

Theorem 4.19 (Pushing smoothly to the quotient)**.** *Let $\pi\colon M \to N$ be a surjective smooth submersion. If P is a smooth manifold and if $F\colon M \to P$ is a smooth map that is constant on the fibers of π, then there exists a unique smooth map $\widetilde{F}\colon N \to P$ such that $\widetilde{F} \circ \pi = F$.*

$$\begin{tikzcd}
M \arrow[d, "\pi"'] \arrow[dr, "F"] & \\
N \arrow[r, dashed, "\widetilde{F}"'] & P
\end{tikzcd}$$

Proof. See [*Exercise Sheet* 6, *Exercise* 5]. □

Theorem 4.20 (Uniqueness of smooth quotients)**.** *Let $\pi_1\colon M \to N_1$ and $\pi_2\colon M \to N_2$ be surjective smooth submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism $F\colon N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$.*

$$\begin{tikzcd}
& M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
N_1 \arrow[rr, dashed, "F"'] & & N_2
\end{tikzcd}$$

Proof. See [*Exercise Sheet* 6, *Exercise* 6]. □

CHAPTER 5

SUBMANIFOLDS

Many familiar manifolds appear naturally as subsets of other manifolds. We have already seen that open subsets of smooth manifolds can be viewed as smooth manifolds in their own right. However, there are many interesting examples beyond the open ones. In this chapter we explore *smooth submanifolds*, which are smooth manifolds that are subsets of other smooth manifolds.

5.1 Embedded Submanifolds

Definition 5.1. Let M be a smooth manifold. An *embedded submanifold of* M is a subset $S \subseteq M$ that is a topological manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth embedding.

If S is an embedded submanifold of M, then the difference $\dim M - \dim S$ is called *the codimension of* S *in* M, and the containing manifold M is called *the ambient manifold for* S. For instance, an embedded submanifold of codimension 1 is called an *embedded hypersurface*. The empty set $\varnothing$ is an embedded submanifold of any dimension.

The easiest embedded submanifolds to understand are those of codimension 0, as the following result demonstrates.

Proposition 5.2 (Open submanifolds)**.** *Let* M *be a smooth manifold. The embedded submanifolds of codimension* 0 *in* M *are exactly the open submanifolds.*

Proof. If $U \subseteq M$ is an open submanifold, then we have already seen that U is a smooth manifold of $\dim U = \dim M$ (Example 1.10(4)) and that the inclusion map $\iota\colon U \hookrightarrow M$ is a smooth embedding (Example 4.4(3)). Therefore, $U \subseteq M$ is an embedded manifold of codimension 0.

Conversely, if $U \subseteq M$ is an embedded submanifold of codimension 0, then the inclusion map $\iota\colon U \hookrightarrow M$ is a smooth embedding. Since $\dim U = \dim M$, it is actually a local diffeomorphism by Proposition 4.10(b), and thus an open map by Proposition 4.9(c). Therefore, U is an open subset of M. □

Proposition 5.3 (Images of embeddings as submanifolds)**.** *Let* $F\colon N \to M$ *be a smooth embedding and set* $S := F(N)$*. With the subspace topology,* S *is a topological manifold,* *and it has a unique smooth structure making it into an embedded submanifold of M with the property that F is a diffeomorphism onto its image.*

Proof. If we give S the subspace topology that it inherits from M, then the assumption that F is an embedding means that F can be considered as a homeomorphism from N onto S, and thus S is a topological manifold. In view of Proposition 1.8(a), we now give S a smooth structure by taking the smooth charts to be those of the form $\big(F(U),\, \varphi \circ F^{-1}\big)$, where (U, φ) is a smooth chart for N. Note that the smooth compatibility of these charts follows from the smooth compatibility of the corresponding charts for N. With this smooth structure on S, the map F is a diffeomorphism onto its image (essentially by definition), and this is obviously the only smooth structure with this property [why?]. The inclusion map $\iota\colon S \hookrightarrow M$ is equal to the composition of a diffeomorphism followed by a smooth embedding

$$S \xrightarrow{F^{-1}} N \xrightarrow{F} M\,,$$

so it is a smooth embedding itself by [*Exercise Sheet* 5*, Exercise* 3(*a*)]. ☐

Since every embedded submanifold is the image of a smooth embedding (namely, its own inclusion map), Proposition 5.3 shows that *embedded submanifolds are exactly the images of smooth embeddings* of smooth manifolds.

Exercise 5.4 (Slice of the product manifold)**:** If M and N are smooth manifolds, then for each $q \in N$ the subset $M \times \{q\}$, called a *slice of the product manifold*, is an embedded submanifold of $M \times N$ diffeomorphic to M.

Solution:

1st proof: The assertion follows immediately from Proposition 5.3 by observing that the set $M \times \{q\}$ is the image of the smooth embedding $M \to M \times N,\ x \mapsto (x, q)$.

2nd proof: The assertion follows immediately from Example 5.5 by considering the constant function

$$f\colon M \to N,\ x \mapsto q \in N,$$

which is smooth by Proposition 2.11(a).

Example 5.5 (Graphs as embedded submanifolds)**.** Let M be a smooth m-manifold and let N be a smooth n-manifold. Let $U \subseteq M$ be an open subset and let $f\colon U \to N$ be a smooth map. Then *the graph of* f,

$$\Gamma(f) := \big\{(x,y) \in M \times N \mid x \in U,\ y = f(x)\big\},$$

is an embedded m-dimensional submanifold of $M \times N$ diffeomorphic to U. Indeed, consider the map

$$\gamma_f\colon U \to M \times N,\ x \mapsto \big(x, f(x)\big).$$

It is a smooth map by [*Exercise Sheet* 3*, Exercise* 4(*b*)] whose image is $\Gamma(f)$. Since the projection $\pi_M\colon M \times N \to M$ satisfies

$$\big(\pi_M \circ \gamma_f\big)(x) = x = \mathrm{Id}_U(x) \quad \text{for all } x \in U,$$

the composition $\mathrm{d}(\pi_M)_{(x,f(x))} \circ \mathrm{d}(\gamma_f)_x$ is the identity on $\mathrm{T}_xU \cong \mathrm{T}_xM$ for each $x \in U$. Thus, $\mathrm{d}(\gamma_f)_x$ is injective, so γ_f is a smooth immersion. It is also a homeomorphism onto its image, since $\pi_M|_{\Gamma(f)}$ is a continuous inverse for it. Therefore, $\Gamma(f)$ is an embedded submanifold of $M \times N$ diffeomorphic to U by Proposition 5.3.

Definition 5.6. An embedded submanifold S of a smooth manifold M is said to be *properly embedded* if the inclusion $S \hookrightarrow M$ is a proper map.

Proposition 5.7. *An embedded submanifold S of a smooth manifold M is properly embedded if and only if S is a closed subset of M*

Proof. See [*Exercise Sheet* 7*, Exercise* 1(*b*)]. ☐

⇝ For instance, every compact embedded submanifold is properly embedded, since compact subspaces of Hausdorff spaces are closed.

⇝ We will encounter many more examples of (properly) embedded submanifolds later in the course and in the exercise sheets as well.

5.1.1 Slice Charts for Embedded Submanifolds

Definition 5.8.

(a) Given an open subset $U \subseteq \mathbb{R}^n$ and an integer $k \in \{0, \dots, n\}$, a *k-dimensional slice of U* (or simply a *k-slice*) is any subset of the form

$$S = \left\{(x^1, \dots, x^k, x^{k+1}, \dots, x^n) \in U \;\middle|\; x^{k+1} = c^{k+1}, \dots, x^n = c^n\right\}$$

for some constants $c^{k+1}, \dots, c^n \in \mathbb{R}$ (often taken to be zero). (When $k = 0$, we have $S = \{\text{point}\} \subseteq U$, while if $k = n$, then $S = U$.)

Note that every k-slice is homeomorphic to an open subset of $\mathbb{R}^k$. (Sometimes it is convenient to consider slices defined by setting some subset of the coordinates other than the last ones equal to constants.)

(b) Let M be a smooth manifold and let (U, φ) be a smooth chart for M. If S is a subset of U such that $\varphi(S)$ is a k-slice of $\varphi(U) \subseteq \mathbb{R}^n$, then we say that S is a *k-slice of U*.

(c) Given a smooth manifold M, a subset $S \subseteq M$ and an integer $k \in \mathbb{N}$, we say that S satisfies the *local k-slice condition* if each point of S is contained in the domain of a smooth chart (U, φ) for M such that $S \cap U$ is a single k-slice in U. Any such chart is called *slice chart for S in M*, and the corresponding coordinates $(x^1, \dots, x^n)$ are called *slice coordinates.*

The following theorem shows that embedded submanifolds are locally modeled on the standard embedding of $\mathbb{R}^k$ into $\mathbb{R}^n$, identifying $\mathbb{R}^k$ with the subspace

$$\left\{(x^1, \dots, x^k, x^{k+1}, \dots, x^n) \in \mathbb{R}^n \mid x^{k+1} = \cdots = x^n = 0\right\} \subseteq \mathbb{R}^n.$$

Theorem 5.9 (Local slice criterion for embedded submanifolds)**.** *Let M be a smooth n-manifold. If S is an embedded k-dimensional submanifold of M, then S satisfies the local k-slice condition. Conversely, if $S \subseteq M$ is a subset that satisfies the local k-slice condition, then with the subspace topology, S is a topological manifold of dimension k, and it has a smooth structure making it into a k-dimensional embedded submanifold of M.*

Proof.

"$\Rightarrow$": Fix $p \in S$. Since the inclusion map $\iota\colon S \hookrightarrow M$ is in particular a smooth immersion, by the rank theorem there are smooth charts (U, φ) for S (in its given smooth manifold structure) and (V, ψ) for M, both centered at p, in which the inclusion map $\iota|_U\colon U \hookrightarrow V$ has the coordinate representation

$$(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0).$$

Now, choose $0 < \varepsilon \ll 0$ so that both U and V contain coordinate balls $U_0 \subseteq U$ and $V_0 \subseteq V$ of radius $\varepsilon > 0$ centered at p. It follows that $U_0 \cong \iota(U_0)$ is exactly a single slice in V_0 (using the above local description). Since $S \subseteq M$ has the subspace topology and since U_0 is open in S, there is an open subset $W \subseteq M$ such that $U_0 = W \cap S$. Setting $V_1 := W \cap V_0$, we obtain a smooth chart $(V_1, \psi|_{V_1})$ for M containing p such that $V_1 \cap S = U_0 \cap V_0 = U_0$, which is a single slice of V_1 (as U_0 is a single slice of V_0).

"$\Leftarrow$": With the subspace topology, S is Hausdorff and second-countable, because both properties are inherited by subspaces. To show that S is locally Euclidean, we construct an atlas. The basic idea of the construction is that if $(x^1, \dots, x^n)$ are slice coordinates for S in M, then we can use $(x^1, \dots, x^k)$ as local coordinates for S.

Figure 5.1: A chart for a subset satisfying the k-slice condition

Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the first k-coordinates. Let (U, φ) be a slice chart for S in M, and define

$$V = U \cap S, \quad \widehat{V} = (\pi \circ \varphi)(V), \quad \psi = (\pi \circ \varphi)|_V\colon V \to \widehat{V}.$$

By definition of slice charts, $\varphi(V)$ is the intersection of $\varphi(U)$ with a certain k-slice $A \subseteq \mathbb{R}^n$ defined by setting $x^{k+1} = c^{k+1}, \dots, x^n = c^n$, and therefore $\varphi(V)$ is open in A. Since $\pi|_A\colon A \to \mathbb{R}^k$ is a diffeomorphism, it follows that $\widehat{V}$ is open in $\mathbb{R}^k$. Moreover, ψ is a homeomorphism, because it has a continuous inverse given by $(\varphi^{-1} \circ j)|_{\widehat{V}}$, where

$$j\colon \mathbb{R}^k \to \mathbb{R}^n,\ j(x^1, \dots, x^k) = (x^1, \dots, x^k, c^{k+1}, \dots, c^n).$$

Thus, S is a topological k-manifold.

Figure 5.2: Smooth compatibility of slice charts

We now check that the charts constructed above are smoothly compatible. Let (U, φ) and (U', φ') be two slice charts for S in M and let (V, ψ) and (V', ψ') be the corresponding charts for S. The transition map is given by

$$\psi' \circ \psi^{-1} = \pi \circ \varphi' \circ \varphi^{-1} \circ j,$$

which is smooth by Proposition 2.11(d) as a composite of four smooth maps. Hence, the atlas we have constructed is actually a smooth atlas (see Remark 1.5), and it defines a smooth structure on S by Proposition 1.8(a).

In terms of a slice chart (U, φ) for S in M and the corresponding chart (V, ψ) for S, the inclusion map $\iota\colon S \hookrightarrow M$ has a coordinate representation of the form

$$(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, c^{k+1}, \dots, c^n),$$

which is a smooth immersion. Since the inclusion map is also a topological embedding, we are done. $\square$

Notice that the local slice condition for $S \subseteq M$ is a condition on the *subset* S only; it does not presuppose any particular topology or smooth structure on S. According to [*Exercise Sheet* 8, *Exercise* 1], the smooth manifold structure constructed in Theorem 5.9 is the *unique* one in which S can be considered as a submanifold, so a subset satisfying the local slice condition is an embedded submanifold in only one way.

5.1.2 Level Sets

Recall: Let $\Phi\colon M \to N$ be a map between sets. If $c \in N$, then $\Phi^{-1}(c) \subseteq M$ is called a *level set* of Φ. (In the special case when $N = \mathbb{R}^k$ and $c = 0 \in \mathbb{R}^k$, the level set $\Phi^{-1}(c)$ is usually called the *zero set* of Φ.)

Definition 5.10. Let $\Phi\colon M \to N$ be a smooth map.

- A point $p \in M$ is called *regular point* of Φ if $\mathrm{d}\Phi_p\colon \mathrm{T}_pM \to \mathrm{T}_{\Phi(p)}N$ is surjective; otherwise, we say that $p \in M$ is a *critical point* of Φ.
- A point $c \in N$ is called a *regular value* of Φ if every point of the level set $\Phi^{-1}(c)$ is a regular point; otherwise, we say that $c \in N$ is a *critical value* of Φ. (In particular, if $\Phi^{-1}(c) = \varnothing$, then c is a regular value.)
- A level set $\Phi^{-1}(c)$ is called a *regular level set* if $c \in N$ is a regular value of Φ.

Remark 5.11. Let $\Phi\colon M \to N$ be a smooth map.

(1) If $\dim M < \dim N$, then every point of M is critical point of Φ.

(2) Every point of M is regular if and only if Φ is a smooth submersion.

(3) According to Lemma 4.3, the set of regular points of Φ is an open subset of M (but may well be empty).

Consider the three smooth functions

$$\begin{aligned} \Theta\colon \mathbb{R}^2 \to \mathbb{R}, &\quad (x, y) \mapsto x^2 - y, \\ \Phi\colon \mathbb{R}^2 \to \mathbb{R}, &\quad (x, y) \mapsto x^2 - y^2, \\ \Psi\colon \mathbb{R}^2 \to \mathbb{R}, &\quad (x, y) \mapsto x^2 - y^3. \end{aligned}$$

Although the zero set $\Theta^{-1}(0)$ of Θ is an embedded submanifold of $\mathbb{R}^2$, because it is the graph of the smooth function $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, it will be shown in [*Exercise Sheet* 7, *Exercise* 4(*b*)] and [*Exercise Sheet* 8, *Exercise* 5(*b*)], respectively, that neither the zero set $\Phi^{-1}(0)$ of Φ nor the zero set $\Psi^{-1}(0)$ of Ψ is an embedded submanifold of $\mathbb{R}^2$.

Figure 5.3: Level sets may or may not be embedded submanifolds

Hence, it is fairly easy to find level sets of smooth functions that are *not* smooth submanifolds (cf. Theorem B.1). In fact, without further assumptions on the smooth function, the situation is about as bad as could be imagined; namely, according to Theorem 2.23, *every* closed subset of M can be expressed as the zero set of a smooth, non-negative, real-valued function. However, using the rank theorem, we can prove the following result:

Theorem 5.12 (Constant-rank level set theorem)**.** *Let $\Phi\colon M \to N$ be a smooth map of constant rank r. Each level set of Φ is a properly embedded submanifold of codimension r in M.*

In particular, if Φ is a smooth submersion, then each level set of Φ is a properly embedded submanifold of M of codimension $r = \dim N$.

Proof. Set $m = \dim M$, $n = \dim N$ and $k = m - r$. Pick $c \in N$ and set $S = \Phi^{-1}(c)$. By Theorem 4.12, for each $p \in S$, there are smooth charts (U, φ) centered at p and (V, ψ) centered at $c = \Phi(p)$ in which Φ has a coordinate representation of the form

$$\widetilde{\Phi}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0).$$

Therefore, $S \cap U = \Phi^{-1}(c) \cap U$ is the single slice

$$\big\{(x^1, \dots, x^r, x^{r+1}, \dots, x^m) \in U \mid x^1 = \dots = x^r = 0\big\}.$$

In conclusion, S satisfies the local $(k = m - r)$-slice condition, so it is an embedded submanifold of dimension k by Theorem 5.9. It is also closed in M by the continuity of Φ, so it is actually properly embedded in M by Proposition 5.7. □

Corollary 5.13 (Regular level set theorem)**.** *Every regular level set of a smooth map between smooth manifolds is a properly embedded submanifold of the domain whose codimension is equal to the dimension of the codomain.*

Proof. Let $\Phi\colon M \to N$ be a smooth map and let $c \in N$ be a regular value of Φ. By Remark 5.11(3) we know that the set

$$U = \big\{p \in M \mid \operatorname{rk}(\mathrm{d}\Phi_p) = \dim N\big\} \subseteq M$$

is open in M, and contains $\Phi^{-1}(c)$ by assumption. Therefore, $\Phi|_U\colon U \to N$ is a smooth submersion, and thus $\Phi^{-1}(c)$ is an embedded submanifold of U by Theorem 5.12. It follows now from Proposition 5.2 and [*Exercise Sheet* 5*, Exercise* 3(*a*)] that

$$\Phi^{-1}(c) \hookrightarrow U \hookrightarrow M$$

is a smooth embedding (as a composite of smooth embeddings), so $\Phi^{-1}(c)$ is an embedded submanifold of M. It is also closed in M by the continuity of Φ, so it is actually properly embedded in M by Proposition 5.7. □

Not all embedded submanifolds can be expressed as level sets of smooth submersions. However, the next proposition shows that every embedded submanifold is at least locally of this form.

Proposition 5.14. *Let S be a subset of a smooth m-manifold M. Then S is an embedded k-submanifold of M if and only if every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion $\Phi\colon U \to \mathbb{R}^{m-k}$.*

Proof. Assume that S is an embedded k-submanifold of M. Then S satisfies the local k-slice criterion by Theorem 5.9. Given $p \in S$, if $(x^1, \dots, x^m)$ are slice coordinates for S in an open neighborhood U of p in M, then there are constants $c^{k+1}, \dots, c^m \in \mathbb{R}$ such that (in coordinates we have)

$$U \cap S = \big\{(x^1, \dots, x^k, x^{k+1}, \dots, x^m) \in U \mid x^{k+1} = c^{k+1}, \dots, x^m = c^m\big\}.$$

Moreover, the map $\Phi\colon U \to \mathbb{R}^{m-k}$ given in coordinates by

$$\Phi(x^1, \dots, x^k, x^{k+1}, \dots, x^m) = (x^{k+1}, \dots, x^m)$$

is a smooth submersion, since its Jacobian is the $(m-k) \times m$ matrix

$$\begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

of rank $m-k$, and clearly we have

$$U \cap S = \Phi^{-1}\big(c^{k+1}, \dots, c^m\big).$$

In conclusion, every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion $\Phi\colon U \to \mathbb{R}^{m-k}$.

Conversely, assume that every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion $\Phi\colon U \to \mathbb{R}^{m-k}$. By Corollary 5.13, $U \cap S$ is a properly embedded k-submanifold of U, so it satisfies the local k-slice criterion by Theorem 5.9. Therefore, S itself satisfies the local k-slice criterion, and hence it is an embedded k-submanifold of M by Theorem 5.9. ☐

Figure 5.4: An embedded submanifold is locally a level set

If $S \subseteq M$ is an embedded k-submanifold, then a smooth map $\Phi\colon M \to N$ such that S is a regular level set of Φ is called a *defining map* for S. In the special case $N = \mathbb{R}^{m-k}$ it is usually called a *defining function* for S. More generally, if $U \subseteq M$ is an open subset and $\Phi\colon U \to N$ is a smooth map such that $S \cap U$ is a regular level set of Φ, then Φ is called a *local defining map* (or *local defining function*) for S. Proposition 5.14 asserts that *every embedded submanifold admits a local defining map in a neighborhood of each of its points.*

5.2 Immersed Submanifolds

Definition 5.15. Let M be a smooth manifold. An *immersed submanifold of M* is a subset $S \subseteq M$ endowed with a topology (not necessarily the subspace topology) with respect to which it is a topological manifold, and with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is (an injective) smooth immersion. The *codimension of S in M* is defined as $\dim M - \dim S$.

Observe that every embedded submanifold is an immersed submanifold, but the converse fails in general; see [*Exercise Sheet* 7*, Exercise* 3] (= Example 5.17) and [*Exercise Sheet* 8*, Exercise* 5(*a*)] for two counterexamples.

Proposition 5.16 (Images of immersions as submanifolds)**.** *Let $F\colon N \to M$ be an injective smooth immersion. Set $S := F(N)$. Then S has a unique topology and smooth structure such that it is an immersed submanifold of M and such that $F\colon N \to S$ is a diffeomorphism onto its image.*

Proof. The proof is very similar to that of Proposition 5.3, except that now we also have to define the topology on S.

We give S a topology by declaring a subset $U \subseteq S$ to be open if and only if $F^{-1}(U) \subseteq N$ is open, and then we give it a smooth structure by taking the smooth charts to be those of the form $\big(F(U), \varphi \circ F^{-1}\big)$, where (U, φ) is a smooth chart for N. (As in the proof of Proposition 5.3, the smooth compatibility of these charts follows from the smooth compatibility of the corresponding charts for N.) With this topology and smooth structure on S, the map F is a diffeomorphism onto its image, and these are the only topology and smooth structure on S with this property. The inclusion map $\iota\colon S \hookrightarrow M$ can be written as the composition

$$S \xrightarrow{F^{-1}} N \xrightarrow{F} M,$$

where the first map is a diffeomorphism and the second map is a smooth immersion, so ι is itself a smooth immersion by [*Exercise Sheet* 5*, Exercise* 1(*a*)]. □

Since every immersed submanifold is the image of an injective smooth immersion (namely its own inclusion map), Proposition 5.16 shows that *immersed submanifolds are exactly the images of injective smooth immersions* of smooth manifolds.

Example 5.17. The figure-eight curve (lemniscate) from Example 4.5(2) is the image of the injective smooth immersion

$$\beta\colon (-\pi, \pi) \to \mathbb{R}^2, \ t \mapsto \big(\sin(2t), \sin t\big)$$

(which is not an embedding, as demonstrated in Example 4.5(2)). Therefore, it is an *immersed* submanifold of $\mathbb{R}^2$ when given an appropriate topology and smooth structure by Proposition 5.16. As a smooth manifold, it is diffeomorphic to $\mathbb{R}$. However, it is *not* an embedded submanifold of $\mathbb{R}^2$, because it does not have the subspace topology; see [*Exercise Sheet* 7*, Exercise* 3]. In fact, the image set $\beta\big((-\pi, \pi)\big)$ cannot be made into an embedded submanifold of $\mathbb{R}^2$ even if we are allowed to change its topology and smooth structure; see [*Exercise Sheet* 8*, Exercise* 1] and [*Exercise Sheet* 8*, Exercise* 2].

Remark 5.18. In general, smooth (immersed) submanifolds can be closed without being embedded (as is, for example, the figure-eight curve from Example 5.17) or embedded without being closed (as is, for example, the open unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$).

The following observation is sometimes useful when thinking about the topology of an immersed submanifold.

Comment: Let M be a smooth manifold and let S be an immersed submanifold of M. Then every subset of S that is open in the subspace topology is also open in its given submanifold topology (that is, the submanifold topology on an immersed submanifold is finer than the subspace topology); and the converse is true if and only if S is embedded.

Given a smooth submanifold that it is only known to be immersed, it is often useful to have simple criteria that guarantee that it is embedded. The next proposition gives several such criteria.

Proposition 5.19. *Let M be a smooth manifold and let S be an immersed submanifold of M. If any of the following conditions holds, then S is embedded.*

(a) *The inclusion map $\iota\colon S \hookrightarrow M$ is proper.*

(b) *S is compact.*

(c) $\operatorname{codim}_M S = 0$.

Proof. Since S is an immersed submanifold of M, the inclusion map $\iota\colon S \hookrightarrow M$ is an injective smooth immersion. If any of the above conditions holds, then Proposition 4.6 implies that ι is a smooth embedding; in particular, $\iota(S) = S$ is endowed with the subspace topology inherited from M. Therefore, in any of these three cases, S is an embedded submanifold of M. $\square$

Even though many immersed submanifolds are not embedded, such as the one from Example 5.17, the next result shows that the *local* structure of an immersed submanifold is the same as that of an embedded one.

Proposition 5.20 (Immersed submanifolds are locally embedded)**.** *If M is a smooth manifold and if S is an immersed submanifold of M, then for each $p \in S$ there exists a neighborhood U of p in S that is an embedded submanifold of M.*

Proof. By assumption, $\iota\colon S \hookrightarrow M$ is a smooth immersion. By Theorem 4.15 every $p \in S$ has a neighborhood U in S such that $\iota|_U\colon U \hookrightarrow M$ is a smooth embedding, so Proposition 5.3 yields the assertion. $\square$

It is important to be clear about what this proposition does and does not say: given an immersed submanifold $S \subseteq M$ and a point $p \in S$, it is possible to find a neighborhood U of p in S such that U is embedded; but it may not be possible to find a neighborhood V of p *in M* such that $V \cap S$ is embedded.

5.3 The Tangent Space to a Submanifold

We discuss here the tangent space to submanifolds. If S is a submanifold of $\mathbb{R}^n$, we intuitively think of the tangent space T_pS at a point $p \in S$ as a subspace of the tangent space $\mathrm{T}_p\mathbb{R}^n$. Similarly, the tangent space to a smooth submanifold of an abstract smooth manifold can be viewed as a subspace of the tangent space to the ambient manifold, once we make appropriate identifications.

Let M be a smooth manifold and let S be an immersed or embedded submanifold of M. Since the inclusion map $\iota\colon S \hookrightarrow M$ is (at least) a smooth immersion, at each point $p \in S$ we have an injective linear map $\mathrm{d}\iota_p\colon \mathrm{T}_pS \hookrightarrow \mathrm{T}_pM$. In terms of derivations, this injection works in the following way: for any vector $v \in \mathrm{T}_pS$, the image vector $\widetilde{v} = \mathrm{d}\iota_p(v) \in \mathrm{T}_pM$ acts on smooth functions f on M by

$$\widetilde{v}f = \mathrm{d}\iota_p(v)(f) = v(f \circ \iota) = v\big(f\big|_S\big).$$

We usually identify T_pS with its image $\mathrm{d}\iota_p(\mathrm{T}_pS)$ under $\mathrm{d}\iota_p$, thereby thinking of T_pS as a certain linear subspace of T_pM. This identification makes sense regardless of whether S is immersed or embedded.

There are several alternative ways of characterizing T_pS as a subspace of T_pM. The first one is the most general; it is just a straightforward generalization of Proposition 3.17.

Proposition 5.21. *Let M be a smooth manifold, let $S \subseteq M$ be an immersed or embedded submanifold, and let $p \in S$. A vector $v \in T_pM$ is in T_pS if and only if there exists a smooth curve $\gamma\colon J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.*

Proof. See [*Exercise Sheet* 8, *Exercise* 3(*a*)]. ☐

The next proposition gives a useful way to characterize T_pS in the embedded case. However, according to [Lee13, Problem 5.20], this does not work in the non-embedded case.

Proposition 5.22. *Let M be a smooth manifold, let S be an embedded submanifold of M and let $p \in S$. As a subspace of T_pM, the tangent space T_pS is characterized by*

$$\mathrm{T}_pS = \big\{v \in \mathrm{T}_pM \mid vf = 0 \text{ whenever } f \in C^\infty(M) \text{ with } f|_S = 0\big\}.$$

Proof. Pick $v \in \mathrm{T}_pS \subseteq \mathrm{T}_pM$. Then $v = \mathrm{d}\iota_p(w)$ for some $w \in \mathrm{T}_pS$, where $\iota\colon S \hookrightarrow M$ is the inclusion map. If $f \in C^\infty(M)$ with $f|_S = 0$, then $vf = \mathrm{d}\iota_p(w)(f) = w\big(f|_S\big) = 0$.

Conversely, if $v \in \mathrm{T}_pM$ satisfies $vf = 0$ whenever f vanishes on S, then we have to show that $v = \mathrm{d}\iota_p(w)$ for some $w \in \mathrm{T}_pS$. Let $(x^1, \dots, x^n)$ be slice coordinates for S in some neighborhood U of p, so that

$$U \cap S = \big\{(x^1, \dots, x^n) \in U \mid x^{k+1} = \ldots = x^n = 0\big\},$$

and $(x^1, \dots, x^k)$ are coordinates for $U \cap S$. Since the inclusion map $\iota\colon U \cap S \hookrightarrow M$ has the coordinate representation

$$\iota(x^1, \dots, x^k) = (x^1, \dots, x^k, 0, \dots, 0)$$

in these coordinates (see the proof of Theorem 5.9), it follows that $\mathrm{T}_pS \cong \mathrm{d}\iota_p(\mathrm{T}_pS)$ is exactly the subspace of T_pM spanned by

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^k}\right|_p.$$

If we write the coordinate representation of $v \in \mathrm{T}_pM$ as

$$v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p,$$

then $v \in \mathrm{T}_pS$ if and only if $v^j = 0$ for all $j > k$.

Let φ be a smooth bump function supported in U that is equal to 1 in a neighborhood of $p \in S$. Choose an index $j > k$ and consider the function $f(x) = \varphi(x)\, x^j$, extended to be zero on $M \setminus \operatorname{supp} \varphi$. Then f vanishes identically on S, so

$$0 = vf = \sum_{i=1}^n v^i \frac{\partial(\varphi(x)\, x^j)}{\partial x^i}(p) \overset{\text{product rule}}{\underset{\text{+properties}}{=\!=\!=\!=}} v^j.$$

Thus, $v \in \mathrm{T}_pS$, as desired. ☐

Finally, if an embedded submanifold is characterized by a defining map (recall also Proposition 5.14), then this map gives a concise characterization of its tangent space at each point.

Proposition 5.23. *Let M be a smooth manifold and let S be an embedded submanifold of M. If $\Phi\colon U \to N$ is a local defining map for S, then*

$$T_pS \cong \ker\big(d\Phi_p\colon T_pM \to T_{\Phi(p)}N\big) \quad \textit{for every } p \in S \cap U.$$

Proof. See [*Exercise Sheet* 8, *Exercise* 4]. ☐

Given a smooth manifold M and a subset S of M, it is important to bear in mind that there are two very different questions one can ask. The simplest question is whether S is an embedded submanifold of M. Since embedded submanifolds are exactly those subsets satisfying the local slice condition, this is simply a question about the subset S itself: either it is an embedded submanifold or it is not, and if so, then the topology and smooth structure making it into an embedded submanifold are uniquely determined according to [*Exercise Sheet* 8, *Exercise* 1].

A more subtle question is whether S can be an immersed submanifold of M. In this case, neither the topology nor the smooth structure is known in advance, so one needs to ask whether there exist *any* topology and smooth structure on S making it into an immersed submanifold. This question is not always straightforward to answer, and it can be especially tricky to prove that S is *not* an immersed submanifold. A typical approach is to assume that it is, and then use one or more of the following phenomena to derive a contradiction:

- At each $p \in S$, the tangent space T_pS is a subspace of T_pM, with the same dimension at each point.

- Each vector tangent to S is the velocity vector of some smooth curve in S.
- Each vector tangent to S annihilates every smooth function that is constant on S.

Here is an example of how this can be done; another one will be given in [*Exercise Sheet 8, Exercise 5(b)*].

Example 5.24. Consider the subset

$$S = \big\{(x, y) \in \mathbb{R}^2 \mid y = |x|\big\} \subseteq \mathbb{R}^2.$$

It is easy to check that $S \setminus \{(0,0)\}$ is an embedded 1-dimensional submanifold of $\mathbb{R}^2$, so if S itself is an immersed submanifold at all, then it must be 1-dimensional. Suppose there were some smooth manifold structure on S making it into an immersed submanifold. Then $\mathrm{T}_{(0,0)}S$ would be a 1-dimensional subspace of $\mathrm{T}_{(0,0)}\mathbb{R}^2$, so by Proposition 5.21 there would be a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ whose image is in S, and that satisfies $\gamma(0) = (0, 0)$ and $\gamma'(0) \neq 0$. Writing $\gamma(t) = \big(x(t), y(t)\big)$, we see that $y(t)$ takes a global minimum at $t = 0$, so $y'(0) = 0$. On the other hand, since every point $(x, y) \in S$ satisfies $x^2 = y^2$, we have $x^2(t) = y^2(t)$ for all $t \in (-\varepsilon, \varepsilon)$. Differentiating twice and setting $t = 0$, we conclude that $2x'(0)^2 = 2y'(0)^2 = 0$, which is a contradiction. Thus, there is no such smooth manifold structure on S (cf. the comment after the proof of Proposition 1.8).

CHAPTER 6

VECTOR BUNDLES

In Section 3.4 we saw that the tangent bundle $\mathrm{T}M$ of a smooth manifold M has a natural structure as a smooth manifold in its own right. The natural coordinates we constructed on $\mathrm{T}M$ make it look locally like the Cartesian product of an open subset of M^n with $\mathbb{R}^n$. This kind of structure arises quite frequently – a collection of vector spaces, one for each point in M, glued together in a way that looks *locally* like the Cartesian product of an open subset of M^n with $\mathbb{R}^n$, but *globally* may be "twisted". Such structures are called *vector bundles* and will be briefly discussed in this chapter. For more information about vector bundles we refer to [Lee13, Chapter 10] and [Lee09, Chapter 6].

There is a deep and extensive body of theory about vector bundles on manifolds, which we will not touch in this course. We introduce them primarily in order to have a convenient language for talking about the tangent bundle and structures like it; see Chapter 7 and Chapter 8.

6.1 Vector Bundles

Definition 6.1. Let M be a topological space. A *(real) vector bundle of rank k over M* is a topological space E together with a continuous surjective map $\pi\colon E \to M$ satisfying the following conditions:

(i) For each $p \in M$, the *fiber* $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional $\mathbb{R}$-vector space.

(ii) For each $p \in M$, there exists a neighborhood U of p in M and a homeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$, which is called a *local trivialization of E over U*, satisfying the following conditions (see Figure 6.1):

  - $\pi_U \circ \Phi = \pi$, where $\pi_U\colon U \times \mathbb{R}^k \to U$ is the projection.
  - For each $q \in U$, the restriction of Φ to E_q is an $\mathbb{R}$-vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

The space E is called *the total space* of the bundle, M is called its *base*, and π is called its *projection*.

If M and E are smooth manifolds, π is a smooth map, and the local trivializations can be chosen to be diffeomorphisms, then E is called a *smooth vector bundle over* M. In this case, any local trivialization that is a diffeomorphism onto its image is called a *smooth local trivialization*.

Figure 6.1: A local trivialization of a vector bundle

Exercise 6.2: Let $\pi\colon E \to M$ be a smooth vector bundle over a smooth manifold M. Show that π is a surjective smooth submersion.

Solution: By definition of a smooth vector bundle, the map π is smooth and surjective, so it remains to check that its differential is everywhere surjective. Let $q \in E$ and set $p := \pi(q) \in M$. Again by definition of a smooth vector bundle, there exists an open neighborhood U of p in M and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ (assuming that $\pi\colon E \to M$ is of rank k) such that $\pi_U \circ \Phi = \pi|_{\pi^{-1}(U)}$, where $\pi_U\colon U \times \mathbb{R}^k \to U$ is the projection to the first factor, which is a smooth submersion according to [*Exercise Sheet* 5, *Exercise* 4(a)]. It follows immediately from [*Exercise Sheet* 5, *Exercise* 3(a)(i)] and [*Exercise Sheet* 5, *Exercise* 3(c)(i)] that $\pi|_{\pi^{-1}(U)}$ is itself a smooth submersion, that is, its differential is surjective at every point of $\pi^{-1}(U)$, which is an open neighborhood of q in E. Since $q \in E$ was arbitrary, we conclude that π is a smooth submersion.

Definition 6.3. If there exists a local trivialization of E over all of M, called a *global trivialization of* E, then E is called a *trivial bundle*. In this case, E is homeomorphic to the product space $M \times \mathbb{R}^k$.

If $E \to M$ is a smooth vector bundle that admits a smooth global trivialization, then we say that E is *smoothly trivial*. In this case, E is diffeomorphic to $M \times \mathbb{R}^k$, not just homeomorphic (as in previous case).

Example 6.4.

(1) *Product bundles*: Given any topological space M, the product space $E = M \times \mathbb{R}^k$ with the map $\pi = \pi_M\colon M \times \mathbb{R}^k \to M$ as its projection is a rank-k vector bundle over M. Any such bundle, called a *product bundle*, is clearly trivial (with the identity map $\Phi = \mathrm{Id}_E\colon M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ as a global trivialization). If M is a smooth manifold, then the smooth product bundle $M \times \mathbb{R}^k$ is smoothly trivial.

(2) The *Möbius bundle*: see [Lee13, Example 10.3 and Problem 10.1]. This is a non-trivial smooth line bundle (i.e., a smooth vector bundle of rank 1) over $\mathbb{S}^1$.

Proposition 6.5 (The tangent bundle as a vector bundle)**.** *Let M be a smooth n-manifold and let* $\mathrm{T}M$ *be its tangent bundle. With its standard projection map* $\pi\colon \mathrm{T}M \to M$, *its natural vector space structure on each fiber, and the topology and smooth structure constructed in Proposition 3.12,* $\pi\colon \mathrm{T}M \to M$ *is a smooth vector bundle of rank n over* M.

Proof. Given any smooth chart (U, φ) for M with coordinate functions (x^i), define a map

$$\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n,\ v^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto \big(p, (v^1, \dots, v^n)\big).$$

This is linear on the fibers and satisfies $\pi_U \circ \Phi = \pi$. The composite map

$$\pi^{-1}(U) \xrightarrow{\quad\Phi\quad} U \times \mathbb{R}^n \xrightarrow{\phi\circ \mathrm{Id}_{\mathbb{R}^n}} \varphi(U) \times \mathbb{R}^n\,,$$

is equal to the coordinate map $\widetilde{\varphi}\colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$ constructed in Proposition 3.12. Since both $\widetilde{\varphi}$ and $\varphi \times \mathrm{Id}_{\mathbb{R}^n}$ are diffeomorphisms, so is Φ. Therefore, Φ satisfies all the conditions for a smooth local trivialization. $\square$

Any bundle that is not trivial requires more than one local trivialization. The next lemma shows that the composition of two smooth local trivializations has a simple form where they overlap.

Lemma 6.6. *Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over M. Suppose that*

$$\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k \quad \textit{and} \quad \Psi\colon \pi^{-1}(V) \to V \times \mathbb{R}^k$$

are two smooth local trivializations of E with $U \cap V \neq \varnothing$. Then there exists a smooth map

$$\tau\colon U \cap V \to \mathrm{GL}(k, \mathbb{R}),$$

called the transition function between the smooth local trivializations Φ and Ψ, *such that the composition*

$$\Phi \circ \Psi^{-1}\colon (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$$

has the form

$$\big(\Phi \circ \Psi^{-1}\big)(p, v) = \big(p, \tau(p) \cdot v\big).$$

Proof. Note that the following diagram commutes:

$$\begin{tikzcd}
(U \cap V) \times \mathbb{R}^k \arrow[ddrr, "\pi_1"'] & & \pi^{-1}(U \cap V) \arrow[ll, "\Psi"'] \arrow[rr, "\Phi"] \arrow[dd, "\pi"] & & (U \cap V) \times \mathbb{R}^k \arrow[ddll, "\pi_1 = \pi_{U \cap V}"] \\
& \circlearrowright & & \circlearrowright & \\
& & U \cap V & &
\end{tikzcd}$$

and thus $\pi_1 \circ (\Phi \circ \Psi^{-1}) = \pi_1$, which means that

$$\big(\Phi \circ \Psi^{-1}\big)(p, v) = \big(p, \sigma(p, v)\big)$$

for some smooth map $\sigma\colon (U \cap V) \times \mathbb{R}^k \to \mathbb{R}^k$. Moreover, for each fixed $p \in U \cap V$, the map $v \mapsto \sigma(p, v)$ is an invertible linear map (since both $\Phi|_{E_p}$ and $\Psi|_{E_p}$ are $\mathbb{R}$-linear isomorphisms), so there exists an invertible $k \times k$ matrix $\tau(p)$ such that $\sigma(p, v) = \tau(p) \cdot v$. It remains to show that the map $\tau\colon U \cap V \to \mathrm{GL}(k, \mathbb{R})$ is smooth; this is done in [*Exercise Sheet* 9*, Exercise* 1(*a*)]. $\square$

Vector bundles are often most easily described by giving a collection of vector spaces, one for each point of the base manifold. In order to make such a set into a vector bundle, we would first have to construct a manifold topology and a smooth structure on the disjoint union of all the vector spaces, and then construct the local trivializations and show that they have the requisite properties. The next lemma provides a shortcut (cf. Lemma 1.11) by showing that it is sufficient to construct the local trivializations, as long as they overlap with smooth transition functions. (See also [*Exercise Sheet* 9*, Exercise* 2] for a stronger form of the result.)

Lemma 6.7 (Vector bundle chart lemma)**.** *Let M be a smooth manifold. Suppose that for each $p \in M$ we are given an $\mathbb{R}$-vector space E_p of some fixed dimension k. Set $E := \bigsqcup_{p\in M} E_p$ and consider the map $\pi\colon E \to M$, $v \in E_p \mapsto p \in M$. Suppose furthermore that we are given the following data:*

(i) *an open cover $\{U_\alpha\}_{\alpha\in A}$ of M,*

(ii) *for each $\alpha \in A$, a bijective map $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is an $\mathbb{R}$-vector space isomorphism from E_p to $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$,*

(iii) *for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \varnothing$, a smooth map $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that the composition*

$$\Phi_\alpha \circ \Phi_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

has the form

$$\big(\Phi_\alpha \circ \Phi_\beta^{-1}\big)(p, v) = \big(p, \tau_{\alpha\beta}(p) \cdot v\big).$$

Then E has a unique topology and smooth structure making it into a smooth manifold and a smooth vector bundle of rank k over M, with π as projection and $\big\{(U_\alpha, \Phi_\alpha)\big\}_{\alpha\in A}$ as smooth local trivializations.

Proof. For the details of the proof, which relies essentially on Lemma 1.11, we refer to [Lee13, Lemma 10.6]. ☐

Here are some examples showing how the vector bundle chart lemma can be used to construct new vector bundles from old ones.

Example 6.8 (Whitney sums)**.** Let M be a smooth manifold. Let $\pi'\colon E' \to M$ and $\pi''\colon E'' \to M$ be two smooth vector bundles of ranks k' and k'', respectively, over M. We will construct a new smooth vector bundle $\pi\colon E \to M$ of rank $k' + k''$ over M, denoted by $E' \oplus E''$ and called *the Whitney sum of E' and E''*, whose fiber over each $p \in M$ is the direct sum $E_p := E'_p \oplus E''_p$, which is a $(k' + k'')$-dimensional $\mathbb{R}$-vector space. For each $p \in M$ choose a small enough neighborhood U of p so that there exist local trivializations (U, Φ') of E' and (U, Φ'') of E'', and define the map

$$\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^{k'+k''},\ \ \Phi(v', v'') := \big(\pi'(v'), \big(\pi_{\mathbb{R}^{k'}} \circ \Phi'(v'),\ \pi_{\mathbb{R}^{k''}} \circ \Phi''(v'')\big)\big)\,.$$

Suppose that we are given another such pair of local trivializations $\big(\widetilde{U}, \widetilde{\Phi}'\big)$ and $\big(\widetilde{U}, \widetilde{\Phi}''\big)$. Let $\tau'\colon U \cap \widetilde{U} \to GL(k', \mathbb{R})$ and $\tau''\colon U \cap \widetilde{U} \to GL(k'', \mathbb{R})$ be the corresponding transition functions. Then the transition function for $E' \oplus E''$ has the form

$$\widetilde{\Phi} \circ \Phi^{-1}\big(p, (v', v'')\big) = \big(p,\, \tau(p)(v', v'')\big),$$

where $\tau(p) := \tau'(p) \oplus \tau''(p) \in GL(k' + k'', \mathbb{R})$ is the block diagonal matrix

$$\begin{pmatrix} \tau'(p) & \mathbb{O} \\ \mathbb{O} & \tau''(p) \end{pmatrix}.$$

Since this depends smoothly on p, it follows from Lemma 6.7 that $E' \oplus E''$ is a smooth vector bundle over M.

Example 6.9 (Restriction of a vector bundle)**.** Let $\pi\colon E \to M$ be a rank-k vector bundle and let $S \subseteq M$ be any subset. We define *the restriction of E to S* to be the set $E|_S := \bigcup_{p\in S} E_p$, with the projection $E|_S \to S$ obtained by restricting π. If $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a local trivialization of E over $U \subseteq M$, it restricts to a bijective map from $(\pi|_S)^{-1}(U \cap S)$ to $(U \cap S) \times \mathbb{R}^k$, and it is easy to check that these form local trivializations for a vector bundle structure on $E|_S$.

- If E is a smooth vector bundle over M and $S \subseteq M$ is an embedded submanifold, then it follows easily from Lemma 6.7 that $E|_S$ is a smooth vector bundle over M.

- If E is a smooth vector bundle over M, but $S \subseteq M$ is merely immersed, then we give $E|_S$ a topology and a smooth structure making it into a smooth rank-k vector bundle over S as follows: For each $p \in S$, choose a neighborhood U of p in M over which there is a smooth local trivialization Φ of E, and a neighborhood V of p in S that is embedded in M and contained in U (see Proposition 5.20). Then the restriction of Φ to $\pi^{-1}(V)$ is a bijection from $\pi^{-1}(V)$ to $V \times \mathbb{R}^k$, and we can apply Lemma 6.7 to these bijections to yield the desired structure.

In particular, if $S \subseteq M$ is a smooth (immersed or embedded) submanifold, then $\mathrm{T}M|_S$ is called *the ambient tangent bundle over S*.

6.2 Sections of Vector Bundles

Definition 6.10. Let $\pi\colon E \to M$ be a (topological) vector bundle. A *local section of E* is a continuous map $\sigma\colon U \to E$ defined on some open subset $U \subseteq M$ and satisfying $\pi \circ \sigma = \mathrm{Id}_U$ (see Figure 6.2). This means that $\sigma(p) \in E_p = \pi^{-1}(p)$ for every $p \in U$. A *global section of E* is a section of E defined on all of M, i.e., a continuous map $\sigma\colon M \to E$ such that $\pi \circ \sigma = \mathrm{Id}_M$. In particular, the *zero section of E* is the global section $\zeta\colon M \to E$ of E defined by $\zeta(p) = 0 \in E_p$ for each $p \in M$. Moreover, a *rough (local or global) section of E* over an open subset $U \subseteq M$ is defined to be a (not necessarily continuous) map $\sigma\colon U \to E$ such that $\pi \circ \sigma = \mathrm{Id}_U$.

Finally, if E is a smooth vector bundle over a smooth manifold M, then a *smooth (local or global) section of E* is one that is a smooth map from its domain to E.

- ⇝ A (rough) local section of E over U is the same as a (rough) global section of the restricted bundle $E|_U$.
- ⇝ The zero section ζ of a vector bundle $E \to M$ is continuous, and if $E \to M$ is a smooth vector bundle, then ζ is smooth; see [*Exercise Sheet* 9, *Exercise* 3(*a*)].

If $E \to M$ is a smooth vector bundle, then the set of all smooth global sections of E is an $\mathbb{R}$-vector space under pointwise addition and scalar multiplication:

$$(c_1\sigma_1 + c_2\sigma_2)(p) := c_1\sigma_1(p) + c_2\sigma_2(p).$$

This vector space is usually denoted by $\Gamma(E)$ – but for particular smooth vector bundles we often introduce specialized notation for their spaces of smooth global sections, see for instance p. 69 or p. 87 – and it is actually infinite-dimensional; see [*Exercise Sheet* 9,

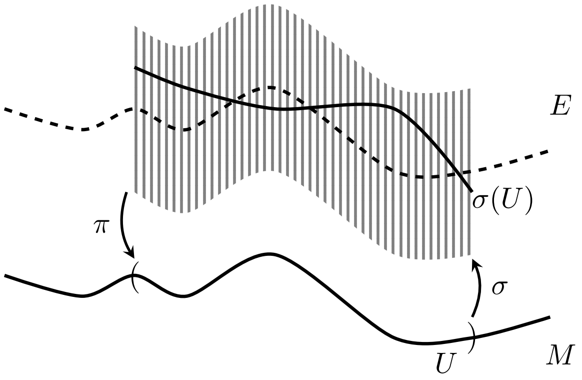


Figure 6.2: A local section of a vector bundle

Exercise 3], Exercise 2.21 and Exercise 6.12. Moreover, smooth global sections of $E \to M$ can be multiplied by smooth real-valued functions: If $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$, then we obtain a new smooth section $f\sigma \in \Gamma(E)$ defined by

$$(f\,\sigma)(p) := \underset{\mathbb{R}}{\underset{\cap}{f(p)}}\,\underset{E_p}{\underset{\cap}{\sigma(p)}}. \tag{6.1}$$

Therefore, $\Gamma(E)$ is a module over the ring $C^\infty(M)$.

- ⇝ The various smoothness claims made above are demonstrated in [*Exercise Sheet* 9, *Exercise* 3(*b*)].
- ⇝ The (smooth) global sections of a (smooth) product bundle are thoroughly discussed in [*Exercise Sheet* 9, *Exercise* 3(*c*)].

Lemma 6.11 (Extension lemma for smooth vector bundles)**.** *Let $\pi\colon E \to M$ be a smooth vector bundle. Let $A \subseteq M$ be a closed subset and let $\sigma\colon A \to E$ be a section of $E|_A$ that is* smooth *in the sense that σ extends to a smooth local section of E in a neighborhood of each point. Then for each open subset $U \subseteq M$ containing A, there exists a smooth global section $\widetilde{\sigma} \in \Gamma(E)$ such that $\widetilde{\sigma}|_A = \sigma$ and*

$$\operatorname{supp}\widetilde{\sigma} := \overline{\{p \in M \mid \widetilde{\sigma}(p) \neq 0\}} \subseteq U.$$

Proof. Exercise! (Similar to the proof of Lemma 2.22). □

Exercise 6.12: Let $\pi\colon E \to M$ be a smooth vector bundle. Show that each element of E is in the image of a smooth global section of E.

Solution (1st way)**:** Fix $q \in E$ and set $p := \pi(q) \in M$. Consider the closed subset $A := \{p\} \subseteq M$ and the section

$$\sigma\colon A \to E,\ p \mapsto q \in E_p$$

of $E|_A = E_p$. We claim that σ extends to a smooth local section of E over some open neighborhood of p. Granting this claim for a moment, by Lemma 6.11 there exists a smooth global section $\widetilde{\sigma}$ of E such that $\widetilde{\sigma}|_A = \sigma$; in particular, we have $\widetilde{\sigma}(p) = \sigma(p) = q$, which shows that $q \in E$ lies in the image of the smooth global section $\widetilde{\sigma} \in \Gamma(E)$.

We now prove the above claim. By definition of a smooth vector bundle, there exists an open neighborhood U of p in M and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that

$$\pi_U \circ \Phi = \pi|_{\pi^{-1}(U)},$$

where $\pi_U\colon U \times \mathbb{R}^k \to U$ is the projection to the first factor. Since $q \in \pi^{-1}(U)$, its image under Φ is a pair $(p, v_q) \in U \times \mathbb{R}^k$ for some vector $v_q \in \mathbb{R}^k$. Consider now the map

$$t\colon U \to U \times \mathbb{R}^k,\ x \mapsto (x, v_q),$$

which is smooth by [*Exercise Sheet* 3, *Exercise* 4(*b*)], and the composite map

$$s := \Phi^{-1} \circ t\colon U \to \pi^{-1}(U),\ x \mapsto \Phi^{-1}(x, v_q),$$

which is also smooth by Proposition 2.11(d) and satisfies

$$s(p) = \Phi^{-1}(p, v_q) = q = \sigma(p).$$

Moreover, we have

$$(\pi \circ s)(x) = \big((\pi \circ \Phi^{-1}) \circ t\big)(x) = (\pi_U \circ t)(x) = x = \mathrm{Id}_U(x) \quad \text{for every } x \in U.$$

Therefore, $s\colon U \to E$ is a smooth section of E over U and may also be regarded as a smooth extension of $\sigma\colon A \to E$. This proves the claim and completes the proof.

Solution (2nd way)**:** Fix $q \in E$ and set $p := \pi(q) \in M$. Consider the closed subset $A := \{p\} \subseteq M$ and the section

$$\sigma\colon A \to E,\ p \mapsto q \in E_p$$

of $E|_A = E_p$. There exists a smooth local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ of E over an open neighborhood U of p, and hence a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over U (associated with Φ) by [*Exercise Sheet* 9, *Exercise* 5(*a*)]. We may thus write

$$\sigma(p) = \sum_{i=1}^{k} v^i\, \sigma_i(p) \in E_p$$

for some uniquely determined constants $v^i \in \mathbb{R}$, $1 \leq i \leq k$. We now define the map

$$s\colon U \to E,\ x \mapsto \sum_{i=1}^{k} v^i\, \sigma_i(x) \in E_x.$$

Note that s is a rough section of π, since $\big(\pi \circ s\big)(x) = x = \mathrm{Id}_U(x)$, and it is actually smooth by Proposition 6.15, since its component functions with respect to the smooth local frame $(\sigma_1, \dots, \sigma_k)$ are constant (namely, the constants $v^i \in \mathbb{R}$). Since we clearly have $s(p) = \sigma(p)$, the section s is a smooth extension of $\sigma\colon A \to E$ over U. Thus, the statement follows readily from Lemma 6.11 (as before).

6.2.1 Local and Global Frames

Definition 6.13. Let $E \to M$ be a vector bundle. If $U \subseteq M$ is an open subset, then a k-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ of E over U is said to be *linearly independent* if their values $\big(\sigma_1(p), \dots, \sigma_k(p)\big)$ form a linearly independent k-tuple in E_p for each $p \in U$. Similarly, they are said to *span* E if their values span E_p for each $p \in U$.

A *local frame for E over U* is an ordered k-tuple $(\sigma_1, \dots, \sigma_k)$ of linearly independent local sections of E over U that span E; thus, $\big(\sigma_1(p), \dots, \sigma_k(p)\big)$ is a basis for the fiber E_p for each $p \in U$. It is called a *global frame* if $U = M$. We often denote a frame $(\sigma_1, \dots, \sigma_k)$ by (σ_i).

If, moreover, $E \to M$ is a smooth vector bundle, then a *local or global frame* for E is said to be *smooth* if each σ_i is a smooth section of E.

⇝ For the *completion of smooth local frames* for smooth vector bundles see [*Exercise Sheet* 9, *Exercise* 4].

Example 6.14 (Global frame for a product bundle)**.** If $E = M \times \mathbb{R}^k \to M$ is a (smooth) product bundle over a (smooth) manifold M, then the standard basis $(e_1, \dots, e_k)$ for $\mathbb{R}^k$ yields a (smooth) global frame $\widetilde{e}_i$ for E, defined by

$$\widetilde{e}_i \colon M \to E,\ p \mapsto (p, e_i).$$

⇝ For the correspondence between (smooth) local frames and (smooth) local trivializations see [*Exercise Sheet* 9, *Exercise* 5]. This also settles the question of the existence of smooth local frames.

We conclude this section with the important observation that smoothness of sections of smooth vector bundles can be characterized in terms of local frames.

Assume that (σ_i) is a smooth local frame for a smooth vector bundle $E \to M$ over some open subset $U \subseteq M$. If $\tau \colon M \to E$ is a rough section, then the value of τ at an arbitrary point $p \in U$ can be written as

$$\tau(p) = \tau^i(p)\, \sigma_i(p)$$

for some uniquely determined numbers $\big(\tau^1(p), \dots, \tau^k(p)\big)$. This clearly defines k-functions $\tau^i \colon U \to \mathbb{R}$, called *the component functions of τ with respect to the given local frame (σ_i).*

Proposition 6.15 (Local frame criterion for continuity/smoothness)**.** *Let $\pi \colon E \to M$ be a continuous (respectively smooth) vector bundle and let $\tau \colon M \to E$ be a rough section. If (σ_i) is a continuous (respectively smooth) local frame for E over an open subset $U \subseteq M$, then τ is continuous (respectively smooth) if and only if its component functions with respect to (σ_i) are continuous (respectively smooth).*

Proof. We prove the statement in the smooth case; the topological case can be treated similarly. Let $\Phi \colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ be the smooth local trivialization associated with the smooth local frame (σ_i), see [*Exercise Sheet* 9, *Exercise* 5(*b*)]. By the construction of Φ in [*Exercise Sheet* 9, *Exercise* 5(*b*)] we know that

$$\big(\Phi \circ \sigma_i\big)(q) = (q, e_i) \quad \text{for all } q \in U,$$

which yields

$$\big(\Phi\circ\tau\big)(p)=\Phi\big(\tau^i(p)\,\sigma_i(p)\big)=\big(p,\tau^i(p)e_i\big)=\Big(p,\big(\tau^1(p),\dots,\tau^k(p)\big)\Big),$$

where (τ^i) are the component functions of τ with respect to (σ_i). Since Φ is a diffeomorphism, τ is smooth on U if and only if $\Phi\circ\tau$ is smooth on U. Therefore, according to [*Exercise Sheet* 3*, Exercise* 4(*b*)], $\Phi\circ\tau$ is smooth if and only if the component functions τ^i are smooth. □

Note that Proposition 6.15 applies equally well to local sections, since a local section of E over an open subset $V\subset M$ is a global section of the restricted bundle $E|_V$.

6.3 Subbundles

Definition 6.16. Given a vector bundle $\pi_E\colon E\to M$, a *subbundle of* E is a vector bundle $\pi_D\colon D\to M$, in which D is a topological subspace of E and π_D is the restriction of π_E to D, such that for each $p\in M$, the subset $D_p=D\cap E_p$ is a linear subspace of E_p, and the vector space structure on D_p is the one inherited from E_p.

If $E\to M$ is a smooth vector bundle, then a subbundle of E is called a *smooth subbundle* if it is a smooth vector bundle and an embedded submanifold of E.

Note that the condition that D be a vector bundle over M implies that all of the fibers D_p must be non-empty and have the same dimension.

The following lemma gives a convenient condition for checking that a union of subspaces $\big\{D_p\subseteq E_p \mid p\in M\big\}$ is a smooth subbundle.

Lemma 6.17 (Local frame criterion for subbundles)**.** *Let $\pi\colon E\to M$ be a smooth vector bundle of rank k. Suppose that for each $p\in M$ we are given an m-dimensional linear subspace $D_p\subseteq E_p$. Then $D=\bigcup_{p\in M}D_p\subseteq E$ is a smooth subbundle of E if and only if the following condition is satisfied:*

"Each point of M has a neighborhood U on which there exist smooth local sections $\sigma_1,\dots,\sigma_m\colon U\to E$ with the property that $\sigma_1(q),\dots,\sigma_m(q)$ form a basis for D_q at each $q\in U$."

Proof. If $D\to M$ is a smooth subbundle of $E\to M$, then by definition each $p\in M$ has a neighborhood U over which there exists a smooth local trivialization of D, and [*Exercise Sheet* 9*, Exercise* 5(*a*)] shows that there exists a smooth local frame for D over U, namely a collection of smooth sections $\tau_1,\dots,\tau_m\colon U\to D$ whose images form a basis for D_q at each point $q\in U$. The smooth sections of E that we seek are obtained by composing with the inclusion map $\iota\colon D\hookrightarrow E$; that is, $\sigma_j=\iota\circ\tau_j$ for $j\in\{1,\dots,m\}$.

For the details of the proof of the converse direction, which uses [*Exercise Sheet* 9*, Exercise* 4(*a*)] and [*Exercise Sheet* 9*, Exercise* 5(*b*)], we refer to [Lee13, Lemma 10.32]. □

Example 6.18 (Subbundles)**.**

(1) Let M be a smooth manifold and let $S\subseteq M$ be an immersed k-submanifold. Then the tangent bundle $\mathrm{T}S$ is a smooth rank-k subbundle of the ambient tangent bundle $\mathrm{T}M|_S$; see [Lee13, Problem 10-14].

(2) Let $E \to M$ be any smoothly trivial vector bundle of rank k and let $(E_1, \dots, E_k)$ be a smooth global frame for E. If $m \in \{0, \dots, k\}$, then the subset $D \subseteq E$ defined by $D_p = \operatorname{span}\bigl(E_1|_p, \dots, E_m|_p\bigr)$ for each $p \in M$ is a smooth rank-m subbundle of E.

(3) If M is a smooth manifold and if V is a nowhere-vanishing smooth vector field on M (see Chapter 7), then the set $D \subseteq \mathrm{T}M$ whose fiber at each $p \in M$ is the linear span of $V_p \in \mathrm{T}_pM \setminus \{0\}$ is a smooth 1-dimensional subbundle of $\mathrm{T}M$.

CHAPTER 7

VECTOR FIELDS AND FLOWS

7.1 Vector Fields

Definition 7.1. A *rough* (resp. *continuous*, *smooth*) *vector field* on a smooth manifold M is a rough (resp. continuous, smooth) global section of the tangent bundle $\pi\colon \mathrm{T}M \to M$.

More concretely, a vector field is a map $X\colon M \to \mathrm{T}M$, usually written $p \mapsto X_p$, with the property that $\pi \circ X = \mathrm{Id}_M$ or, equivalently, $X_p \in \mathrm{T}_pM$ for each $p \in M$. *The support of* X is defined as the closure of the set $\{p \in M \mid X_p \neq 0\}$. In particular, we say that X is *compactly supported* if its support is a compact set.

If $U \subseteq M$ is open, then the fact that T_pU is naturally identified with T_pM for each $p \in U$ (see Proposition 3.9) allows us to identify $\mathrm{T}U$ with the open subset $\pi^{-1}(U) \subseteq \mathrm{T}M$. Therefore, a vector field on U can be thought of either as a map $U \to \mathrm{T}U$ or as a map $U \to \mathrm{T}M$. If X is a vector field on M, then its restriction $X|_U$ is a vector field on U, which is smooth if X itself smooth.

A (continuous) vector field on an open subset $U \subseteq \mathbb{R}^n$ is simply a continuous map $U \to \mathbb{R}^n$, which can be visualized as attaching an "arrow" to each point of U. Similarly, we think of a (continuous) vector field on an abstract smooth manifold M as an arrow attached to each point of M, chosen to be tangent to M and to vary continuously from point to point (see Figure 7.1).

Figure 7.1: A vector field

⇝ *Extension lemma for vector fields*: this is a special case of Lemma 6.11.

⇝ Exercise 6.12 for $E = \mathrm{T}M$ yields the following *application*: Any tangent vector at a point can be extended to a smooth vector field on the entire manifold.

⇝ The set $\mathfrak{X}(M)$ of all smooth (global) vector fields on a smooth manifold M is an infinite-dimensional $\mathbb{R}$-vector space and a module over the ring $C^\infty(M)$: this is a special case of [*Exercise Sheet* 9*, Exercise* 3], taking also Exercise 2.21 and Exercise 6.12 (in the above special case) into account.

- ⇝ *Local* (resp. *global*) *frame* for M = local (resp. global) frame for $\mathrm{T}M$, see Definition 6.13.
- ⇝ *Completion of smooth local frames for* M: this is a special case of [*Exercise Sheet* 9, *Exercise* 4].

Let M be a smooth n-manifold and let $X\colon M \to \mathrm{T}M$ be a rough vector field on M. If $\big(U,(x^i)\big)$ is a smooth coordinate chart for M, then we can write the value of X at any point $p \in U$ in terms of the coordinate basis vectors:

$$X_p = X^i(p)\,\frac{\partial}{\partial x^i}\bigg|_p\,. \tag{7.1}$$

This defines n functions $X^i\colon U \to \mathbb{R}$, called *the component functions of* X *in the given chart*.

Proposition 7.2 (Smoothness criterion for vector fields)**.** *Let M be a smooth manifold and let $X\colon M \to \mathrm{T}M$ be a rough vector field on M. If $\big(U,(x^i)\big)$ is a smooth coordinate chart for M, then the restriction of X to U is smooth if and only if its components functions with respect to this chart are smooth.*

Proof. If $\big(U,\,\varphi=(x^i)\big)$ is a smooth coordinate chart for M, then $\big(\pi^{-1}(U),\,\widetilde{\varphi}=(x^i,v^i)\big)$ are the corresponding natural coordinates on $\mathrm{T}M$ (see Proposition 3.12), and the coordinate representation of X with respect to these charts is

$$\begin{aligned}\widehat{X}(x^1,\dots,x^n) &= \widetilde{\varphi}\left(X^i\big(\varphi^{-1}(x)\big)\,\frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(x)}\right)\\ &= \Big(x^1,\dots,x^n,X^1\big(\varphi^{-1}(x)\big),\dots,X^n\big(\varphi^{-1}(x)\big)\Big).\end{aligned}$$

Therefore, X is smooth on U if and only if its component functions X^i, $i \in \{1,\dots,n\}$, are smooth on U. □

Example 7.3 (Examples of smooth vector fields)**.**

(1) *Coordinate vector fields*: If $\big(U,(x^i)\big)$ is any smooth chart for M, then the assignment

$$p \mapsto \frac{\partial}{\partial x^i}\bigg|_p$$

determines a vector field on U, called *the i-th coordinate vector field* and denoted by $\frac{\partial}{\partial x^i}$. It is smooth by Proposition 7.2, because its component functions are constant.

In particular, the coordinate vector fields form a smooth local frame $\big(\frac{\partial}{\partial x^i}\big)$ for $\mathrm{T}M$, called a *coordinate frame*. Every point of M is in the domain of such a local frame.

(2) *The Euler vector field*: The vector field V on $\mathbb{R}^n$ whose value at a point $x = (x^1,\dots,x^n) \in \mathbb{R}^n$ is

$$V_x = x^1\,\frac{\partial}{\partial x_1}\bigg|_x + \ldots + x^n\,\frac{\partial}{\partial x_n}\bigg|_x$$

is clearly smooth, because its coordinate functions are linear. It vanishes at the origin $(0,\dots,0) \in \mathbb{R}^n$, and points radially outward everywhere else. It is called the *Euler vector field* because of its appearance in *Euler's homogeneous function theorem* (see Exercise 7.4).

Exercise 7.4: Given $c \in \mathbb{R}$, let $f\colon \mathbb{R}^n \smallsetminus \{0\} \to \mathbb{R}$ be a smooth function which is *positively homogeneous of degree* c, i.e., $f(\lambda x) = \lambda^c f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Prove that $Vf = cf$, where V is Euler's vector field on $\mathbb{R}^n$.

[Hint: Differentiate the above relation with respect to both x^i and λ.]

Comment: The statement from Exercise 7.4 is referred to as the *Euler's homogeneous function theorem* in the literature. In fact, it can also be shown that the *converse* to Euler's homogeneous function theorem holds; namely, *if* $f \in C^\infty\big(\mathbb{R}^n \setminus \{(0,0)\}\big)$ *satisfies* $Vf = cf$*, where* V *is the Euler vector field on* $\mathbb{R}^n$ *and* $c \in \mathbb{R}$*, then* f *is positively homogeneous of degree* c*.*

$\rightsquigarrow$ We will encounter many more examples of vector fields (especially on $\mathbb{R}^n$) later in the course and in the exercise sheets as well.

7.1.1 Vector Fields as Derivations of $C^\infty(M)$

An essential property of smooth vector fields is that they define operators on the space of smooth real-valued functions. If $X \in \mathfrak{X}(M)$ and $f \in C^\infty(U)$, where $U \subseteq M$ is open, then we obtain a new function

$$Xf\colon U \to \mathbb{R},\ p \mapsto (Xf)(p) \coloneqq X_p f.$$

(Caution: Do not confuse the notations fX and Xf: the former is a smooth vector field on U obtained by multiplying X by f, that is, $(fX)(p) = f(p)X_p$, while the latter is the real-valued function on U obtained by applying the vector field X to the smooth function f.) Since the action of a tangent vector on a function is determined by the values of the function in any arbitrary small neighborhood (see Proposition 3.8), it follows that Xf is locally determined. In particular, for any open subset $V \subseteq U$, we have

$$(Xf)|_V = X\big(f|_V\big).$$

This construction yields another useful smoothness criterion for vector fields.

Proposition 7.5 (Smoothness criterion for vector fields)**.** *Let* M *be a smooth manifold and let* $X\colon M \to \mathrm{T}M$ *be a rough vector field on* M*. The following are equivalent:*

(a) X *is smooth.*

(b) *For every* $f \in C^\infty(M)$*, the function* $Xf\colon M \to \mathbb{R}$ *is smooth.*

(c) *For every open subset* $U \subseteq M$ *and every* $f \in C^\infty(U)$*, the function* $Xf\colon U \to \mathbb{R}$ *is smooth.*

Proof.

(a) $\Rightarrow$ (b): Given $p \in M$, pick a smooth chart $\big(U, (x^i)\big)$ for M containing p. For $x \in U$ we may write

$$(Xf)(x) = \left(X^i(x) \left.\frac{\partial}{\partial x^i}\right|_x\right) f = X^i(x) \frac{\partial \widehat{f}}{\partial x^i}(\widehat{x}).$$

Since the component functions X^i of X are smooth on U by assumption and by Proposition 7.2, it follows that Xf is smooth on U. We conclude by Proposition 2.9(a).

(b) ⇒ (c): Fix an open subset $U \subseteq M$ and $f \in C^\infty(U)$. For any $p \in U$, let ψ be a smooth bump function that is equal to 1 in a neighborhood of p and supported in U (see Proposition 2.20), and define $\widetilde{f} = \psi f$, extended to be zero on $M \setminus \operatorname{supp}\psi$. Then $X\widetilde{f}$ is smooth by assumption, and equal to Xf in a neighborhood of p by construction (and by the above discussion). We conclude by Proposition 2.9(a).

(c) ⇒ (a): If (x^i) are smooth local coordinates on an open subset $U \subseteq M$, then we think of each coordinate x^i as a smooth function on U, and we have

$$X(x^i) = \Big(X^j \frac{\partial}{\partial x^j}\Big)(x^i) \overset{\frac{\partial x^i}{\partial x^j} = \delta^i_j}{=\!=\!=\!=} X^i\,,$$

which is smooth by assumption. We conclude by Proposition 2.9(a) and Proposition 7.2. □

One direct consequence of Proposition 7.5 is that a smooth vector field $X \in \mathfrak{X}(M)$ defines a map

$$C^\infty(M) \to C^\infty(M),\ f \mapsto Xf,$$

which (as can be checked pointwise) is $\mathbb{R}$-linear and satisfies the following product rule for vector fields:

$$X(fg) = f\,Xg + g\,Xf;$$

in other words, this map is a *derivation of* $C^\infty(M)$.

The next proposition shows that derivations of $C^\infty(M)$ can be identified with smooth vector fields. Due to this result, we sometimes identify smooth vector fields on M with derivations of $C^\infty(M)$, using the same letter for both the vector field (thought of as a smooth map $M \to \mathrm{T}M$) and the derivation (thought of as a linear map $C^\infty(M) \to C^\infty(M)$).

Proposition 7.6. *Let M be a smooth manifold. A map $D\colon C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some $X \in \mathfrak{X}(M)$.*

Proof. We already showed above that any smooth vector field on M induces a derivation of $C^\infty(M)$. Let us prove the converse below.

Let $p \in M$ and consider the map

$$X_p\colon C^\infty(M) \to \mathbb{R},\ f \mapsto (Df)(p).$$

Since D is $\mathbb{R}$-linear, X_p is also $\mathbb{R}$-linear, and since D is a derivation, we have

$$\begin{aligned} X_p(fg) = D(fg)(p) &= \big(f\,D(g) + g\,D(f)\big)(p) \\ &= f(p)\,D(g)(p) + g(p)\,D(f)(p) \\ &= f(p)\,X_p g + g(p)\,X_p f. \end{aligned}$$

Hence, X_p is a derivation at $p \in M$, i.e., $X_p \in \mathrm{T}_pM$. We obtain thus a rough vector field

$$X\colon M \to \mathrm{T}M,\ p \mapsto X_p,$$

but since $Xf = Df$ is smooth for every $f \in C^\infty(M)$, X is actually smooth by Proposition 7.5, so we are done. □

7.1.2 Vector Fields and Smooth Maps

If $F\colon M \to N$ is a smooth map and if X is a (rough) vector field on M, then for each point $p \in M$ we obtain a tangent vector $dF_p(X_p) \in \mathrm{T}_{F(p)}N$ by applying the differential of F at p to the tangent vector $X_p \in \mathrm{T}_pM$. However, this does not define a (rough) vector field on N in general. For example, if F is not surjective, there is no way to decide what tangent vector to assign to a point $q \in N \setminus F(M)$, while if F is not injective, then for some points of N there may be several different tangent vectors obtained by applying dF to X at different points of M.

Let $F\colon M \to N$ be a smooth map and let X be a (rough) vector field on X. If there exists a (rough) vector field Y on N such that $dF_p(X_p) = Y_{F(p)}$ for each $p \in M$, then X and Y are said to be *F-related*.

Lemma 7.7. *Let $F\colon M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then X and Y are F-related if and only if for every smooth real-valued function f defined on an open subset of N, we have*

$$X(f \circ F) = (Yf) \circ F.$$

Proof. See [*Exercise Sheet* 10, *Exercise* 1(*a*)]. ☐

It is important to remember that for a given smooth map $F\colon M \to N$ and vector field $X \in \mathfrak{X}(M)$, there may not be any vector field on N that is F-related to X. There is one special case, however, in which there is always such a vector field, as the next proposition shows.

Proposition 7.8. *Let $F\colon M \to N$ be a diffeomorphism. For every smooth vector field X on M there exists a unique smooth vector field Y on N that is F-related to X. The smooth vector field Y is denoted by F_*X and is called the* pushforward of X by F.

Proof. See [*Exercise Sheet* 10, *Exercise* 1(*c*)]. ☐

7.1.3 Vector Fields and Submanifolds

If $S \subseteq M$ is an immersed or embedded submanifold, a (rough) vector field X on M does not necessarily restrict to a (rough) vector field on S, because $X_p \in \mathrm{T}_pM$ may not lie in the subspace $\mathrm{T}_pS \subseteq \mathrm{T}_pM$ at a point $p \in S$; we refer to Subsection 5.3 for various characterizations of T_pS. Given a point $p \in S$, a (rough) vector field X on M is said to be *tangent to S at p* if $X_p \in \mathrm{T}_pS \subseteq \mathrm{T}_pM$. It is called *tangent to S* if it is tangent to S at all points of S (see Figure 7.2).

Figure 7.2: A vector field tangent to a submanifold

Proposition 7.9. *Let M be a smooth manifold, $S \subseteq M$ be an embedded submanifold, and $X \in \mathfrak{X}(M)$. Then X is tangent to S if and only if $(Xf)|_S = 0$ for every $f \in C^\infty(M)$ such that $f|_S \equiv 0$.*

Proof. This is an immediate consequence of Proposition 5.22. ☐

Proposition 7.10 (Restricting smooth vector fields to submanifolds)**.** *Let M be a smooth manifold, let S be an immersed submanifold of M, and let $\iota\colon S \hookrightarrow M$ be the inclusion map. The following statements hold:*

(a) *If $Y \in \mathfrak{X}(M)$ and if there is $X \in \mathfrak{X}(S)$ that is ι-related to Y, then $Y \in \mathfrak{X}(M)$ is tangent to S.*

(b) *If $Y \in \mathfrak{X}(M)$ is tangent to S, then there is a unique smooth vector field on S, denoted by $Y|_S$, which is ι-related to Y.*

Proof. See [*Exercise Sheet* 10, *Exercise* 5(*a*)]. □

7.1.4 The Lie Bracket

We now introduce an important way of combining two smooth vector fields to obtain another smooth vector field.

Let M be a smooth manifold and let $X, Y \in \mathfrak{X}(M)$. Given $f \in C^\infty(M)$, we can apply X to f to obtain $Xf \in C^\infty(M)$ (see Proposition 7.5) and we can now apply Y to Xf to obtain $Y(Xf) \in C^\infty(M)$. The operation $f \mapsto YXf$ does not satisfy the product rule in general, and thus cannot be a vector field (see Proposition 7.6), as the following example shows:

Example 7.11. Consider the smooth vector fields

$$X = \frac{\partial}{\partial x} \quad \text{and} \quad Y = x\,\frac{\partial}{\partial y}$$

and the smooth functions

$$f(x,y) = x \quad \text{and} \quad g(x,y) = y$$

on $\mathbb{R}^2$. We compute

$$(XY)(fg) = X\left(x\,\frac{\partial(xy)}{\partial y}\right) = X(x^2) = \frac{\partial(x^2)}{\partial x} = 2x$$

and

$$fXYg + gXYf = x\,X\left(x\,\cancelto{1}{\frac{\partial y}{\partial y}}\right) + y\,X\left(x\,\cancelto{0}{\frac{\partial x}{\partial y}}\right) = x\,\cancelto{1}{\frac{\partial x}{\partial x}} = x,$$

so XY is not a derivation of $C^\infty(\mathbb{R}^2)$.

However, we can also apply the same two vector fields in the opposite order, obtaining a (usually different) smooth function $YXf \in C^\infty(M)$. Applying both of these operators to $f \in C^\infty(M)$ and subtracting, we obtain the operator

$$[X,Y]\colon C^\infty(M) \to C^\infty(M),\ f \mapsto XYf - YXf,$$

called *the Lie bracket of X and Y*. The key fact, following readily from Proposition 7.6, is that this operator *is* a vector field.

Lemma 7.12. *The Lie bracket of any pair of smooth vector fields on a smooth manifold is a smooth vector field.*

Proof. Fix $X, Y \in \mathfrak{X}(M)$. The $\mathbb{R}$-linearity of $[X, Y]$ follows immediately from the $\mathbb{R}$-linearity of both X and Y. Let us now verify the product rule:

$$\begin{aligned}
[X, Y](fg) &= XY(fg) - YX(fg) \\
&= X\big(fYg + gYf\big) - Y\big(fXg + gXf\big) \\
&= \big(\cancel{Xf\,Yg} + f\,XYg + Xg\,Yf + g\,XYf\big) - \\
&\qquad \big(\cancel{Yf\,Xg} + f\,YXg + Yg\,Xf + g\,YXf\big) \\
&= f\big(XYg - YXg\big) + g\big(XYf - YXf\big) \\
&= f\,[X, Y]g + g\,[X, Y]f.
\end{aligned}$$

In conclusion, $[X, Y]$ is a smooth vector field on M by Proposition 7.6. ☐

We mention below the basic properties of the Lie bracket and we refer to *Exercise Sheet 10* for their proofs. The geometric interpretation of the Lie bracket will not be covered in this course, but we refer to [Lee13, Chapter 9, Lie derivatives] for some details.

Proposition 7.13 (Coordinate formula for the Lie bracket)**.** *Let M be a smooth n-manifold and let $X, Y \in \mathfrak{X}(M)$. Let*

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \quad \textit{and} \quad Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$$

be the coordinate expressions for X and Y, respectively, in terms of some smooth local coordinates (x^i) for M. Then the Lie bracket $[X, Y]$ has the following coordinate expression:

$$[X, Y] = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

Proof. See [*Exercise Sheet* 10, *Exercise* 2(*a*)] ☐

Example 7.14. We compute here the Lie bracket $[X, Y]$ of the smooth vector fields

$$X = x\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\,\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial z}$$

on $\mathbb{R}^3$. Specifically, using Proposition 7.13 we obtain

$$\begin{aligned}
[X, Y] &= \big((x \cdot 0 - 1 \cdot 1) + (1 \cdot 0 - 0 \cdot 0) + (x(y+1) \cdot 0 - 1 \cdot 0)\big)\,\frac{\partial}{\partial x} \\
&\qquad + \big((x \cdot 0 - 1 \cdot 0) + (x \cdot 0 - 0 \cdot 0) + (x(y+1) \cdot 0 - y \cdot 0)\big)\,\frac{\partial}{\partial y} \\
&\qquad + \big((x \cdot 0 - 1 \cdot (y+1)) + (1 \cdot 1 - 0 \cdot x) + (x(y+1) \cdot 0 - y \cdot 0)\big)\,\frac{\partial}{\partial z} \\
&= -\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial z}.
\end{aligned}$$

Proposition 7.15 (Properties of the Lie bracket)**.** *Let M be a smooth manifold. The Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:*

(a) Bilinearity*: For all $a, b \in \mathbb{R}$ we have*

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

(b) Antisymmetry*:*

$$[X, Y] = -[Y, X].$$

(c) Jacobi identity*:*

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0.$$

(d) For all $f, g \in C^\infty(M)$ we have

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X.$$

Proof. See [*Exercise Sheet* 10*, Exercise* 3]. ☐

Proposition 7.16 (Naturality of the Lie bracket)**.** *Let $F\colon M \to N$ be a smooth map. Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be smooth vector fields such that X_i is F-related to Y_i for $i \in \{1, 2\}$. Then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.*

Proof. See [*Exercise Sheet* 10*, Exercise* 4(*a*)]. ☐

Corollary 7.17 (Pushforwards of Lie brackets)**.** *Let $F\colon M \to N$ be a diffeomorphism. For any $X_1, X_2 \in \mathfrak{X}(M)$ we have $F_*[X_1, X_2] = [F_*X_1, F_*X_2]$.*

Proof. See [*Exercise Sheet* 10*, Exercise* 4(*b*)]. ☐

Corollary 7.18 (Lie brackets of smooth vector fields tangent to submanifolds)**.** *Let M be a smooth manifold and let S be an immersed submanifold of M. If Y_1 and Y_2 are smooth vector fields on M that are tangent to S, then their Lie bracket $[Y_1, Y_2]$ is tangent to S as well.*

Proof. See [*Exercise Sheet* 10*, Exercise* 5(*b*)]. ☐

7.2 Integral Curves

Let M be a smooth manifold. If $\gamma\colon J \subseteq \mathbb{R} \to M$ is a smooth curve, then for each $t \in J$ the velocity vector $\gamma'(t)$ is an element of $\mathrm{T}_{\gamma(t)}M$. We describe next a way to work backwards: given a tangent vector at each point, we seek a curve whose velocity at each point is equal to the given vector there.

Definition 7.19. Let M be a smooth manifold and let V be a (rough) vector field on M.

(a) An *integral curve of V* is a differentiable curve $\gamma\colon J \to M$ whose velocity at each point is equal to the value of V at that point:

$$\gamma'(t) = V_{\gamma(t)}, \ \forall\, t \in J\,.$$

If $0 \in J$, then $\gamma(0) \in M$ is called *the starting point of γ*.

(b) An integral curve of V is called *maximal* if it cannot be extended to an integral curve of V on any larger open interval.

Finding integral curves of smooth vector fields boils down to solving a system of ODEs in a smooth chart. Suppose that $V \in \mathfrak{X}(M)$ and that $\gamma \colon J \subseteq \mathbb{R} \to M$ is a smooth curve. On a smooth coordinate domain $U \subseteq M$ we can write γ in local coordinates as

$$\gamma(t) = \big(\gamma^1(t), \dots, \gamma^n(t)\big).$$

Then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V can be written as

$$\dot{\gamma}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} = V^i\big(\gamma(t)\big) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)},$$

see (3.9) and (7.1), which reduces to the following autonomous system of ODEs:

$$\begin{cases} \dot{\gamma}^1(t) & = V^1\big(\gamma^1(t), \dots, \gamma^n(t)\big) \\ & \vdots \\ \dot{\gamma}^n(t) & = V^n\big(\gamma^1(t), \dots, \gamma^n(t)\big) \end{cases}. \tag{7.2}$$

The fundamental fact about such systems is the following existence, uniqueness and smoothness theorem. (This is the reason for the terminology “integral curves”, because solving a system of ODEs is often referred to as “integrating” the system.)

Theorem 7.20 (Fundamental theorem for autonomous ODEs)**.** *Let $V \colon U \to \mathbb{R}^n$ be a smooth vector-valued function, where $U \subseteq \mathbb{R}^n$ is open. Consider the initial value problem*

$$\dot{y}^i(t) = V^i\big(y^1(t), \dots, y^n(t)\big), \quad 1 \le i \le n \tag{1}$$

$$y^i(t_0) = c^i, \qquad 1 \le i \le n \tag{2}$$

for arbitrary $t_0 \in \mathbb{R}$ and $c = (c^1, \dots, c^n) \in U$.

(a) Existence*: For any $t_0 \in \mathbb{R}$ and $x_0 \in U$, there exists an open interval $J_0 \ni t_0$ and an open subset $x_0 \in U_0 \subseteq U$ such that for each $c \in U_0$, there is a C^1 map $y \colon J_o \to U$ that solves* (1) *-* (2)*.*

(b) Uniqueness*: Any two differentiable solutions to* (1) *-* (2) *defined on intervals containing t_0 agree on their common domain.*

(c) Smoothness*: Let J_0 and U_0 be as in* (a)*, and consider the map $\theta \colon J_0 \times U_0 \to U$, $(t, x) \mapsto y(t)$, where $y \colon J_0 \to U$ is the unique solution to* (1) *with initial condition $y(t_0) = x$. Then θ is smooth.*

Proposition 7.21. *Let V be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exists $\varepsilon > 0$ and a smooth curve $\gamma \colon (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at $p \in M$.*

Proof. Follows immediately from the existence and smoothness part of Theorem 7.20. □

Comment: Given M and V as in Proposition 7.21, it is a consequence of the uniquness part of Theorem 7.20 that for each $p \in M$ there actually exists a *unique*, *maximal* integral curve of V starting at $p \in M$; see Theorem 7.28(a).

The next two lemmas show how affine reparametrizations affect integral curves.

Lemma 7.22 (Rescaling lemma). *Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma\colon J \to M$ be an integral curve of V. For any $a \in \mathbb{R}$, the curve*

$$\widetilde{\gamma}\colon \widetilde{J} \to M,\ t \mapsto \gamma(at)$$

is an integral curve of the vector field $\widetilde{V} := aV$ on M, where $\widetilde{J} := \{t \in \mathbb{R} \mid at \in J\}$.

Proof. See [*Exercise Sheet* 11, *Exercise* 1(*a*)]. ☐

Lemma 7.23 (Translation lemma). *Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma\colon J \to M$ be an integral curve of V. For any $b \in \mathbb{R}$, the curve*

$$\widehat{\gamma}\colon \widehat{J} \to M,\ t \mapsto \gamma(t+b)$$

is also an integral curve of V on M, where $\widehat{J} := \{t \in \mathbb{R} \mid t+b \in J\}$.

Proof. See [*Exercise Sheet* 11, *Exercise* 1(*b*)]. ☐

Proposition 7.24 (Naturality of integral curves). *Let $F\colon M \to N$ be a smooth map. Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related if and only if F takes integral curves of X to integral curves of Y.*

Proof. See [*Exercise Sheet* 11, *Exercise* 4(*a*)]. ☐

Example 7.25. Let (x, y) be the standard coordinates on $\mathbb{R}^2$.

(1) Let

$$V = \frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$$

be the first coordinate vector field. Note that the integral curves of V are precisely the straight lines parallel to the x-axis (see Figure 7.3a), with parametrization of the form $\gamma(t) = (a + t, b)$ for constants $a, b \in \mathbb{R}$. Thus, there is a unique integral curve starting at each point of the plane, and the images of different integral curves are either identical or disjoint.

(2) Let

$$W = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2).$$

To determine the integral curves of W we proceed as follows (see p. 77):

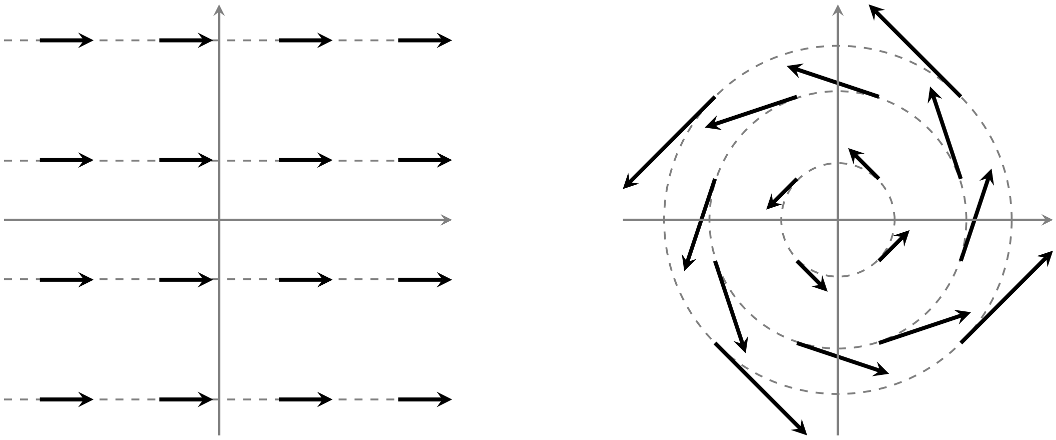
$$\gamma(t) = \big(\gamma^1(t), \gamma^2(t)\big) \quad\Longrightarrow\quad \dot\gamma(t) = W_{\gamma(t)} \quad\Longrightarrow\quad \begin{cases}\dot\gamma_1(t) = -\gamma_2(t)\\ \dot\gamma_2(t) = \gamma_1(t)\end{cases}$$

$$\xRightarrow{\ddot\gamma_1(t)+\gamma_1(t)=0} \quad \begin{cases}\gamma_1(t) = a\cos t - b\sin t\\ \gamma_2(t) = a\sin t + b\cos t\ \big(= -\dot\gamma_1(t)\big)\end{cases}$$

for constants $a, b \in \mathbb{R}$. Thus, each curve of the form

$$\gamma(t) = (a\cos t - b\sin t,\ a\sin t + b\cos t),\ \ t \in \mathbb{R},$$

is an integral curve of W. When $(a, b) = (0, 0)$, this is the constant curve $\gamma(t) = (0, 0)$; otherwise, it is a circle traversed clockwise (see Figure 7.3b). Since $\gamma(0) = (a, b)$, we see again that there is a unique integral curve staring at each point $(a, b) \in \mathbb{R}^2$, and the images of the various integral curves are either identical or disjoint.



(a) Integral curves of V

(b) Integral curves of W

7.3 Flows

Definition 7.26. Let M be a smooth manifold.

(a) A *flow domain* for M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set

$$\mathcal{D}^{(p)} := \big\{t \in \mathbb{R} \mid (t, p) \in \mathcal{D}\big\}$$

is an open interval containing $0 \in \mathbb{R}$.

(b) A *(local) flow* on M is a continuous map $\theta\colon \mathcal{D} \to M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, which satisfies the following group laws:

- $\forall\, p \in M :\ \ \theta(0, p) = p$.
- $\forall\, s \in \mathcal{D}^{(p)}\ \ \forall\, t \in \mathcal{D}^{(\theta(s,p))}$ such that $s + t \in \mathcal{D}^{(p)}$, we have

$$\theta\big(t, \theta(s, p)\big) = \theta(t + s, p).$$

When $\mathcal{D} = \mathbb{R} \times M$ (and hence $\theta\colon \mathbb{R} \times M \to M$ is a continuous left $\mathbb{R}$-action on M) we say that θ is a *global flow* on M (or a *one-parameter group action*).

(c) A *maximal flow* on M is a flow that admits no extension to a flow on a larger flow domain.

Let $\theta\colon \mathcal{D} \to M$ be a flow on M.

- For each $p \in M$ we define a map
$$\theta^{(p)} \colon \mathcal{D}^{(p)} \to M,\ \theta^{(p)}(t) = \theta(t,p).$$
- For each $t \in \mathbb{R}$ we define a set
$$M_t := \big\{ p \in M \mid (t,p) \in \mathcal{D} \big\}$$
and a map
$$\theta_t \colon M_t \to M,\ \theta_t(p) = \theta(t,p) \ \big(= \theta^{(p)}(t) \big).$$
These maps satisfy
$$\theta_t \circ \theta_s = \theta_{t+s} \quad \text{and} \quad \theta_0 = \mathrm{Id}_M,$$
so each θ_t is a homeomorphism, and if θ is smooth, then each θ_t is a diffeomorphism.
- Note also that
$$p \in M_t \iff (t,p) \in \mathcal{D} \iff t \in \mathcal{D}^{(p)}.$$

Proposition 7.27. *If $\theta \colon \mathcal{D} \to M$ is a smooth flow on M, then the* infinitesimal generator *V of θ, defined as*
$$V \colon M \to \mathrm{T}M,\ p \mapsto V_p := \theta^{(p)\prime}(0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta^{(p)}(t)\,,$$
is a smooth vector field on M, and each curve $\theta^{(p)}$ is an integral curve of V starting at $p \in M$.

Proof. If $\mathcal{D} = \mathbb{R} \times M$, then this is shown in [*Exercise Sheet* 11, *Exercise* 3]. The proof of the general case is essentially identical to the proof for global flows (after verifying that all the expressions involved make sense). ☐

The term "infinitesimal generator" comes from the following picture: in a smooth chart, a good approximation to an integral curve can be obtained by composing many small straight-line motions, with the direction and length of each motion determined by the value of the vector field at the point arrived at in the previous step. Intuitively, one can think of a flow as a sequence of infinitely many infinitesimally small linear steps.

Theorem 7.28 (Fundamental theorem on flows)**.** *Let V be a smooth vector field on a smooth manifold M. There exists a unique smooth maximal flow $\theta \colon \mathcal{D} \to M$ whose infinitesimal generator is V. This flow, which is called the* flow generated by V*, or just the* flow of V*, has the following properties:*

(a) *For each $p \in M$, the curve $\theta^{(p)} \colon \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p.*

(b) *If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval*
$$\mathcal{D}^{(\theta(s,p))} = \mathcal{D}^{(p)} - s = \big\{ t - s \mid t \in \mathcal{D}^{(p)} \big\}.$$

(c) *For each $t \in \mathbb{R}$, the set M_t is open in M, and the map $\theta_t \colon M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.*

Proof.

(a) Proposition 7.21 shows that there exists an integral curve of V starting at each point $p \in M$. Suppose that γ and $\widetilde{\gamma}$ are two integral curves of V defined on the same open interval J such that $\gamma(t_0) = \widetilde{\gamma}(t_0)$ for some $t_0 \in J$. Consider the set

$$\mathcal{S} := \big\{ t \in J \mid \gamma(t) = \widetilde{\gamma}(t) \big\}$$

and observe that $\mathcal{S} \neq \emptyset$, because $t_0 \in \mathcal{S}$ by hypothesis, and also that it is closed in J by continuity. On the other hand, pick $t_1 \in \mathcal{S}$. Then in a smooth coordinate neighborhood around the point $p = \gamma(t_1)$, γ and $\widetilde{\gamma}$ are both solutions to the same ODE with the same initial condition $\gamma(t_1) = \widetilde{\gamma}(t_1) = p$. By the uniqueness part of Theorem 7.20, $\gamma \equiv \widetilde{\gamma}$ on an open interval containing t_1, which implies that $\mathcal{S}$ is open in J. Since J is connected, we infer that $\mathcal{S} = J$, which in turn shows that $\gamma = \widetilde{\gamma}$ on all of J. Thus, any two integral curves that agree at one point agree on their common domain.

For each $p \in M$, let $\mathcal{D}^{(p)}$ be the union of all open intervals $J \subseteq \mathbb{R}$ containing 0 on which an integral curve of V starting at p is defined. Define $\theta^{(p)} \colon \mathcal{D}^{(p)} \to M$ by letting $\theta^{(p)}(t) = \gamma(t)$, where γ is any integral curve starting at p and defined on an open interval containing 0 and t. Since all such integral curves agree at t by the argument above, $\theta^{(p)}$ is well defined, and it is obviously the unique maximal integral curve of V starting at p.

Next, for the verification that the set

$$\mathcal{D} := \big\{ (t,p) \in \mathbb{R} \times M \mid t \in \mathcal{D}^{(p)} \big\}$$

is open (so that it is a flow domain) and that the map

$$\theta \colon \mathcal{D} \to M, \ \theta(t,p) := \theta^{(p)}(t)$$

satisfies the claimed properties, as well as for the proof of (b), we refer to [Lee13, Theorem 9.12], which makes heavy use of Theorem 7.20.

(c) The fact that M_t is open in M is an immediate consequence of the fact that $\mathcal{D}$ is open. We have

$$\begin{aligned} p \in M_t \quad &\Longrightarrow \quad t \in \mathcal{D}^{(p)} \quad &&\overset{\text{(b)}}{\Longrightarrow} \quad \mathcal{D}^{(\theta(t,p))} = \mathcal{D}^{(p)} - t \\ &\overset{\text{dfn}}{\Longrightarrow} \quad -t \in \mathcal{D}^{(\theta(t,p))} \quad &&\Longrightarrow \quad \theta_t(p) \in M_{-t}\,, \end{aligned}$$

which shows that θ_t maps M_t to M_{-t} for any (fixed) $t \in \mathbb{R}$. Moreover, the group laws then show that $\theta_{-t} \circ \theta_t$ is equal to the identity on M_t. Reversing the roles of t and $-t$ shows that $\theta_t \circ \theta_{-t}$ is equal to the identity on M_{-t}. This completes the proof of (c). ☐

The naturality of integral curves (see Proposition 7.24) translates into the following naturality statement for flows.

Proposition 7.29 (Naturality of flows)**.** *Let $F \colon M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η be the flow of Y. If X and Y are F-related, then for each $t \in \mathbb{R}$ it holds that $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:*

$$\begin{tikzcd}
M_t \arrow[r, "F"] \arrow[d, "\theta_t"'] & N_t \arrow[d, "\eta_t"] \\
M_{-t} \arrow[r, "F"] & N_{-t}
\end{tikzcd}$$

Proof. See [*Exercise Sheet* 11, *Exercise* 4(*b*)]. ☐

Corollary 7.30 (Diffeomorphism invariance of flows)**.** *Let $F\colon M \to N$ be a diffeomorphism. If $X \in \mathfrak{X}(M)$ and if θ is the flow of X, then the flow of $F_*X \in \mathfrak{X}(N)$ is $\eta_t = F \circ \theta_t \circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.*

Proof. See [*Exercise Sheet* 11, *Exercise* 4(*c*)]. ☐

7.3.1 Complete Vector Fields

Example 7.31 (Global flows)**.** The two smooth vector fields on the plane described in Example 7.25 both had integral curves defined for all $t \in \mathbb{R}$, so they generate global flows. We can write them down explicitly:

(1) $\theta_V\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,\ \big(t,(x,y)\big) \mapsto (x+t,\, y)$.

For each $t \in \mathbb{R} \setminus \{0\}$, $(\theta_V)_t$ translates the plane to the left ($t<0$) or to the right ($t>0$) by a distance $|t|$.

(2) $\theta_W\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,\ \big(t,(x,y)\big) \mapsto (x\cos t - y\sin t,\, x\sin t + y\cos t)$.

For each $t \in \mathbb{R}$, $(\theta_W)_t$ rotates the plane through an angle t about the origin.

There are also smooth vector fields whose integral curves are not defined for all $t \in \mathbb{R}$. Here are two such examples:

Example 7.32. Let (x,y) be the standard coordinates on $\mathbb{R}^2$.

(1) Consider $M = \mathbb{R}^2 \setminus \{(0,0)\}$ and $V = \dfrac{\partial}{\partial x} \in \mathfrak{X}(M)$.

The unique integral curve of V starting at $(-1,0) \in M$ is the curve $\gamma(t) = (t-1,0)$, cf. Example 7.25(1). However, it cannot be extended continuously past $t=1$. This is intuitively evident because of the "hole" in M at the origin.

(To prove it rigorously, suppose that $\widetilde{\gamma}$ is a continuous extension of γ past $t=1$. Then $\gamma(t) \to \widetilde{\gamma}(1) \in \mathbb{R}^2 \setminus \{(0,0)\}$ as $t \nearrow 1$. But we may also consider γ as a map into $\mathbb{R}^2$ by composing with the inclusion $M \hookrightarrow \mathbb{R}^2$, and it is obvious from the formula that $\gamma(t) \to (0,0)$ as $t \nearrow 1$. Since limits in $\mathbb{R}^2$ are unique, this is a contradiction.)

(2) Consider $M = \mathbb{R}^2$ and $W = x^2 \dfrac{\partial}{\partial x} \in \mathfrak{X}(M)$.

The unique integral curve of W starting at $(1,0)$ is $\gamma(t) = \big(\frac{1}{1-t}, 0\big)$. It cannot be extended past $t=1$, because its x-coordinate is unbounded as $t \nearrow 1$.

Definition 7.33. A smooth vector field V on a smooth manifold M is called *complete* if it generates a global flow or, equivalently, if each of its maximal integral curves is defined for all $t \in \mathbb{R}$.

It is not always easy to determine by looking at a vector field whether it is complete or not. If one can solve the ODE explicitly to find all of the integral curves, and they all exist for all time (as we did for the vector fields of Example 7.31), then the vector field is complete. On the other hand, if one can find one single integral curve that cannot be extended to all of $\mathbb{R}$ (as we did for the vector fields of Example 7.32), then it is not complete. However, it is often impossible to solve the ODE explicitly, so it is useful to have some general criteria for determining when a vector field is complete. The following theorem provides such a criterion. For the details of its proof we refer to [Lee13, Lemma 9.15 and Theorem 9.16].

Theorem 7.34. *Every compactly supported smooth vector field on a smooth manifold is complete.*

Corollary 7.35. *Every smooth vector field on a compact smooth manifold is complete.*

Exercise 7.36 (The escape lemma)**:** Let M be a smooth manifold and let V be a smooth vector field on M. Show that if $\gamma\colon J \to M$ is a maximal integral curve of V whose domain J has a finite least upper bound $b \in \mathbb{R}$, then for any $t_0 \in J$ the image $\gamma\big([t_0, b)\big)$ of the interval $[t_0, b)$ under γ is not contained in any compact subset of M.

CHAPTER 8

DIFFERENTIAL FORMS

In this chapter we transfer the algebra of alternating tensors on a finite-dimensional real vector space (see Appendix C) to smooth manifolds and begin to explore the basic properties of *differential forms.* The heart of the chapter is the introduction of the most important operation on differential forms, called the *exterior derivative.* It is one of the very few differential operators that are naturally defined on every smooth manifold without any arbitrary choices.

8.1 Differential 1-Forms

8.1.1 Covectors

Definition 8.1. Let M be a smooth manifold. For each $p \in M$ we define *the cotangent space at* p, denoted by T_p^*M, to be the dual space of T_pM:

$$\mathrm{T}_p^*M := (\mathrm{T}_pM)^*.$$

Elements of T_p^*M are called *(tangent) covectors at* $p \in M$.

Given smooth local coordinates (x^i) on an open subset $U \subseteq M$, for each $p \in U$ the coordinate basis $\left(\frac{\partial}{\partial x^i}\big|_p\right)$ for T_pM gives rise to a dual basis for T_p^*M, which we denote temporarily by $\left(\lambda^i|_p\right)$. Any covector $\omega \in \mathrm{T}_p^*M$ can thus be written uniquely as

$$\omega = \omega_i\, \lambda^i\big|_p\,, \text{ where } \omega_i = \omega\left(\frac{\partial}{\partial x^i}\bigg|_p\right).$$

Given now another set of smooth local coordinates $(\widetilde{x}^j)$ whose domain contains $p \in U$, denote by $\left(\widetilde{\lambda}^j|_p\right)$ the basis for T_p^*M dual to $\left(\frac{\partial}{\partial \widetilde{x}^j}\big|_p\right)$. We can compute the components of the same covector $\omega \in \mathrm{T}_p^*M$ with respect to the new coordinate system as follows. According to (3.6), the coordinate vector fields transform as follows:

$$\frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\, \frac{\partial}{\partial \widetilde{x}^j}\bigg|_p \tag{8.1}$$

Writing ω in both systems as

$$\omega = \omega_i \, \lambda^i\big|_p = \widetilde{\omega}_j \, \widetilde{\lambda}^j\big|_p \, ,$$

we can use (8.1) to compute ω_i in terms of $\widetilde{\omega}_j$:

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \omega\left(\frac{\partial \widetilde{x}^j}{\partial x^i}(p)\, \frac{\partial}{\partial \widetilde{x}^j}\bigg|_p\right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\, \omega\left(\frac{\partial}{\partial \widetilde{x}^j}\bigg|_p\right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\, \widetilde{\omega}_j \, . \tag{8.2}$$

8.1.2 The Cotangent Bundle

Definition 8.2. Let M be a smooth manifold. *The cotangent bundle of* M is denoted by T^*M and is defined as the disjoint union

$$\mathrm{T}^*M = \bigsqcup_{p \in M} \mathrm{T}^*_p M.$$

It has a natural projection map

$$\pi \colon \mathrm{T}^*M \to M, \ \omega \in \mathrm{T}^*_p M \mapsto p.$$

As in Subsection 8.1.1, given any smooth local coordinates (x^i) on an open subset $U \subseteq M$, for each $p \in M$ we denote by $\big(\lambda^i|_p\big)$ the basis for $\mathrm{T}^*_p M$ dual to $\big(\frac{\partial}{\partial x^i}\big|_p\big)$. This defines n maps

$$\lambda^1, \dots, \lambda^n \colon U \to \mathrm{T}^*M$$

(to be denoted differently soon), and λ^i is called the *i-th coordinate covector field*.

Proposition 8.3 (The cotangent bundle as vector bundle)**.** *Let M be a smooth n-manifold. With its standard projection map and the natural vector space structure on each fiber, the cotangent bundle T^*M has a unique topology and smooth structure making it into a smooth vector bundle of rank n over M for which all coordinate covector fields are smooth local sections.*

Proof. (Similar to the proof of Proposition 6.5.) Given any smooth chart (U, φ) for M with coordinate functions (x^i), define a map

$$\begin{aligned} \Phi \colon \pi^{-1}(U) &\to U \times \mathbb{R}^n, \\ \xi_i \, \lambda^i\big|_p &\mapsto \big(p, (\xi_1, \dots, \xi_n)\big), \end{aligned}$$

where λ^i is the i-th coordinate covector field associated with (x^i). Suppose that $(\widetilde{U}, \widetilde{\varphi})$ is another smooth chart for M with coordinate functions $(\widetilde{x}^j)$, and let $\widetilde{\Phi} \colon \pi^{-1}(\widetilde{U}) \to \widetilde{U} \times \mathbb{R}^n$ be defined analogously. On $\pi^{-1}(U \cap \widetilde{U})$, it follows from (8.2) that

$$\big(\Phi \circ \widetilde{\Phi}^{-1}\big)\big(p, (\widetilde{\xi}^1, \dots, \widetilde{\xi}^n)\big) = \left(p, \left(\frac{\partial \widetilde{x}^j}{\partial x^1}(p)\, \widetilde{\xi}_j, \dots, \frac{\partial \widetilde{x}^j}{\partial x^n}(p)\, \widetilde{\xi}_j\right)\right).$$

The $\mathrm{GL}(n, \mathbb{R})$-valued function $\big(\frac{\partial \widetilde{x}^j}{\partial x^i}\big)$ is smooth, so it follows from the vector bundle chart lemma (see Lemma 6.7) that T^*M has a smooth structure making it into a smooth vector bundle for which the maps Φ are smooth local trivializations. Uniqueness follows as in the proof of [*Exercise Sheet* 9, *Exercise* 6]. $\square$

As in the case of the tangent bundle (see the proof of Proposition 3.12), smooth local coordinates for M yield smooth local coordinates for its cotangent bundle. If (x^i) are smooth coordinates on an open subset $U \subseteq M$, then [*Exercise Sheet* 9*, Exercise* 5(*d*)] shows that the map

$$\pi^{-1}(U) \to \mathbb{R}^{2n}, \ \xi_i \, \lambda^i\big|_p \mapsto \big(x^1(p), \dots, x^n(p), \xi_1, \dots, \xi_n\big),$$

is a smooth coordinate chart for T^*M. We call (x^i, ξ_i) *the natural coordinates for* T^*M *associated with* (x^i).

8.1.3 Covector Fields

Definition 8.4. A rough (resp. continuous, smooth) local or global section of T^*M is called a *rough* (resp. *continuous*, *smooth*) *covector field* or a *(differential)* 1*-form* on the smooth manifold M.

- ⇝ *Extension lemma for covector fields*: this is a special case of Lemma 6.11.
- ⇝ Exercise 6.12 for $E = \mathrm{T}^*M$ yields the following *application*: Any tangent covector at a point can be extended to a smooth covector field on the entire manifold.
- ⇝ The set $\mathfrak{X}^*(M)$ of all smooth (global) covector fields on a smooth manifold M is an infinite-dimensional $\mathbb{R}$-vector space and a module over the ring $C^\infty(M)$: this is a special case of [*Exercise Sheet* 9*, Exercise* 3], taking also Exercise 2.21 and Exercise 6.12 (in the above special case) into account.
- ⇝ *Local/global coframe* for M = local/global frame for T^*M, see Definition 6.13.
- ⇝ *Completion of smooth local coframes for M*: this is a special case of [*Exercise Sheet* 9*, Exercise* 4].

In any smooth local coordinates (x^i) on an open subset $U \subseteq M$, a (rough) covector field ω can be written in terms of the coordinate covector fields (λ^i) as $\omega = \omega_i \, \lambda^i$ for n functions $\omega_i \colon U \to \mathbb{R}$, called *the component functions of ω in the given chart* and characterized by

$$\omega_i(p) = \omega_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right).$$

If ω is a (rough) covector field and if X is a (rough) vector field on M, then we can form the function

$$\omega(X) \colon M \to \mathbb{R}, \ p \mapsto \omega_p(X_p).$$

If we write $\omega = \omega_i \, \lambda^i$ and $X = X^i \frac{\partial}{\partial x^i}$ in terms of local coordinates, then $\omega(X)$ has the local coordinate representation

$$\omega(X) = \omega_i \, X^i \, .$$

Just as in the case of vector fields (see Proposition 7.2 and Proposition 7.5), there are several ways to check smoothness of a covector field (see also Proposition 6.15).

Proposition 8.5 (Smoothness criteria for covector fields)**.** *Let M be a smooth manifold and let $\omega \colon M \to \mathrm{T}^*M$ be a rough covector field on M. The following are equivalent:*

(a) *ω is smooth.*

(b) *In every smooth chart, the component functions of ω are smooth.*

(c) *Each point of M is contained in some coordinate chart in which ω has smooth component functions.*

(d) *For every $X \in \mathfrak{X}(M)$, the function $\omega(X)\colon M \to \mathbb{R}$ is smooth.*

(e) *For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X)\colon U \to \mathbb{R}$ is smooth.*

Proof. See [*Exercise Sheet* 12*, Exercise* 1]. □

Of course, since any open subset of a smooth manifold is again a smooth manifold, Proposition 8.5 applies equally well to covector fields defined only on some open subset of M.

Example 8.6. For any smooth chart $\big(U, (x^i)\big)$, the coordinate covector fields (λ^i) defined above constitute a local coframe over U, called a *coordinate coframe*. By Proposition 8.5, every coordinate coframe is smooth, because its component functions in the given chart are constants.

More generally, if (E_i) is a (rough) local frame for $\mathrm{T}M$ over an open subset $U \subseteq M$, then there is a uniquely determined (rough) local coframe (ε^i) over U such that $\big(\varepsilon^i|_p\big)$ is the dual basis to $(E_i|_p)$ for each $p \in U$, or equivalently $\varepsilon^i(E_j) = \delta^i_j$. This coframe is called *the coframe dual to* (E_i). Conversely, if (ε^i) is a (rough) local coframe over an open subset $U \subseteq M$, then there is a uniquely determined (rough) local frame (E_i) for $\mathrm{T}M$ over U, called *the frame dual to* (ε^i) and determined by $\varepsilon^i(E_j) = \delta^i_j$. For example, in a smooth chart, the coordinate frame $\big(\frac{\partial}{\partial x^i}\big)$ and the coordinate coframe (λ^i) are dual to each other.

Lemma 8.7. *Let M be a smooth manifold. If (E_i) is a rough local frame over an open subset $U \subseteq M$ and if (ε^i) is its dual coframe, then (E_i) is smooth if and only if (ε^i) is smooth.*

Proof. It suffices to show that for each $p \in U$, the frame (E_i) is smooth in a neighborhood of p if and only if (ε^i) is. Given $p \in U$, let $\big(V, (x^i)\big)$ be a smooth coordinate chart such that $p \in V \subseteq U$ and write

$$E_i = a_i^k \frac{\partial}{\partial x^k} \qquad \text{and} \qquad \varepsilon^j = b^j_\ell \, \lambda^\ell$$

for some matrices of real-valued functions a_i^k and b^j_ℓ defined on V. By virtue of Propositions 7.2 and 8.5, the vector fields E_i are smooth on V if and only if the functions a_i^k are smooth, and the covector fields ε^j are smooth on V if and only if the functions b^ℓ_j are smooth. The fact that $\varepsilon^j(E_i) = \delta^j_i$ implies that the matrices (a_i^k) and (b^ℓ_j) are inverses to each other. Since matrix inversion is a smooth map $GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$, we conclude that either one of these matrix-valued functions is smooth if and only if the other one is smooth. □

8.1.4 The Differential of a Smooth Function

The most important application of covector fields is that they enable us to interpret in a coordinate-independent way the partial derivatives of a smooth function as the components of a covector field.

Let $f \in C^\infty(M)$. We define a (rough) covector field

$$\mathrm{d}f\colon M \to \mathrm{T}^*M,$$

called *the differential of* f, by

$$p \in M \mapsto \mathrm{d}f_p \in \mathrm{T}_p^*M,$$

where *the differential* $\mathrm{d}f_p$ *of* f *at* p is defined as

$$\begin{aligned} \mathrm{d}f_p\colon \mathrm{T}_pM &\to \mathbb{R} \\ v \in \mathrm{T}_pM &\mapsto \mathrm{d}f_p(v) := vf \in \mathbb{R}. \end{aligned}$$

It is straightforward to check that the map $\mathrm{d}f_p$ is $\mathbb{R}$-linear, i.e., $\mathrm{d}f_p \in \mathrm{T}_p^*M$ for all $p \in M$. Moreover, we have:

Proposition 8.8. *The differential of a smooth function is a smooth covector field.*

Proof. To verify that $\mathrm{d}f$ is smooth we apply Proposition 8.5(d): for any $X \in \mathfrak{X}(M)$, the function $\mathrm{d}f(X)\colon M \to \mathbb{R}$ is smooth, because it is equal to Xf (see Proposition 7.5). ☐

For a smooth real-valued function $f\colon M \to \mathbb{R}$ on a smooth manifold M, we now have two different definitions for the differential of f at $p \in M$. In Chapter 3 we defined $\mathrm{d}f_p$ as a linear map $\mathrm{T}_pM \to \mathrm{T}_{f(p)}\mathbb{R}$, while here we defined $\mathrm{d}f_p$ as a covector at $p \in M$, i.e., a linear map $\mathrm{T}_pM \to \mathbb{R}$. These are really the same object, once we take into account the canonical identification between $\mathrm{T}_{f(p)}\mathbb{R}$ and $\mathbb{R}$; one easy way to see this is to note that both are represented in coordinates by the row matrix whose components are the partial derivatives of f. (Let us verify this below for $\mathrm{d}f$ defined as above.)

Let us compute the coordinate representation of $\mathrm{d}f$. Let (x^i) be smooth coordinates on an open subset $U \subseteq M$ and let (λ^i) be the corresponding coordinate coframe on U. Write $\mathrm{d}f$ in coordinates as $\mathrm{d}f_p = A_i(p)\,\lambda^i|_p$ for some functions $A_i\colon U \to \mathbb{R}$. Then the definition of $\mathrm{d}f$ implies

$$A_i(p) = \mathrm{d}f_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p)\,,$$

which yields the following formula for the coordinate representation of $\mathrm{d}f$:

$$\mathrm{d}f_p = \frac{\partial f}{\partial x^i}(p)\,\lambda^i\big|_p\,. \tag{8.3}$$

Thus, the component functions of $\mathrm{d}f$ in any smooth coordinate chart are the partial derivatives of (the coordinate representation of) f with respect to those coordinates. Due to this, we can think of $\mathrm{d}f$ as an analogue of the classical *gradient* (the vector field in $\mathbb{R}^n$ whose components are the partial derivatives of the function), reinterpreted in a way that makes coordinate-independent sense on a manifold.

If we apply (8.3) to the special case in which f is one of the coordinate functions $x^j : U \to \mathbb{R}$, then we obtain

$$\mathrm{d}x^j\big|_p = \frac{\partial x^j}{\partial x^i}(p)\,\lambda^i\big|_p = \delta^j_i\,\lambda^i\big|_p = \lambda^j\big|_p\,;$$

in other words, the coordinate vector field λ^j is none other than the differential $\mathrm{d}x^j$. Therefore, (8.3) can be rewritten as

$$\mathrm{d}f_p = \frac{\partial f}{\partial x^i}(p)\,\mathrm{d}x^i\big|_p\,, \quad p \in U,$$

or as an equation between covector fields instead of covectors

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i\,. \tag{8.4}$$

In particular, in the 1-dimensional case, this reduces to

$$\mathrm{d}f = \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x\,.$$

Thus, we have recovered the familiar classical expression for the differential of a function f in coordinates. Henceforth, we *abandon* the notation λ^i for the coordinate coframe, and use $\mathrm{d}x^i$ instead.

Example 8.9. If

$$f : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 y \cos x,$$

then

$$\begin{aligned}\mathrm{d}f &= \frac{\partial(x^2 y\cos x)}{\partial x}\,\mathrm{d}x + \frac{\partial(x^2 y\cos x)}{\partial y}\,\mathrm{d}y\\ &= (2xy\cos x - x^2 y\sin x)\,\mathrm{d}x + (x^2\cos x)\,\mathrm{d}y\,.\end{aligned}$$

Proposition 8.10 (Properties of the differential)**.** *Let M be a smooth manifold and let $f, g \in C^\infty(M)$. The following statements hold:*

(a) *If $a, b \in \mathbb{R}$, then $\mathrm{d}(af + bg) = a\,\mathrm{d}f + b\,\mathrm{d}g$.*

(b) $\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f$.

(c) $\mathrm{d}(f/g) = (g\,\mathrm{d}f - f\,\mathrm{d}g)/g^2$ *on the set where $g \neq 0$.*

(d) *If $J \subseteq \mathbb{R}$ is an interval containing the image of f and if $h : J \to \mathbb{R}$ is a smooth function, then $\mathrm{d}(h \circ f) = (h' \circ f)\,\mathrm{d}f$.*

(e) *If f is constant, then $\mathrm{d}f = 0$. Conversely, if $\mathrm{d}f = 0$, then f is constant on each connected component of M.*

Proof. See [*Exercise Sheet* 12, *Exercise* 3]. ☐

Proposition 8.11 (Derivative of a function along a curve)**.** *Let M be a smooth manifold, $\gamma\colon J \to M$ be a smooth curve, and $f\colon M \to \mathbb{R}$ be a smooth function. Then the derivative of $f \circ \gamma\colon J \to \mathbb{R}$ is given by*

$$(f \circ \gamma)'(t) = df_{\gamma(t)}\big(\gamma'(t)\big).$$

Proof. See [*Exercise Sheet* 12, *Exercise* 4(*a*)]. □

Let M be a smooth manifold and let $f \in C^\infty(M)$. If γ is a smooth curve in M, then we have two different meanings for the expression $(f\circ\gamma)'(t)$. On the one hand, $f\circ\gamma$ can be interpreted as a smooth curve in $\mathbb{R}$, and thus $(f\circ\gamma)'(t)$ is its velocity (vector) at the point $(f \circ \gamma)(t)$, which is an element of the tangent space $T_{(f\circ\gamma)(t)}\mathbb{R}$. Proposition 3.18 shows that this tangent vector is equal to $df_{\gamma(t)}\big(\gamma'(t)\big)$, thought of as an element of $T_{(f\circ\gamma)(t)}\mathbb{R}$. On the other hand, $f \circ \gamma$ can also be considered simply as a real-valued function of one real variable, and then $(f \circ \gamma)'(t)$ is just its ordinary derivative. Proposition 8.11 shows that this derivative is equal to $df_{\gamma(t)}\big(\gamma'(t)\big)$, thought of as a real number.

8.1.5 Pullback of Covector Fields

Definition 8.12. Let $F\colon M \to N$ be a smooth map and let $p \in M$. The differential (or *tangent map*) $\mathrm{d}F_p\colon \mathrm{T}_pM \to \mathrm{T}_{F(p)}N$ yields a dual linear map $\mathrm{d}F_p^*\colon \mathrm{T}^*_{F(p)}N \to \mathrm{T}^*_pM$, called *the (pointwise) pullback by F at p* (or *the cotangent map of F at p*) and characterized by

$$\mathrm{d}F_p^*(\omega)(v) = \omega\big(\mathrm{d}F_p(v)\big),\ \ \omega \in \mathrm{T}^*_{F(p)}N,\ \ v \in \mathrm{T}_pM.$$

Unlike vector fields, whose pushforwards are defined only in certain special cases (see for instance Subsection 7.1.2), *covector fields always pullback to covector fields.*

Definition 8.13. Let $F\colon M \to N$ be a smooth map and let $\omega\colon N \to \mathrm{T}^*N$ be a rough covector field on N. We define a rough covector field $F^*\omega\colon M \to \mathrm{T}^*M$ on M, called *the pullback of ω by F*, by

$$(F^*\omega)_p = \mathrm{d}F_p^*(\omega_{F(p)}). \tag{8.5}$$

It acts on a vector $v \in \mathrm{T}_pM$ by

$$(F^*\omega)_p(v) = \omega_{F(p)}\big(\mathrm{d}F_p(v)\big)\,.$$

Proposition 8.14. *Let $F\colon M \to N$ be a smooth map and let ω be a (continuous) covector field on N. If $u\colon N \to \mathbb{R}$ is a continuous function, then*

$$F^*(u\omega) = (u \circ F)F^*\omega.$$

If additionally $u\colon N \to \mathbb{R}$ is smooth, then

$$F^*(\mathrm{d}u) = \mathrm{d}(u \circ F).$$

Proof. Regarding the first statement, for any $p \in M$ we have

$$\begin{aligned} F^*(u\omega)_p &\overset{(8.5)}{=\!=} \mathrm{d}F_p^*\big((u\omega)_{F(p)}\big) = \mathrm{d}F_p^*\big(u(F(p))\,\omega_{F(p)}\big) \\ &\overset{\text{lin.}}{=\!=} u(F(p))\,\mathrm{d}F_p^*\big(\omega_{F(p)}\big) \overset{(8.5)}{=\!=} (u\circ F)(p)\,(F^*\omega)_p \\ &=\!= \big((u\circ F)(F^*\omega)\big)_p\,, \end{aligned}$$

which implies the assertion.

Regarding the second statement, if $p \in M$ and $v \in \mathrm{T}_pM$, then

$$
\begin{aligned}
& & \big(F^*(\mathrm{d}u)\big)_p(v) &\overset{(8.5)}{=\!=\!=} \big(\mathrm{d}F_p^*(\mathrm{d}u_{F(p)})\big)(v)\\
& & &\overset{\text{dfn}}{=\!=\!=} \mathrm{d}u_{F(p)}\big(\mathrm{d}F_p(v)\big)\\
&\text{immediately using Prop. 3.7(b) and the identification} \;\Big\downarrow & &\overset{\text{dfn of } \mathrm{d}u}{=\!=\!=} \big(\mathrm{d}F_p(v)\big)u\\
& & &\overset{\text{dfn of } \mathrm{d}F_p}{=\!=\!=} v(u \circ F)\\
& & &\overset{\text{dfn of } \mathrm{d}(u \circ F)}{=\!=\!=} \mathrm{d}(u \circ F)_p(v)\,,
\end{aligned}
$$

which yields the assertion. ☐

Proposition 8.15. *Let $F\colon M \to N$ be a smooth map and let ω be a (continuous) covector field on N. Then $F^*\omega$ is a (continuous) covector field on M, and if ω is smooth, then so is $F^*\omega$.*

Proof. Fix $p \in M$ and choose smooth coordinates (y^j) for N in a neighborhood V of $F(p)$. Set $U = F^{-1}(V)$ and observe that U is a neighborhood of p in M. Writing ω in coordinates as $\omega = \omega_j\, \mathrm{d}y^j$ for (continuous) functions on V and using Proposition 8.14 twice (for $F|_U$), we compute that

$$
F^*\omega = F^*(\omega_j\, \mathrm{d}y^j) = (\omega_j \circ F)F^*\, \mathrm{d}y^j = (\omega_j \circ F)\, \mathrm{d}(y^j \circ F)\,. \tag{8.6}
$$

In view of Proposition 8.8, this expression is continuous, and it is smooth when ω is smooth, so we are done. ☐

Formula (8.6) for the pullback of a covector field can also be written in the following way:

$$
F^*\omega = (\omega_j \circ F)\, \mathrm{d}(y^j \circ F) = (\omega_j \circ F)\, \mathrm{d}F^j\,,
$$

where F^j is the j-th component function of F in these coordinates. Using either of these formulas, the computation of pullbacks in coordinates is quite simple.

Example 8.16. Consider the smooth map

$$
F\colon \mathbb{R}^3 \to \mathbb{R}^2,\;\; (x, y, z) \mapsto (x^2y,\, y\sin z) = (u, v)
$$

and the smooth covector field

$$
\omega = u\, \mathrm{d}v + v\, \mathrm{d}u \in \mathfrak{X}^*(\mathbb{R}^2).
$$

According to (8.6), we have

$$
\begin{aligned}
F^*\omega &= (u \circ F)\, \mathrm{d}(v \circ F) + (v \circ F)\, \mathrm{d}(u \circ F)\\
&= (x^2y)\, \mathrm{d}(y\sin z) + (y\sin z)\, \mathrm{d}(x^2y)\\
&= (x^2y)(\sin z\, \mathrm{d}y + y\cos z\, \mathrm{d}z) + y\sin z\, (2xy\, \mathrm{d}x + x^2\, \mathrm{d}y)\\
&= (2xy^2\sin z)\, \mathrm{d}x + (2x^2y\sin z)\, \mathrm{d}y + (x^2y^2\cos z)\, \mathrm{d}z.
\end{aligned}
$$

In other words, to compute $F^*\omega$, all we need to do is substitute the component functions of F for the coordinate functions of N everywhere they appear in ω.

Remark 8.17. Let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps between smooth manifolds (with or without boundary) and let $\eta \in \mathfrak{X}^*(P)$. Then

$$(G\circ F)^*\,\eta = F^*(G^*\eta).$$

Indeed, given $p \in M$ and $v \in T_pM$, using Proposition 3.7(b) we obtain

$$\begin{aligned}\big(F^*(G^*\eta)\big)_p(v) &= (G^*\eta)_{F(p)}\big(dF_p(v)\big) = \eta_{G(F(p))}\Big(dG_{F(p)}\big(dF_p(v)\big)\Big)\\ &= \eta_{(G\circ F)(p)}\Big(d(G\circ F)_p(v)\Big) = \Big((G\circ F)^*\eta\Big)_p(v),\end{aligned}$$

which yields the assertion.

8.1.6 Covector Fields and Submanifolds

In Subsection 7.1.3 we considered the conditions under which a (smooth) vector field restricts to a submanifold. The restriction of a (smooth) covector field to a submanifold is much simpler and will be briefly discussed below.

Let M be a smooth manifold, let $S \subseteq M$ be an immersed submanifold and let $\iota\colon S \hookrightarrow M$ be the inclusion map. If $\omega \in \mathfrak{X}^*(M)$, then $\iota^*\omega \in \mathfrak{X}^*(S)$. More precisely, given $p \in S$ and $v \in \mathrm{T}_pS$, we have

$$(\iota^*\omega)_p\, v = \omega_p\big(\mathrm{d}\iota_p(v)\big) = \omega_p(v),$$

since $\mathrm{d}\iota_p\colon \mathrm{T}_pS \to \mathrm{T}_pM$ is just the inclusion map under our usual identification of T_pS with the subspace $\mathrm{d}\iota_p(\mathrm{T}_pS)$ of T_pM. Thus, $\iota^*\omega$ is just the restriction of ω to vectors tangent to S. For this reason, $\iota^*\omega$ is often called *the restriction of ω to S*. Note, however, that $\iota^*\omega$ might equal zero at a given point of S, even though considered as a covector field on M, ω might not vanish there. For example:

Example 8.18. Consider $\omega = \mathrm{d}y \in \mathfrak{X}^*(\mathbb{R}^2)$ and let $S\colon (y = 0)$ be the x-axis, considered as an embedded submanifold of $\mathbb{R}^2$. As a covector field on $\mathbb{R}^2$, ω is clearly nonzero everywhere, because one of its components is always equal to 1. However, the restriction $\iota^*\omega$ of ω to S is identically zero, because y vanishes identically on S:

$$\iota^*\omega = \iota^*\,\mathrm{d}y = \mathrm{d}(y\circ\iota) = 0.$$

To distinguish the two ways in which we might interpret the statement "ω vanishes on S", one usually says that ω *vanishes along S* (or *vanishes at points of S*) if $\omega_p = 0$ for every $p \in S$. The weaker condition that $\iota^*\omega = 0$ is expressed by saying that *the restriction of ω to S vanishes* (or *the pullback of ω to S vanishes*).

Exercise 8.19: Let M be a smooth manifold, let S be an immersed submanifold of M, and let $\iota\colon S \hookrightarrow M$ be the inclusion map. For any $f \in C^\infty(M)$, show that $d\big(f|_S\big) = \iota^*(df)$. Conclude that the pullback of df to S is zero if and only if f is constant on each connected component of S.

8.2 Differential k-Forms

Definition 8.20. Let M be a smooth n-manifold and fix $k \in \mathbb{N}$.

(a) We define *the bundle of covariant k-tensors on M* by

$$\mathrm{T}^k(\mathrm{T}^*M) \coloneqq \bigsqcup_{p\in M} \mathrm{T}^k(\mathrm{T}_p^*M)$$

with the obvious projection map. It can be shown (exercise!) that it is a smooth vector bundle of rank n^k over M. Its (smooth) sections are called (smooth) *covariant k-tensor fields on M.*

(b) The subset of $\mathrm{T}^k(\mathrm{T}^*M)$ consisting of alternating k-tensors is defined as:

$$\bigwedge\nolimits^k(\mathrm{T}^*M) \coloneqq \bigsqcup_{p\in M} \bigwedge\nolimits^k(\mathrm{T}_p^*M).$$

It can be shown (exercise!) that $\bigwedge^k(\mathrm{T}^*M)$ is a smooth subbundle of $\mathrm{T}^k(\mathrm{T}^*M)$, and thus it is a smooth vector bundle of rank $\binom{n}{k}$ over M. Its sections are called *(differential) k-forms* on M; they are (continuous) tensor fields on M whose value at each point of M is an alternating k-tensor. The integer k is called the *degree* of the form.

We denote the $\mathbb{R}$-vector space of *smooth* global (differential) k-forms by

$$\Omega^k(M) \coloneqq \Gamma\big(\bigwedge\nolimits^k(\mathrm{T}^*M)\big).$$

$\rightsquigarrow$ A 0-form is a continuous real-valued function on M, because

$$\bigwedge\nolimits^0(\mathrm{T}^*M) = \bigsqcup_{p\in M} \bigwedge\nolimits^0(\mathrm{T}_p^*M) \cong \bigsqcup_{p\in M} \mathbb{R} = M \times \mathbb{R},$$

see [*Exercise Sheet* 9*, Exercise* 3(*c*)].

$\rightsquigarrow$ A 1-form is a continuous covector field on M, because

$$\bigwedge\nolimits^1(\mathrm{T}^*M) = \bigsqcup_{p\in M} \bigwedge\nolimits^1(\mathrm{T}_p^*M) \cong \bigsqcup_{p\in M} \mathrm{T}_p^*M = \mathrm{T}^*M.$$

The *wedge product* of two differential forms is defined pointwise:

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p \,.$$

Thus, the wedge product of a k-form with an ℓ-form is a $(k+\ell)$-form. In particular, if f is a 0-form and if η is a k-form, then we interpret the wedge product $f \wedge \eta$ to mean the ordinary product $f\eta$; see (6.1).

Comment: If we define

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M),$$

then the wedge product turns $\Omega^*(M)$ into an associative, anti-commutative, graded $\mathbb{R}$-algebra.

In any smooth chart $\big(U,(x^i)\big)$, a k-form ω can be written as

$$\omega = \sum_I{}' \omega_I \, \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} = \sum_I{}' \omega_I \, \mathrm{d}x^I ,$$

where the coefficients ω_I are continuous functions defined on the coordinate domain U, we use $\mathrm{d}x^I$ as an abbreviation for $\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}$ (where $I = (i_1, \ldots, i_n)$) and the primed summation sign denotes a sum over only *increasing* multi-indices. According to Proposition 6.15, ω is smooth if and only if the component functions ω_I are smooth. Since

$$\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} \left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}} \right) = \delta^I_J$$

by Lemma C.20 and Proposition C.25(c), the component functions ω_I of ω are determined by

$$\omega_I = \omega \left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}} \right) .$$

8.2.1 Pullback of k-Forms

If $F\colon M \to N$ is a smooth map and if ω is a differential k-form on N, then $F^*\omega$ is a differential k-form on M, defined as follows:

$$(F^*\omega)_p(v_1, \ldots, v_k) := \omega_{F(p)}\big(\mathrm{d}F_p(v_1), \ldots, \mathrm{d}F_p(v_k)\big).$$

Lemma 8.21. *The following statements hold:*

(a) $F^*\colon \Omega^k(N) \to \Omega^k(M)$ *is linear over* $\mathbb{R}$.

(b) $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$.

(c) *In any smooth chart* $\big(V,(y^i)\big)$ *for* N*, we have*

$$F^*\Big(\sum_I{}' \omega_I \, \mathrm{d}y^{i_1} \wedge \ldots \wedge \mathrm{d}y^{i_k}\Big) = \sum_I{}' (\omega_I \circ F) \, \mathrm{d}(y^{i_1} \circ F) \wedge \ldots \wedge \mathrm{d}(y^{i_k} \circ F).$$

Proof. See [*Exercise Sheet* 13, *Exercise* 1]. □

This lemma gives a computational rule for pullbacks of differential forms similar to the one we developed earlier for covector fields, see (8.6).

Example 8.22. Consider the smooth function

$$F\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto (u, v, u^2 - v^2) = (x, y, z)$$

and the smooth 2-form

$$\omega = y \, \mathrm{d}x \wedge \mathrm{d}z + x \, \mathrm{d}y \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^3) .$$

Then

$$\begin{aligned}F^*\omega &= F^*\big(y\,\mathrm{d}x\wedge\mathrm{d}z + x\,\mathrm{d}y\wedge\mathrm{d}z\big)\\ &= v\,\mathrm{d}u\wedge\mathrm{d}(u^2-v^2) + u\,\mathrm{d}v\wedge\mathrm{d}(u^2-v^2)\\ &= v\,\mathrm{d}u\wedge(2u\,\mathrm{d}u - 2v\,\mathrm{d}v) + u\,\mathrm{d}v\wedge(2u\,\mathrm{d}u-2v\,\mathrm{d}v) \underset{\mathrm{d}v\wedge\mathrm{d}v=0}{\overset{\mathrm{d}u\wedge\mathrm{d}u=0}{=\!=\!=}}\\ &= -2v^2\,\mathrm{d}u\wedge\mathrm{d}v + 2u^2\,\mathrm{d}v\wedge\mathrm{d}u \underset{-\mathrm{d}v\wedge\mathrm{d}u}{\overset{\mathrm{d}u\wedge\mathrm{d}v=}{=\!=\!=}}\\ &= -2(u^2+v^2)\,\mathrm{d}u\wedge\mathrm{d}v.\end{aligned}$$

Proposition 8.23 (Pullback formula for top forms)**.** *Let $F\colon M\to N$ be a smooth map between smooth n-manifolds. If (x^i) and (y^j) are smooth coordinates on open subsets $U\subseteq M$ and $V\subseteq N$, respectively, and if u is a real-valued function on V, then the following holds on $U\cap F^{-1}(V)$:*

$$F^*\big(u\,\mathrm{d}y^1\wedge\ldots\wedge\mathrm{d}y^n\big) = (u\circ F)\big(\det \mathrm{D}F\big)\,\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n, \tag{8.7}$$

where $\mathrm{D}F$ represents the Jacobian matrix of F in these coordinates.

Proof. Since the fiber of $\bigwedge^n(\mathrm{T}^*M)$ is spanned by $\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n$ at each point, it suffices to show that both sides of (8.7) agree when evaluated on $\big(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\big)$. We have

$$\begin{aligned}&F^*\big(u\,\mathrm{d}y^1\wedge\ldots\wedge\mathrm{d}y^n\big)\Big(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\Big) \overset{\textit{Lemma 8.21(c)}}{=\!=\!=\!=\!=}\\ &\quad= (u\circ F)\,\mathrm{d}(\underbrace{y^1\circ F}_{F^1})\wedge\ldots\wedge\mathrm{d}(\underbrace{y^n\circ F}_{F^n})\Big(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\Big)\\ &\quad= (u\circ F)\,\big(\mathrm{d}F^1\wedge\ldots\wedge\mathrm{d}F^n\big)\Big(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\Big) \overset{\textit{Proposition C.25(c)(d)}}{=\!=\!=\!=\!=\!=}\\ &\quad= (u\circ F)\det\left(\mathrm{d}F^j\Big(\frac{\partial}{\partial x^i}\Big)\right) = (u\circ F)\det\Big(\frac{\partial F^j}{\partial x^i}\Big)\\ &\quad= (u\circ F)\det\mathrm{D}F\,\underbrace{\big(\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n\big)\Big(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\Big)}_{=1},\end{aligned}$$

as desired. □

Corollary 8.24. *If $\big(U,(x^i)\big)$ and $\big(\widetilde{U},(\widetilde{x}^j)\big)$ are overlapping smooth coordinate charts on a smooth manifold M, then the following identity holds on $U\cap\widetilde{U}$:*

$$\mathrm{d}\widetilde{x}^1\wedge\ldots\wedge\mathrm{d}\widetilde{x}^n = \det\Big(\frac{\partial\widetilde{x}^j}{\partial x^i}\Big)\,\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n\,.$$

Proof. Apply Proposition 8.23 for $F=\mathrm{Id}_{U\cap\widetilde{U}}$, but using coordinates (x^i) in the domain and $(\widetilde{x}^j)$ in the codomain. □

8.2.2 The Exterior Derivative

We now define a natural differential operator on smooth forms, called the exterior derivative, which is a generalization of the differential of a function. More precisely, for each smooth manifold M, we will show that there is a differential operator $\mathrm{d}\colon \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying $\mathrm{d}(\mathrm{d}\omega) = 0$ for all smooth forms ω.

The definition of d on Euclidean space is straightforward: if

$$\omega = \sum_J{}' \omega_J \,\mathrm{d}x^J$$

is a smooth k-form on an open subset $U \subseteq \mathbb{R}^n$, then its *exterior derivative* $\mathrm{d}\omega$ is defined to be the following $(k+1)$-form

$$\mathrm{d}\Big(\sum_J{}' \omega_J \,\mathrm{d}x^J\Big) := \sum_J{}' \mathrm{d}\omega_J \wedge \mathrm{d}x^J, \tag{8.8}$$

where $\mathrm{d}\omega_J$ is the differential of the smooth function ω_J, see Subsection 8.1.4. In somewhat more detail, this is

$$\mathrm{d}\Big(\sum_J{}' \omega_J \,\mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_k}\Big) = \sum_J \sum_i \frac{\partial \omega_J}{\partial x^i}\,\mathrm{d}x^i \wedge \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_k}.$$

For instance, for a smooth 0-form f we have

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i\,,$$

which is just the differential of f, see (8.4), while for a smooth 1-form $\omega = \omega_j \,\mathrm{d}x^j$ we compute that

$$\mathrm{d}\omega = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

Example 8.25. Let us compute the exterior derivatives of arbitrary 1-forms and 2-forms on $\mathbb{R}^3$.

- Any smooth 1-form ω on $\mathbb{R}^3$ can be written as

$$\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

for some smooth functions P, Q, R on $\mathbb{R}^3$. Using (8.8) and the fact that the wedge product of any 1-form with itself is zero, we compute

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}P \wedge \mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}y + \mathrm{d}R \wedge \mathrm{d}z \\
&= \left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y + \frac{\partial P}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x + \\
&\qquad + \left(\frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y + \frac{\partial Q}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \;+\; \left(\frac{\partial R}{\partial x}\mathrm{d}x + \frac{\partial R}{\partial y}\mathrm{d}y + \frac{\partial R}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \;+\; \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}z \;+\; \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z\,.
\end{aligned}$$

- Any smooth 2-form η on $\mathbb{R}^3$ can be written as

$$\eta = u\,\mathrm{d}x \wedge \mathrm{d}y + v\,\mathrm{d}x \wedge \mathrm{d}z + w\,\mathrm{d}y \wedge \mathrm{d}z$$

for some smooth functions u, v, w on $\mathbb{R}^3$. Similarly, we compute

$$\mathrm{d}\eta = \left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,.$$

In order to transfer the previous definition to manifolds, we first need to check that it satisfies the following properties.

Proposition 8.26 (Properties of the exterior derivative on $\mathbb{R}^n$)**.**

(a) d *is* $\mathbb{R}$*-linear.*

(b) *If* ω *is a smooth* k*-form and* η *is a smooth* ℓ*-form on an open subset* $U \subseteq \mathbb{R}^n$*, then*

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k\, \omega \wedge \mathrm{d}\eta.$$

(c) $\mathrm{d} \circ \mathrm{d} \equiv 0$.

(d) d *commutes with pullbacks: If* $F\colon U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ *is a smooth map between open subsets of Euclidean spaces, and if* $\omega \in \Omega^k(V)$*, then*

$$F^*(\mathrm{d}\omega) = \mathrm{d}(F^*\omega).$$

Proof.

(a) Follows immediately from the definition.

(b) Due to (a), it suffices to consider terms of the form $\omega = u\,\mathrm{d}x^I \in \Omega^k(U)$ and $\eta = v\,\mathrm{d}x^J \in \Omega^\ell(U)$, where $u, v \in C^\infty(U)$.

– Claim: For any multi-index I we have

$$\mathrm{d}(u\,\mathrm{d}x^I) = \mathrm{d}u \wedge \mathrm{d}x^I.$$

– Proof: If I has repeated indices, then clearly $\mathrm{d}(u\,\mathrm{d}x^I) = 0 = \mathrm{d}u \wedge \mathrm{d}x^I$. Otherwise, let σ be a permutation sending I to an increasing multi-index J. Then

$$\mathrm{d}(u\,\mathrm{d}x^I) = \mathrm{sgn}(\sigma)\,\mathrm{d}(u\,\mathrm{d}x^J) = \mathrm{sgn}(\sigma)\,\mathrm{d}u \wedge \mathrm{d}x^J = \mathrm{d}u \wedge \mathrm{d}x^I.$$

Using the claim, we compute

$$\begin{aligned}
\mathrm{d}(\omega \wedge \eta) &=\!=\!= \mathrm{d}\big((u\,\mathrm{d}x^I) \wedge (v\,\mathrm{d}x^J)\big) \\
&=\!=\!= \mathrm{d}(uv\,\mathrm{d}x^I \wedge \mathrm{d}x^J) \\
&\overset{\text{dfn}}{=\!=\!=} (v\,\mathrm{d}u + u\,\mathrm{d}v) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \xlongequal[(-1)^k \mathrm{d}x^I \wedge \mathrm{d}v]{\mathrm{d}v \wedge \mathrm{d}x^I =} \\
&=\!=\!= (\mathrm{d}u \wedge \mathrm{d}x^I) \wedge (v\,\mathrm{d}x^J) + (-1)^k (u\,\mathrm{d}x^I) \wedge (\mathrm{d}v \wedge \mathrm{d}x^J) \\
&\overset{\text{Claim}}{=\!=\!=} \mathrm{d}(\underbrace{u\,\mathrm{d}x^I}_{=\omega}) \wedge (\underbrace{v\,\mathrm{d}x^J}_{=\eta}) + (-1)^k (\underbrace{u\,\mathrm{d}x^I}_{=\omega}) \wedge \mathrm{d}(\underbrace{v\,\mathrm{d}x^J}_{=\eta})\,.
\end{aligned}$$

(c) We first deal with the case of a smooth 0-form u:

$$\begin{aligned}
\mathrm{d}(\mathrm{d}u) &= \mathrm{d}\Big(\frac{\partial u}{\partial x^i}\,\mathrm{d}x^i\Big) = \frac{\partial^2 u}{\partial x^i \partial x^j}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \overset{\mathrm{d}x^i\wedge \mathrm{d}x^i=0}{=\!=\!=\!=} \\
&= \sum_{i<j}\Big(\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i}\Big)\,\mathrm{d}x^i \wedge \mathrm{d}x^j \\
&= 0
\end{aligned}$$

Let us now deal with the general case ($u = \sum_J' \omega_J\,\mathrm{d}x^J \in \Omega^k(U)$):

$$\begin{aligned}
\mathrm{d}(\mathrm{d}u) &= \mathrm{d}\Big(\sum_J{}' \mathrm{d}\omega_J \wedge \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_k}\Big) \\
&\overset{\text{(a)}}{\underset{\text{(b)}}{=\!=}} \sum_J{}' \cancelto{0 \ \text{by case}\ k=0}{\mathrm{d}(\mathrm{d}\omega_J)} \wedge \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_k} \; + \\
&\qquad + \sum_J{}' (-1)\cdot \mathrm{d}\omega_J \wedge \cancelto{\substack{0\ \text{by (b) and} \\ \text{by case}\ k=0}}{\mathrm{d}\big(\mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_k}\big)} \\
&= 0\,.
\end{aligned}$$

(d) Due to (a), it suffices to consider $\omega = u\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}$. We have

$$\begin{aligned}
F^*\big(\mathrm{d}(u\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k})\big) = F^*(\mathrm{d}u \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}) &\overset{\textit{Lemma } 8.21(b)(c)\,\&}{\underset{\textit{Proposition } 8.14}{=\!=\!=\!=\!=}} \\
= \mathrm{d}(u\circ F) \wedge \mathrm{d}(x^{i_1}\circ F) \wedge \ldots \wedge \mathrm{d}(x^{i_k}\circ F) &\overset{(*)^1}{=\!=} \\
= \mathrm{d}\big((u\circ F)\,\mathrm{d}(x^{i_1}\circ F) \wedge \ldots \wedge \mathrm{d}(x^{i_k}\circ F)\big) &\overset{\textit{Lemma } 8.21(c)}{=\!=\!=\!=\!=} \\
= \mathrm{d}\big(F^*(u\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k})\big).
\end{aligned}$$

□

These results allow us to transplant the definition of the exterior derivative to smooth manifolds.

Theorem 8.27 (Existence and uniqueness of exterior differentiation)**.** *Let M be a smooth manifold. For each $k \in \mathbb{N}$ there are unique operators*

$$\mathrm{d}\colon \Omega^k(M) \to \Omega^{k+1}(M)\,,$$

called exterior differentiation*, satisfying the following properties:*

---

[1] $(*)$: We have an expression of the form $\mathrm{d}f \wedge \eta$, where $\eta = \mathrm{d}g_1 \wedge \ldots \wedge \mathrm{d}g_k$ (with $f = u\circ F$ and $g_\ell = x^{i_\ell}\circ F$), so

$$\mathrm{d}(f\eta) \overset{\text{p. 94}}{=\!=} \mathrm{d}(f \wedge \eta) \overset{\text{(b)}}{=\!=} \mathrm{d}f \wedge \eta + (-1)^0 f \wedge \mathrm{d}\eta = \mathrm{d}f \wedge \eta,$$

since $\mathrm{d}\eta = 0$ by (b) and (c).

(a) d *is* $\mathbb{R}$*-linear.*

(b) *If* $\omega \in \Omega^k(M)$ *and* $\eta \in \Omega^\ell(M)$*, then*

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k\, \omega \wedge \mathrm{d}\eta.$$

(c) $\mathrm{d} \circ \mathrm{d} \equiv 0$.

(d) *For* $f \in \Omega^0(M) = C^\infty(M)$, $\mathrm{d}f$ *is the differential of* f*, given by* $\mathrm{d}f(X) = Xf$.

In any smooth chart, d *is given by* (8.8)*.*

Proof.

— *Existence*: Given $\omega \in \Omega^k(M)$, for each smooth chart (U, φ) for M, we set

$$\mathrm{d}\omega := \varphi^*\, \mathrm{d}\big((\varphi^{-1})^*\omega\big). \tag{8.9}$$

This is well-defined, since for any other smooth chart (V, ψ) for M, the map $\varphi \circ \psi^{-1}$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, so

$$\begin{aligned}
\psi^*\, \mathrm{d}\big((\psi^{-1})^*\omega\big) &= (\underbrace{\varphi^{-1} \circ \varphi}_{\mathrm{Id}})^*\, \psi^*\, \mathrm{d}\big((\psi^{-1})^*\omega\big) \\
&= \varphi^*\, (\varphi^{-1})^*\, \psi^*\, \mathrm{d}\big((\psi^{-1})^*\omega\big) \quad \underset{\&\ \textit{Proposition}\ 8.26(d)}{\overset{(\varphi^{-1})^*\psi^* = (\psi\circ\varphi^{-1})^*}{=\!=\!=\!=\!=\!=\!=\!=}} \\
&= \varphi^*\, \mathrm{d}\Big(\underbrace{(\psi \circ \varphi^{-1})^*(\psi^{-1})^*}_{(\psi^{-1}\circ\psi\circ\varphi^{-1})^* = (\varphi^{-1})^*}\ \omega\Big) \\
&= \varphi^*\, \mathrm{d}\big((\varphi^{-1})^*\omega\big)\,.
\end{aligned}$$

Moreover, d satisfies (a) - (d) by virtue of Proposition 8.26.

— *Uniqueness*: Suppose that d is any operator satisfying (a) – (d). We first show that d is determined locally: if ω_1 and ω_2 are k-forms that agree on an open subset $U \subseteq M$, then $\mathrm{d}\omega_1 = \mathrm{d}\omega_2$ on U. Indeed, let $p \in U$, set $\eta := \omega_1 - \omega_2$ and let $\psi \in C^\infty(M)$ be a smooth bump function that is identically 1 on some neighborhood of p and supported in U. Then $\psi\eta$ is identically zero, so (a) - (d) imply that $0 = \mathrm{d}(\psi\eta) = \mathrm{d}\psi \wedge \eta + \psi\, \mathrm{d}\eta$. Evaluating this at p and using that $\psi(p) = 1$ and $\mathrm{d}\psi_p = 0$, we conclude that $0 = \mathrm{d}\eta_p = \mathrm{d}\omega_1|_p - \mathrm{d}\omega_2|_p$, which proves the assertion.

Now, let $\omega \in \Omega^k(M)$ be arbitrary, and let $\big(U, \varphi = (x^i)\big)$ be a smooth chart for M. Write ω in coordinates as $\sum_I' \omega_I\, \mathrm{d}x^I$. For any $p \in U$ by means of a smooth bump function we construct global smooth functions $\widetilde{\omega}_I$ and $\widetilde{x}^i$ on M that agree with ω_I and x^i in a neighborhood of p. By virtue of (a) – (d), together with the observation in the previous paragraph, it follows that (8.8) holds at $p \in U$. Since p was arbitrary, this d must be equal to the one we defined above. ☐

Comment: The preceding theorem can be summarized by saying that the differential on functions extends uniquely to an anti-derivation of $\Omega^*(M)$ of degree $+1$ whose square is zero.

Proposition 8.28 (Naturality of exterior derivative). *If $F\colon M \to N$ is a smooth map, then for each k the pullback map $F^*\colon \Omega^k(N) \to \Omega^k(M)$ commutes with* d*, i.e.,*

$$F^*(\mathrm{d}\omega) = \mathrm{d}(F^*\omega), \ \forall\, \omega \in \Omega^k(N).$$

Proof. Applying Proposition 8.26(d) to the coordinate representation $\psi \circ F \circ \varphi^{-1}$ of F and using (8.9) together with Remark 8.17, on $U \cap F^{-1}(V)$ we obtain

$$\begin{aligned} F^*(\mathrm{d}\omega) &= F^*\Big(\psi^*\,\mathrm{d}\big((\psi^{-1})^*\omega\big)\Big) \\ &= \varphi^*\,(\psi \circ F \circ \varphi^{-1})^*\,\mathrm{d}\big((\psi^{-1})^*\omega\big) \\ &= \varphi^*\,\mathrm{d}\big((\psi \circ F \circ \varphi^{-1})^*(\psi^{-1})^*\omega\big) \\ &= \varphi^*\,\mathrm{d}\big((\varphi^{-1})^*(F^*\omega)\big) \\ &= \mathrm{d}(F^*\omega). \end{aligned}$$

□

Definition 8.29. Let M be a smooth manifold and let $\omega \in \Omega^k(M)$. We say that ω is *closed* if $\mathrm{d}\omega = 0$, and *exact* if there exists $\eta \in \Omega^{k-1}(M)$ such that $\omega = \mathrm{d}\eta$.

Remark 8.30. Every exact form is closed, since $\mathrm{d} \circ \mathrm{d} \equiv 0$, but the converse does not hold in general, see Example 11.28. However, it can be shown that closed forms are locally exact (but not necessarily globally), so the question of whether a given closed form is exact depends on global properties of the manifold.

CHAPTER 9

MANIFOLDS WITH BOUNDARY

We briefly discuss manifolds with boundary. They play a central role in the theory of integration on manifolds, which will be developed in Chapter 11.

9.1 Topological Manifolds with Boundary

Definition 9.1. The *closed n-dimensional upper half-space* $\mathbb{H}^n \subseteq \mathbb{R}^n$ is defined as

$$\mathbb{H}^n = \big\{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\big\}.$$

The *interior* and the *boundary* of $\mathbb{H}^n$ as a subset of $\mathbb{R}^n$ are denoted by $\operatorname{Int}\mathbb{H}^n$ and $\partial\mathbb{H}^n$, respectively.

If $n > 0$, then

$$\operatorname{Int}\mathbb{H}^n = \big\{(x^1, \dots, x^n) \in \mathbb{R}^n \bigm| x^n > 0\big\},$$

$$\partial\mathbb{H}^n = \big\{(x^1, \dots, x^n) \in \mathbb{R}^n \bigm| x^n = 0\big\},$$

whereas if $n = 0$, then

$$\mathbb{H}^0 = \mathbb{R}^0 = \{0\} \quad \text{and} \quad \partial\mathbb{H}^0 = \varnothing\,.$$

Definition 9.2. An *n-dimensional topological manifold with boundary* is a second-countable, Hausdorff topological space M in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $\mathbb{H}^n$.

An open subset $U \subseteq M$ together with a map $\varphi\colon U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ is called a *chart for M*. When it is necessary to make the distinction, we call (U, φ) an *interior chart for M* if $\varphi(U)$ is an open subset of $\mathbb{R}^n$ (which includes the case of an open subset of $\mathbb{H}^n$ that does not intersect $\partial\mathbb{H}^n$), and a *boundary chart for M* if $\varphi(U)$ is a open subset of $\mathbb{H}^n$ such that $\varphi(U) \cap \partial\mathbb{H}^n \neq \varnothing$.

A point $p \in M$ is called an *interior point of M* if it is in the domain of some interior chart, and a *boundary point of M* if it is in the domain of a boundary chart that sends p to $\partial\mathbb{H}^n$. The set of all boundary points of M is denoted by ∂M and is called *the boundary of M*, while the set of all interior points of M is denoted by $\operatorname{Int} M$ and is called *the interior of M*.

Theorem 9.3 (Topological invariance of the boundary). *If M is a topological manifold with boundary, then each point of M is either a boundary point or an interior point, but not both. Thus, ∂M and $\operatorname{Int} M$ are disjoint sets whose union is M.*

Example 9.4.

(1) Every interval in $\mathbb{R}$ is a connected topological 1-manifold with boundary, whose manifold boundary consists of its endpoints (if any).

(2) The closed unit ball $\overline{\mathbb{B}}^n \subseteq \mathbb{R}^n$ is a connected topological n-manifold with boundary, whose (manifold) boundary is $\mathbb{S}^{n-1}$ and whose interior is $\mathbb{B}^n$; see [Lee13, Problem 1.11].

Proposition 9.5. *Let M be a topological manifold with boundary.*

(a) *$\operatorname{Int} M$ is an open subset of M and a topological n-manifold without boundary.*

(b) *∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.*

(c) *M is a topological manifold (in the sense of Definition 1.1) if and only if $\partial M = \varnothing$.*

Proof. Exercise! ☐

9.2 Smooth Manifolds with Boundary

If U is an open subset of $\mathbb{H}^n$, then a map $F\colon U \to \mathbb{R}^k$ is said to be *smooth* if for each $x \in U$ there exists an open subset $\widetilde{U} \subseteq \mathbb{R}^n$ containing x and a smooth map $\widetilde{F}\colon \widetilde{U} \to \mathbb{R}^k$ that agrees with F on $\widetilde{U} \cap U$. If F is such a map, then the restriction of F to $U \cap \operatorname{Int} \mathbb{H}^n$ is smooth in the usual sense. By continuity, all partial derivatives of F at points of $U \cap \operatorname{Int} \mathbb{H}^n$ are determined by their values in $\operatorname{Int} \mathbb{H}^n$, and thus in particular are independent of the choice of extension.

Definition 9.6. Let M be a topological manifold with boundary. A *smooth structure for M* is defined to be a maximal smooth atlas (a collection of charts whose domains cover M and whose transition maps (and their inverses) are smooth in the sense just described). With such a structure, M is called a *smooth manifold with boundary*.

In the following lengthy remark we collect some basic definitions and facts about smooth manifolds with boundary, referring to [Lee13] for further information.

Remark 9.7.

(1) Compare with Chapter 2: *Smoothness* of a map $F\colon M \to N$ between manifolds with boundary is defined in the same way (see Definition 2.4), with the usual understanding that a map whose domain is a subset of $\mathbb{H}^n$ is smooth if it admits an extension to a smooth map in a neighborhood of each point, and a map whose codomain is a subset of $\mathbb{H}^n$ is smooth if it is smooth as a map into $\mathbb{R}^n$.

Smooth partitions of unity exist on smooth manifolds with boundary.

(2) Compare with Chapter 3: If M is a smooth n-manifold with boundary, then *the tangent space* T_pM *to* M *at* $p \in M$ is defined in the same way (see Definition 3.4), and it is an n-dimensional $\mathbb{R}$-vector space. For any smooth chart $\big(U,(x^i)\big)$ containing p, the coordinate vectors

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p$$

(where $\left.\frac{\partial}{\partial x^n}\right|_p$ should be interpreted as a one-sided derivative when $p \in \partial M$) form a basis for T_pM.

Let M be a smooth manifold with boundary and let $p \in \partial M$. It is intuitively evident that the vectors in T_pM can be separated in three classes: those tangent to the boundary, those pointing inward, and those pointing outward. Formally, we make the following definition:

Definition: If $p \in \partial M$, then the vector $v \in \mathrm{T}_pM \setminus \mathrm{T}_p\partial M$ it said to be *inward-pointing* if for some $\varepsilon > 0$ there exists a smooth curve $\gamma\colon [0,\varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, and it is called *outward-pointing* if there exists such a curve with domain $(-\varepsilon, 0]$.

Proposition: Let M is a smooth manifold with boundary, $p \in \partial M$, and (x^i) be any smooth boundary coordinates defined on a neighborhood of p. The inward-pointing vectors in T_pM are precisely those with positive x^n-component, the outward-pointing ones are those with negative x^n-component, and the ones tangent to ∂M are those with zero x^n-component. Thus, T_pM is the disjoint union of $\mathrm{T}_p\partial M$, the set of inward-pointing vectors, and the set of outward-pointing vectors. Finally, $v \in \mathrm{T}_pM$ is inward-pointing if and only if $-v$ is outward-pointing.

The *differential* of a smooth map $F\colon M \to N$ between manifolds with boundary is defined in the same way (see Definition 3.6) and has the same representation in coordinates bases.

(3) Compare with Chapter 4: *Submersions*, *immersions*, *embeddings* and *local diffeomorphisms* are defined in the same way (see Definitions 4.2 and 4.7), and there is also a version of the rank theorem in this setting (see [Lee13, Theorem 4.15 and Problem 4.3]).

(4) Compare with Chapter 5: *Immersed* and *embedded submanifolds* of smooth manifolds with boundary are defined in the same way (see Definitions 5.1 and 5.15) and are themselves smooth manifolds with (possibly empty) boundary.

- ⇝ For properties of (immersed) submanifolds with boundary, see [Lee13, Chapter 5, Submanifolds with Boundary].
- ⇝ For a version of *the regular level set theorem* in this setting (cf. Theorem 5.12), see [Lee13, Problem 5.23].

Theorem: If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n-1)$-manifold (without boundary), and has a unique smooth structure such that it is a properly embedded submanifold of M.

(5) Compare with Chapter 7: *The tangent bundle* of a smooth n-fold with boundary is defined in the same way (see Definition 3.11) and it is a smooth vector bundle of rank n over the given manifold (see Proposition 6.5). *Vector fields* are also defined in the same way (see Definition 7.1), but *flows* in this setting need to be treated with extra care (see [Lee13, Chapter 9, Flows and Flowouts on Manifolds with Boundary]).

Proposition: If M is a smooth manifold with boundary, then there exists a smooth vector field on M whose restriction to ∂M is everywhere inward-pointing, and one whose restriction to ∂M is everywhere outward-pointing.

(6) Compare with Chapter 8: *The cotangent bundle* T^*M (respectively *the k-th exterior power $\bigwedge^k(\mathrm{T}^*M)$ of the cotangent bundle*) of a smooth n-manifold M with boundary is defined in the same way (see Definition 8.2, respectively Definition 8.20(b)), and it is a smooth vector bundle of rank n (respectively of rank $\binom{n}{k}$) over M (see Proposition 8.3, respectively Definition 8.20(b)). *Differential k-forms* ($0 \leq k \leq n$) are also defined in the same way (see Definition 8.20(b)), and so does their *exterior derivative* (see Theorem 8.27).

CHAPTER 10

ORIENTATIONS

The purpose of this chapter is to introduce a subtle but important property of smooth manifolds, called *orientation.* An orientation of a line or a curve is simply a choice of direction along it. For 2-dimensional manifolds, an orientation is essentially a choice of which rotational direction should be considered "clockwise" and which "counterclockwise". For 3-dimensional ones, it is a choice between "left-handedness" and "right-handedness". The general definition of an orientation is an adaptation of these everyday concepts to arbitrary dimensions.

10.1 Orientations of Vector Spaces

In this section we discuss orientations of vector spaces. We are all familiar with certain informal rules for singling out preferred ordered bases of $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$. We usually choose a basis for $\mathbb{R}^1$ that points to the right (i.e., in the positive direction). A natural family of preferred ordered bases for $\mathbb{R}^2$ consists of those for which the rotation from the first vector to the second is in the counterclockwise direction. And every student of vector calculus encounters "right-handed" bases in $\mathbb{R}^3$: these are the ordered bases (E_1, E_2, E_3) with the property that when the fingers of your right hand curl from E_1 to E_2, your thumb points in the direction of E_3.

Although "to the right", "counterclockwise", and "right-handed" are not mathematical terms, it is easy to translate the rules for selecting preferred bases of $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ into rigorous mathematical language: in all three cases, the preferred bases are the ones whose transition matrices from the standard basis have positive determinants.

In an abstract vector space for which there is no canonical basis, we no longer have any way to determine which bases are "correctly oriented". For example, if V is the vector space of polynomials in one real variable of degree at most 2, who is to say which of the ordered bases $(1, x, x^2)$ and $(x^2, x, 1)$ is "right-handed"? All we can say in general is what it means for two bases to have the "same orientation". Thus, we are led to introduce the following definition.

Definition 10.1. Let V be a real vector space of dimension $n \geq 1$. We say that two ordered bases $(E_1, \dots, E_n)$ and $(\widetilde{E}_1, \dots, \widetilde{E}_n)$ for V are *consistently oriented* if the transition matrix $(B_i^j)_{1\le i,j\le n}$, defined by

$$E_i = \sum_j B_i^j \widetilde{E}_j\,,$$

has positive determinant.

Exercise 10.2: Show that being consistently oriented is an equivalence relation on the set of all ordered bases of V, and show that there are exactly two equivalence classes.

Definition 10.3. Let V be a real vector space.

- If $\dim_{\mathbb{R}} V = n \ge 1$, we define an *orientation for* V as an equivalence class of ordered bases. If $(E_1, \dots, E_n)$ is any ordered basis for V, then we denote the orientation that it determines by $[E_1, \dots, E_n]$, and the opposite orientation by $-[E_1, \dots, E_n]$.
- If $\dim_{\mathbb{R}} V = 0$, we define an orientation for V to be simply a choice of one of the numbers ± 1.

Definition 10.4. A vector space together with a choice of orientation is called an *oriented vector space*. If V is oriented, then any ordered basis $(E_1, \dots, E_n)$ that is in the given orientation is said to be *positively oriented* (or simply *oriented*). Any ordered basis that is not in the given orientation is said to be *negatively oriented*.

Example 10.5. Consider the Euclidean space $V = \mathbb{R}^n$. The orientation $[e_1, \dots, e_n]$ of $\mathbb{R}^n$ determined by the standard basis $\{e_1, \dots, e_n\}$ is called the *standard orientation*. You should convince yourself that, in our usual way of representing the axes graphically, a positively oriented basis for $\mathbb{R}^1$ is one that points to the right; a positively oriented basis for $\mathbb{R}^2$ is one for which the rotation from the first basis vector to the second is counterclockwise; and a positively oriented basis for $\mathbb{R}^3$ is a right-handed one. (These can be taken as mathematical definitions for the words "right", "counterclockwise", and "right-handed".) The standard orientation for $\mathbb{R}^0$ is defined to be $+1$.

There is an important connection between orientations and alternating tensors, which is expressed in the following proposition.

Proposition 10.6. *Let V be a real vector space of dimension n. Each nonzero element $\omega \in \Lambda^n(V^*)$ determines an orientation $\mathcal{O}_\omega$ of V as follows: if $n \ge 1$, then $\mathcal{O}_\omega$ is the set of ordered bases $(E_1, \dots, E_n)$ for V such that $\omega(E_1, \dots, E_n) > 0$, while if $n = 0$, then $\mathcal{O}_\omega$ is $+1$ if $\omega > 0$, and -1 if $\omega < 0$. Moreover, two nonzero n-covectors on V determine the same orientation if and only if each is a positive multiple of the other.*

Proof. The 0-dimensional case is immediate, since a nonzero element of $\Lambda^0(V^*)$ is just a nonzero real number (as it is a function $\mathbb{R}^0 \to \mathbb{R}$). Thus, we may assume that $n \ge 1$. Let ω be a nonzero element of $\Lambda^n(V^*)$, and denote by $\mathcal{O}_\omega$ the set of ordered bases on which ω gives positive values. We need to show that $\mathcal{O}_\omega$ is exactly one equivalence class.

Suppose (E_i) and $(\widetilde{E}_j)$ are any two ordered bases for V, and let $B\colon V \to V$ be the linear map sending E_j to $\widetilde{E}_j$ for all j. This means that the matrix representation of B with respect to (E_i) on the source and $(\widetilde{E}_j)$ on the target is the transition matrix between the two bases. By Proposition C.22 we obtain

$$\omega(\widetilde{E}_1, \dots, \widetilde{E}_n) = \omega(BE_1, \dots, BE_n) = (\det B)\,\omega(E_1, \dots, E_n).$$

It follows that the basis $(\widetilde{E}_j)$ is consistently oriented with (E_i) if and only if $\omega(\widetilde{E}_1, \ldots, \widetilde{E}_n)$ and $\omega(E_1, \ldots, E_n)$ have the same sign, which is the same as saying that $\mathcal{O}_\omega$ is one equivalence class. The last statement then follows easily (and is thus left as an exercise). $\square$

Definition 10.7. If V is an oriented n-dimensional real vector space and if ω is an n-covector that determines the orientation of V as described in Proposition 10.6, then we say that ω is a *(positively) oriented n-covector*.

For example, the n-covector $\varepsilon^{1\ldots n} = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ is positively oriented for the standard orientation on $\mathbb{R}^n$; see Lemma C.20(c).

Recall that if V is an n-dimensional real vector space, then the vector space $\Lambda^n(V^*)$ is 1-dimensional by Proposition C.21. Proposition 10.6 shows that choosing an orientation for V is equivalent to choosing one of the two components of $\Lambda^n(V^*) \setminus \{0\}$. This formulation also works for 0-dimensional vector spaces, and explains why we have defined an orientation of a 0-dimensional space in the way we did.

10.2 Orientations of Smooth Manifolds

In this section we briefly discuss the theory of orientations of smooth manifolds. They have numerous applications, most notably in the theory of integration on manifolds, see Chapter 11.

Definition 10.8. Let M be a smooth manifold with or without boundary. A *pointwise orientation* on M is defined to be a choice of orientation of each tangent space.

By itself, this is not a very useful concept, because the orientations at nearby points may have no relation to each other. For example, a pointwise orientation on $\mathbb{R}^n$ might switch randomly from point to point between the standard orientation and its opposite. In order for pointwise orientations to have some relationship with the smooth structure, we need an extra condition to ensure that the orientations of nearby tangent spaces are consistent with each other.

Definition 10.9. Let M be a smooth manifold with or without boundary, endowed with a pointwise orientation. If (E_i) is a local frame for TM over an open subset $U \subseteq M$, then we say that (E_i) is *positively oriented* (or simply *oriented*) if $(E_1|_p, \ldots, E_n|_p)$ is a positively oriented ordered basis for T_pM at each point $p \in U$; see Definition 10.4. A *negatively oriented* frame for TM over $U \subseteq M$ is defined analogously.

Definition 10.10. Let M be a smooth manifold with or without boundary (of dimension $n \geq 1$).

(a) A pointwise orientation on M is said to be *continuous* if every point of M is in the domain of an oriented local frame for TM.

(b) An *orientation of M* is a continuous pointwise orientation.

(c) We say that M is *orientable* if there exists an orientation for it; otherwise we say that M is *nonorientable*.

Exercise 10.11: Let M be an oriented smooth manifold with or without boundary of dimension $n \geq 1$. Show that every local frame with connected domain is either positively oriented or negatively oriented. Moreover, show that the connectedness assumption is necessary.

Example 10.12. We give here some examples of orientable and nonorientable manifolds.

(1) Every parallelizable[1] manifold is orientable. Indeed, if $(E_1, \dots, E_n)$ is a smooth global frame for M, then we define a pointwise orientation on M by declaring the basis $(E_1|_p, \dots, E_n|_p)$ for T_pM to be positively oriented at each $p \in M$, and it is clear that this pointwise orientation is continuous, because every point of M is in the domain of the oriented smooth global frame (E_i). Therefore, for each $n \in \mathbb{N}$, the Euclidean space $\mathbb{R}^n$ is orientable.

(2) For each $n \in \mathbb{N}$, the unit n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is orientable. Indeed, there are a few ways to check the previous assertion:

- This follows from Proposition 10.21, because $\mathbb{S}^n$ is a hypersurface in $\mathbb{R}^{n+1}$, to which the vector field $N = x^i \partial/\partial x^i$ is nowhere tangent. We define *the standard orientation of* $\mathbb{S}^n$ to be the one determined by N. (The standard orientation of $\mathbb{S}^0$ is the one that assigns the orientation $+1$ to the point $+1 \in \mathbb{S}^0$ and -1 to the point $-1 \in \mathbb{S}^0$.)
- This follows from Corollary 10.22, taking (a) and [*Exercise Sheet* 7*, Exercise* 2(*ii*)] into account.
- This follows from Proposition 10.23, because $\mathbb{S}^n$ is the boundary of the closed unit ball. (It can be checked that the orientation thus induced on $\mathbb{S}^n$ is the standard one.)

(3) The so-called *Möbius band* is nonorientable; see [Lee13, Examples 10.3 and 15.38].

Definition 10.13. An *oriented manifold* (with or without boundary) is an orderer pair $(M, \mathcal{O})$, where M is an orientable smooth manifold (with or without boundary) and $\mathcal{O}$ is a choice of orientation for M. For each $p \in M$, the orientation of T_pM determined by $\mathcal{O}$ is denoted by $\mathcal{O}_p$.

If M is zero-dimensional, then this definition just means that an orientation of M is a choice of ± 1 attached to each of its points. The continuity condition is vacuous in this case, and the notion of oriented frames is not useful. Clearly, every 0-manifold is orientable.

10.2.1 Two Ways of Specifying Orientations

The following two propositions, namely Proposition 10.14 and Proposition 10.18, give ways of specifying orientations on manifolds that are more practical to use than the definition.

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[1] A smooth manifold M with or without boundary which admits a smooth global frame or, equivalently, whose tangent bundle TM is the trivial smooth vector bundle of rank $\dim M$ (see [*Exercise Sheet* 10*, Exercise* 5]) is called *parallelizable*. Note that the Euclidean space $\mathbb{R}^n$ is parallelizable, and it can also be shown that $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ are the only spheres that are parallelizable.

Proposition 10.14 (The orientation determined by an n-form)**.** *Let M be a smooth n-manifold with or without boundary. Any nonvanishing n-form ω on M determines a unique orientation of M for which ω is positively oriented at each point. Conversely, if M is given an orientation, then there is a smooth nonvanishing n-form on M that is positively oriented at each point.*

Proof.

"$\Rightarrow$": Let ω be a nonvanishing n-form on M. By Proposition 10.6, ω defines a pointwise orientation on M, so it remains to show that it is continuous. Since this is trivially true for $n = 0$, we may assume that $n \geq 1$. Given $p \in M$, let (E_i) be any local frame for TM over a connected open neighborhood U of p in M, and let (ε^i) be the dual coframe. The expression for ω in this frame over U is

$$\omega = f\,\varepsilon^1 \wedge \ldots \wedge \varepsilon^n$$

for some continuous function f on U. The fact that ω is nonvanishing means that f is nonvanishing, and thus by Lemma C.20(c) we obtain

$$\omega_p\left(E_1|_p, \ldots, E_n|_p\right) = f(p) \neq 0 \quad \text{for all } p \in U.$$

Since U is connected, it follows that this expression is either always positive or always negative on U, and therefore the given frame is either positively oriented or negatively oriented. If the latter case holds, then we can replace E_1 by $-E_1$ to obtain a new frame that is positively oriented. Hence, the pointwise orientation determined by ω is continuous.

"$\Leftarrow$": We refer to [Lee13, Proposition 15.5] for the details. □

Due to Proposition 10.14, we may now give the following definition.

Definition 10.15. Let M be a smooth n-manifold with or without boundary. Any nonvanishing n-form on M is called an *orientation form*. If M is oriented and if ω is an orientation form determining the given orientation, then we also say that ω is *positively oriented* (or simply *oriented*).

If M is zero-dimensional, then a nonvanishing 0-form (i.e., a nonvanishing smooth real-valued function) on M assigns the orientation $+1$ to points where it is positive and -1 to points where it is negative.

Remark 10.16. It is straightforward to check (see Proposition 10.6) that if ω and $\widetilde{\omega}$ are two positively oriented smooth n-forms on M, then $\widetilde{\omega} = f\omega$ for some strictly positive smooth real-valued function f on M.

Definition 10.17.

(a) A smooth coordinate chart $\left(U, (x^i)\right)$ on an oriented smooth manifold with or without boundary is said to be *positively oriented* (or simply *oriented*) if the coordinate frame $(\partial/\partial x^i)$ is positively oriented, and *negatively oriented* if the coordinate frame $(\partial/\partial x^i)$ is negatively oriented; see Definition 10.9.

(b) A smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ for a smooth manifold M with or without boundary is said to be *consistently oriented* if for each α, β, the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobian determinant everywhere on $\varphi_\alpha(U_\alpha \cap U_\beta)$.

Proposition 10.18 (The orientation determined by a coordinate atlas)**.** *Let M be a smooth manifold with or without boundary of dimension $n \geq 1$. Given any consistently oriented smooth atlas for M, there exists a unique orientation for M with the property that each chart in the given atlas is positively oriented. Conversely, if M is oriented and either $\partial M = \emptyset$ or $n > 1$, then the collection of all oriented smooth charts is a consistently oriented smooth atlas for M.*

Proof. Assume first that M has a consistently oriented smooth atlas. Each chart in the atlas determines a pointwise orientation at each point of its domain. Wherever two of the charts overlap, the transition matrix between their respective coordinate frames is the Jacobian matrix of the transition map (see the bottom of p. 29 and (3.6)), which has positive determinant by assumption, so they determine the same pointwise orientation at each point. The pointwise orientation on M thus determined is continuous, because each point of M is in the domain of an oriented coordinate frame.

Conversely, assume that M is oriented and either $\partial M = \emptyset$ or $n > 1$. Each point is in the domain of a smooth chart with connected domain, and if the chart is negatively oriented (see Exercise 10.11), then we can replace x^1 with $-x^1$ to obtain a new chart that is positively oriented. The fact that all these charts are positively oriented guarantees that their transition maps have positive Jacobian determinants, so they form a consistently oriented smooth atlas.[2] □

Exercise 10.19: Let M be a connected, orientable, smooth manifold with or without boundary. Show that M has exactly two orientations. Moreover, if two orientations of M agree at one point, then they are equal.

10.2.2 Orientations of Hypersurfaces

If M is an oriented smooth manifold and if S is an immersed submanifold of M (with or without boundary), then S might not inherit an orientation from M, even if S is embedded. Clearly, it is not sufficient to restrict an orientation form from M to S, since the restriction of an n-form to a manifold of lower dimension must necessarily be zero. For example, the *Möbius band* (see Example 10.12(3)) is nonorientable, even though it can be embedded in $\mathbb{R}^3$, which is orientable.

However, when S is an immersed or embedded *hypersurface* in M (i.e., a codimension 1-submanifold of M), it is sometimes possible to use an orientation on M to induce an orientation on S; see Proposition 10.21 below for the details. We first need to introduce the following definitions.

Definition 10.20. Let M be a smooth manifold with or without boundary and let $S \subseteq M$ be an immersed submanifold with or without boundary. A *vector field along* S is a section of the ambient tangent bundle $TM|_S$, i.e., a continuous map $N\colon S \to TM$ with the property that $N_p \in T_pM$ for every $p \in S$. Such a vector field is said to be *nowhere tangent to* S if $N_p \in T_pM \setminus T_pS$ for all $p \in S$; cf. Subsection 7.1.3.

Note that any vector field on M restricts to a vector field *along* S (not necessarily *tangent to* S), but in general not every vector field along S is of this form, see Lemma 6.11.

---

[2] This does not work for boundary charts when $\dim M = n = 1$, because of our convention that the last coordinate is nonnegative in a boundary chart.

Proposition 10.21. *Let M be an oriented smooth n-manifold with or without boundary, let S be an immersed hypersurface with or without boundary in M, and let N be a vector field along S which is nowhere tangent to S. Then S has a unique orientation such that for each $p \in S$, $(E_1, \dots, E_{n-1})$ is an oriented basis for T_pS if and only if $(N_p, E_1, \dots, E_{n-1})$ is an oriented basis for T_pM.*

Proof. See [Lee13, Proposition 15.21]. □

Figure 10.1: The orientation induced by a nowhere tangent vector field

We highlight that not every hypersurface admits a nowhere tangent vector field, see for instance [Lee13, Problem 15.6]. However, the following result gives a sufficient condition that holds in many cases.

Corollary 10.22. *If M is an oriented smooth manifold and if $S \subseteq M$ is a regular level set of a smooth function $f\colon M \to \mathbb{R}$, then S is orientable.*

Proof. See [Lee13, Proposition 15.23]. □

10.2.3 Boundary Orientations

If M is a smooth manifold with boundary $\partial M \neq \emptyset$, then ∂M is an embedded hypersurface without boundary in M (see the *Theorem* in Remark 9.7(4)) and there always exists a smooth outward-pointing vector field along ∂M (see the *Proposition* in Remark 9.7(5)). Since such a vector field is nowhere tangent to ∂M (see the *Proposition* in Remark 9.7(2)), it determines an orientation on ∂M by Proposition 10.21, provided that M is oriented. The following proposition shows that this orientation is independent of the choice of an outward-pointing vector field along ∂M, and it is called the *induced orientation* or the *Stokes orientation* on ∂M.

Proposition 10.23 (The induced orientation on a boundary)**.** *Let M be an oriented smooth n-manifold with boundary, where $n \geq 1$. Then ∂M is orientable, and all outward-pointing vector fields along ∂M determine the same orientation on ∂M.*

Proof. See [Lee13, Proposition 15.24]. □

Example 10.24. We determine the induced orientation on $\partial\mathbb{H}^n$ when $\mathbb{H}^n$ itself has the standard orientation inherited from $\mathbb{R}^n$. We can identify $\partial\mathbb{H}^n$ with $\mathbb{R}^{n-1}$ under the correspondence

$$(x^1, \dots, x^{n-1}, 0) \leftrightarrow (x^1, \dots, x^{n-1}).$$

Since the vector field $-\partial/\partial x^n$ is outward-pointing along $\mathbb{H}^n$, the standard coordinate frame for $\mathbb{R}^{n-1}$ is positively oriented for $\partial\mathbb{H}^n$ if and only if $[-\partial/\partial x^n, \partial/\partial x^1, \dots, \partial/\partial x^{n-1}]$ is the standard orientation for $\mathbb{R}^n$; see Proposition 10.21. This orientation satisfies

$$\begin{aligned}\left[-\partial/\partial x^n, \partial/\partial x^1, \dots, \partial/\partial x^{n-1}\right] &= -\left[\partial/\partial x^n, \partial/\partial x^1, \dots, \partial/\partial x^{n-1}\right] \\ &= (-1)^n \left[\partial/\partial x^1, \dots, \partial/\partial x^{n-1}, \partial/\partial x^n\right].\end{aligned}$$

Thus, the induced orientation on $\partial\mathbb{H}^n$ is equal to the standard orientation on $\mathbb{R}^{n-1}$ when n is even, but it is opposite to the standard orientation when n is odd. In particular, the standard coordinates on $\partial\mathbb{H}^n \approx \mathbb{R}^{n-1}$ are positively oriented if and only if n is even.

10.2.4 Orientations and Smooth Maps

Definition 10.25. Let M and N be oriented smooth manifolds with or without boundary and let $F\colon M \to N$ be a local diffeomorphism.

- If both M and N are positive-dimensional, then we say that F is *orientation-preserving* if for each $p \in M$, the isomorphism $\mathrm{d}F_p\colon \mathrm{T}_pM \to \mathrm{T}_{F(p)}N$ takes positively oriented bases of T_pM to positively oriented bases of $\mathrm{T}_{F(p)}N$, and *orientation-reversing* if it takes positively oriented bases of T_pM to negatively oriented bases of $\mathrm{T}_{F(p)}N$.
- If both M and N are zero-dimensional, then we say that F is *orientation-preserving* if for every $p \in M$, the points p and $F(p)$ have the same orientation, and it is *orientation-reversing* if they have opposite orientation; see the paragraph after Definition 10.13.

Remark 10.26. A composition of orientation-preserving maps is also orientation-preserving.

Lemma 10.27. *Let M and N be oriented positive-dimensional smooth manifolds with or without boundary and let $F\colon M \to N$ be a local diffeomorphism. Show that the following are equivalent:*

(a) *F is orientation-preserving.*

(b) *With respect to any positively oriented smooth charts for M and N, the Jacobian matrix of F has positive determinant.*

(c) *If ω is any positively oriented orientation form for N, then $F^*\omega$ is a positively oriented orientation form for M.*

Proof. Exercise! □

Here is another important method for constructing orientations.

Proposition 10.28 (The pullback orientation)**.** *Let M and N be smooth manifolds with or without boundary. If $F\colon M \to N$ is a local diffeomorphism and if N is oriented, then M has a unique orientation, called* the pullback orientation induced by F*, such that F is orientation-preserving.*

Proof. For each $p \in M$ there is a unique orientation on T_pM that makes the isomorphism $\mathrm{d}F_p\colon \mathrm{T}_pM \to \mathrm{T}_{F(p)}N$ orientation-preserving. This defines a pointwise orientation on M; provided that it is continuous, it is the unique orientation on M with respect to which F is orientation-preserving. To see that it is continuous, just choose a smooth orientation form ω of N using Proposition 10.14 (so that ω is positively oriented) and note that $F^*\omega$ is a smooth orientation form for M, determining by construction and by Proposition 10.14 the above pointwise orientation on M, which is thus continuous, as desired. □

CHAPTER 11

INTEGRATION ON MANIFOLDS

11.1 Line Integrals

Another important application of covector fields (cf. Subsection 8.1.4) is to make coordinate-independent sense of the notion of line integrals, which generalize ordinary integrals to the setting of curves in manifolds.

Definition 11.1. Let M be a smooth manifold with or without boundary. A *curve segment in* M is a continuous curve $\gamma\colon [a,b] \to M$ whose domain is a compact interval. It is a *smooth curve segment in* M if it is smooth when $[a,b]$ is considered as a manifold with boundary (or, equivalently, if γ has an extension to a smooth curve defined in a neighborhood of each endpoint). It is a *piecewise smooth curve segment in* M if there exists a finite partition $a_0 = a < a_1 < \cdots < a_{k-1} < a_k = b$ of $[a,b]$ such that $\gamma|_{[a_{i-1},a_i]}$ is smooth[1] for every $1 \leq i \leq k$.

Definition 11.2. Let M be a smooth manifold with or without boundary. Let ω be a smooth covector field on M. If $\gamma\colon [a,b] \to M$ is a piecewise smooth curve segment, then *the line integral of* ω *over* γ is defined to be the real number

$$\int_\gamma \omega := \sum_{i=1}^{k} \int_{[a_{i-1},a_i]} \gamma^*\omega,$$

where $[a_{i-1}, a_i]$, $1 \leq i \leq k$, are subintervals of $[a,b]$ on which γ is smooth. If t denotes the standard coordinate on $\mathbb{R}$, then the smooth covector field $\omega_i := \gamma^*\omega = \big(\gamma|_{[a_{i-1},a_i]}\big)^*\omega$ on $[a_{i-1}, a_i]$ can be written as $\omega_i = f_i(t)\,dt$ for some smooth function $f_i\colon [a_{i-1}, a_i] \to \mathbb{R}$, so the integral of ω_i over $[a_{i-1}, a_i]$ is given by

$$\int_{[a_{i-1},a_i]} \omega_i = \int_{a_{i-1}}^{a_i} f_i(t)\,dt.$$

---

[1] Continuity of γ means that $\gamma(t)$ approaches the same value as t approaches any of the points a_i (other than a_0 or a_k) from the left or the right. Smoothness of γ in each subinterval means that γ has one-sided velocity vectors at each such a_i when approaching from the left or the right, but these one-sided velocities need not be equal.

Therefore,

$$\int_\gamma \omega = \sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} f_i(t)\, dt.$$

Proposition 11.3 (Properties of line integrals)**.** *Let M be a smooth manifold with or without boundary. Let $\gamma\colon [a,b] \to M$ be a piecewise smooth curve segment in M, and let ω, ω_1, $\omega_2 \in \mathfrak{X}^*(M)$. The following statements hold:*

(a) *For any $c_1, c_2 \in \mathbb{R}$ we have*

$$\int_\gamma (c_1\, \omega_1 + c_2\, \omega_2) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2.$$

(b) *If γ is a constant map, then*

$$\int_\gamma \omega = 0.$$

(c) *If $\gamma_1 := \gamma|_{[a,c]}$ and $\gamma_2 := \gamma|_{[c,b]}$, where $a, b, c \in \mathbb{R}$ with $a < c < b$, then*

$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

(d) *If $F\colon M \to N$ is any smooth map and if $\eta \in \mathfrak{X}^*(N)$, then*

$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$$

Proof. See [*Exercise Sheet* 14, *Exercise* 1]. □

Example 11.4. Consider the smooth covector field ω on $M = \mathbb{R}^2 \backslash \{0\}$ given by

$$\omega = \frac{x\, dy - y\, dx}{x^2 + y^2}$$

and the smooth curve segment

$$\gamma\colon [0, 2\pi] \to M,\ t \mapsto (\cos t, \sin t).$$

The line integral of ω over γ equals

$$\int_\gamma \omega = \int_{[0,2\pi]} \gamma^*\omega = \int_0^{2\pi} \frac{\cos t(\cos t\, dt) - \sin t(-\sin t\, dt)}{\sin^2 t + \cos^2 t} = \int_0^{2\pi} dt = 2\pi.$$

Definition 11.5. Let M be a smooth manifold with or without boundary. If $\gamma\colon [a,b] \to M$ and $\widetilde{\gamma}\colon [c,d] \to M$ are piecewise smooth curve segments in M, then we say that $\widetilde{\gamma}$ is a *reparametrization* of γ if $\widetilde{\gamma} = \gamma \circ \varphi$ for some diffeomorphism $\varphi\colon [c,d] \to [a,b]$. If φ is an increasing function (i.e., $t_1 < t_2 \implies \varphi(t_1) < \varphi(t_2)$), then we say that $\widetilde{\gamma}$ is a *forward reparametrization* of γ, while if φ is a decreasing function (i.e., $t_1 < t_2 \implies \varphi(t_1) > \varphi(t_2)$), then we say that $\widetilde{\gamma}$ is a *backward reparametrization* of γ. (More generally, with obvious modifications one can allow φ to be piecewise smooth.)

Lemma 11.6 (Diffeomorphism invariance of the integral)**.** *Let ω be a smooth covector field on the compact interval $[a,b] \subseteq \mathbb{R}$ and let $\varphi\colon [c,d] \to [a,b]$ be a diffeomorphism. We have*

$$\int_{[c,d]} \varphi^*\omega = \begin{cases} \displaystyle\int_{[a,b]} \omega & \text{if } \varphi \text{ is increasing}, \\ \displaystyle -\int_{[a,b]} \omega & \text{if } \varphi \text{ is decreasing}. \end{cases}$$

Proof. Denote by s, resp. t, the standard coordinate on $[c,d]$, resp. $[a,b]$. Then ω can be written as $\omega_t = f(t)\,\mathrm{d}t$ for some smooth function $f\colon [a,b] \to \mathbb{R}$, and now (8.4) and (8.6) show that $\varphi^*\omega$ has the coordinate expression $(\varphi^*\omega)_s = f\big(\varphi(s)\big)\varphi'(s)\,\mathrm{d}s$. Inserting this into the definition of the line integral and using *the change of variables formula* for ordinary integrals, we obtain

$$\int_{[c,d]} \varphi^*\omega = \int_c^d f\big(\varphi(s)\big)\varphi'(s)\,\mathrm{d}s = \begin{cases} \displaystyle\int_a^b f(t)\,\mathrm{d}t & \text{if } \varphi \text{ is increasing}, \\ \displaystyle -\int_a^b f(t)\,\mathrm{d}t & \text{if } \varphi \text{ is decreasing}, \end{cases}$$

which yields the statement. ☐

Proposition 11.7 (Parameter independence of line integrals)**.** *Let M be a smooth manifold with or without boundary, let $\omega \in \mathfrak{X}^*(M)$, and let γ be a piecewise smooth curve segment in M. For any reparametrization $\widetilde{\gamma}$ of γ we have*

$$\int_{\widetilde{\gamma}} \omega = \begin{cases} \displaystyle\int_{\gamma} \omega & \text{if } \widetilde{\gamma} \text{ is a forward reparametrization}, \\ \displaystyle -\int_{\gamma} \omega & \text{if } \widetilde{\gamma} \text{ is a backward reparametrization}. \end{cases}$$

Proof. See [*Exercise Sheet* 14, *Exercise* 2]. ☐

Proposition 11.8. *Let M be a smooth manifold with or without boundary and let $\omega \in \mathfrak{X}^*(M)$. If $\gamma\colon [a,b] \to M$ is a piecewise smooth curve segment in M, then the line integral of ω over γ can also be expressed as the ordinary integral*

$$\int_{\gamma} \omega = \int_a^b \omega_{\gamma(t)}\big(\gamma'(t)\big)\mathrm{d}t.$$

Proof. See [Lee13, Proposition 11.38]. ☐

Theorem 11.9 (Fundamental theorem for line integrals)**.** *Let M be a smooth manifold with or without boundary. Let $f \in C^\infty(M)$ and let γ be a piecewise smooth curve segment in M. Then*

$$\int_{\gamma} \mathrm{d}f = f\big(\gamma(b)\big) - f\big(\gamma(a)\big).$$

Proof. Suppose first that γ is smooth. By combining Proposition 11.8, Proposition 8.11 and *the fundamental theorem of calculus* we obtain

$$\int_\gamma \mathrm{d}f = \int_a^b \mathrm{d}f_{\gamma(t)}\big(\gamma'(t)\big)\mathrm{d}t = \int_a^b \big(f\circ\gamma\big)'(t) = \big(f\circ\gamma\big)(b) - \big(f\circ\gamma\big)(a).$$

Suppose now that γ is merely piecewise smooth and consider a finite partition $a_0 = a < a_1 < \cdots < a_{k-1} < a_k = b$ of $[a,b]$ such that $\gamma|_{[a_{i-1},a_i]}$ is smooth for every $1 \leq i \leq k$. In view of Proposition 11.3(c), applying the above argument on each subinterval and summing, we find that

$$\int_\gamma \mathrm{d}f = \sum_{i=1}^k \Big(f\big(\gamma(a_i)\big) - f\big(\gamma(a_{i-1})\big)\Big) = f\big(\gamma(b)\big) - f\big(\gamma(a)\big),$$

because the contributions from all the interior points cancel. □

Example 11.10. Consider the smooth covector field

$$\omega = 2xy^3\,\mathrm{d}x + 3x^2y^2\,\mathrm{d}y \in \mathfrak{X}^*(\mathbb{R}^2) = \Omega^1(\mathbb{R}^2).$$

Note that ω is exact, since $\omega = \mathrm{d}f$ for the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x^2y^3$. We now compute the line integral of ω along the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$. Since the latter can be parametrized by the smooth curve segment $\gamma\colon [0,1] \to \mathbb{R}^2$, $t \mapsto (t,t^2)$, by Theorem 11.9 we obtain

$$\int_\gamma \omega = \int_\gamma \mathrm{d}f = \cancelto{1}{f(\gamma(1))} - \cancelto{0}{f(\gamma(0))} = 1.$$

This can also be verified by a direct computation as follows: We have

$$\gamma^*\omega = 2t(t^2)^3\,\mathrm{d}t + 3t^2(t^2)^2\,\mathrm{d}(t^2) = 2t^7\,\mathrm{d}t + 6t^7\,\mathrm{d}t = 8t^7\,\mathrm{d}t,$$

and hence

$$\int_\gamma \omega \overset{\text{dfn}}{=\!=} \int_{[0,1]} \gamma^*\omega = \int_0^1 8t^7\,dt = \big[t^8\big]_0^1 = 1.$$

11.2 Integration of Differential Forms

We first define the integral of a differential form over a domain in Euclidean space, and we then show how to use diffeomorphism invariance and smooth partitions of unity to extend this definition to differential n-forms on oriented n-manifolds. The key feature of this definition is that it is invariant under orientation-preserving diffeomorphisms. After developing the general theory of integration of differential forms on oriented manifolds, we state (without giving the complete proof) one of the most important theorems in differential geometry: *Stokes' theorem* (Theorem 11.21). It is a far-reaching generalization of the fundamental theorem of calculus and of the fundamental theorem for line integrals (Theorem 11.9), as well as of the classical theorems of vector calculus, such as Green's theorem (Theorem 11.23).

11.2.1 Integration in $\mathbb{R}^n$

Definition 11.11. Let $D \subseteq \mathbb{R}^n$ be a *domain of integration* (i.e., a bounded subset of $\mathbb{R}^n$ whose boundary ∂D has n-dimensional measure zero, such as a rectangle according to [Lee13, Proposition C.18]), and let ω be a continuous n-form on $\overline{D}$. Since ω can be written as $\omega = f\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ for some continuous function $f\colon \overline{D} \to \mathbb{R}$, we define *the integral of* ω *over* D to be the usual integral

$$\int_D \omega = \int_D f\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n := \int_D f\,\mathrm{d}x^1 \ldots \mathrm{d}x^n = \int_D f\,\mathrm{d}V\,.$$

(In simple terms, to compute the integral of a form such as $f\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$, just "erase the wedges".)

Definition 11.12. Let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and let ω be a compactly supported n-form on U. We define

$$\int_U \omega := \int_D \omega\,,$$

where $D \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$ is any domain of integration (such as a rectangle) containing $\operatorname{supp}\omega$, see Lemma 11.14, and ω is extended to be zero on the complement of its support. Note that Definition 11.12 does not depend on the choice of domain of integration, and the right-hand side reduces to Definition 11.11.

Proposition 11.13. *Let D and E be open domains of integration in $\mathbb{R}^n$ or $\mathbb{H}^n$, and let $G\colon \overline{D} \to \overline{E}$ be a smooth map that restricts to an orientation-preserving or orientation-reversing diffeomorphism $D \to E$. If ω is an n-form on $\overline{E}$, then*

$$\int_D G^*\omega = \begin{cases} \displaystyle\int_E \omega\,, & \text{if } G \text{ is orientation-preserving,} \\ \displaystyle -\int_E \omega\,, & \text{if } G \text{ is orientation-reversing.} \end{cases}$$

Proof. Follows from the (usual) *change of variables formula* (see [Lee13, Theorem C.26]) and the *pullback formula* for n-forms (see Proposition 8.23), taking also Lemma 10.27 into account. □

As we cannot guarantee that arbitrary open or compact subsets are domains of integration, we need the following lemma in order to extend Proposition 11.13 to compactly supported n-forms defined on open subsets.

Lemma 11.14. *If U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and if K is a compact subset of U, then there is an open domain of integration D such that*

$$K \subseteq D \subseteq \overline{D} \subseteq U.$$

Proof. See [Lee13, Lemma 16.2]. □

Proposition 11.15. *Let U and V be open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, and let $G\colon U \to V$ be an orientation-preserving or orientation-reversing diffeomorphism. If ω is a compactly supported n-form on V, then*

$$\int_U G^*\omega = \begin{cases} \displaystyle\int_V \omega\,, & \text{if } G \text{ is orientation-preserving,} \\[2ex] \displaystyle -\int_V \omega\,, & \text{if } G \text{ is orientation-reversing.} \end{cases}$$

Proof. By Lemma 11.14 there is an open domain of integration E such that

$$\operatorname{supp}\omega \subseteq E \subseteq \overline{E} \subseteq V.$$

Since diffeomorphisms take interiors to interiors, boundaries to boundaries, and sets of measure zero to sets of measure zero, we infer that $D := G^{-1}(E) \subseteq U$ is an open domain of integration containing $\operatorname{supp}(G^*\omega)$. We conclude by Proposition 11.13. ☐

Using the above proposition we can now make sense of the integral of a differential n-form over an oriented n-manifold.

11.2.2 Integration on Manifolds

Definition 11.16. Let M be an oriented smooth n-manifold with or without boundary and let ω be an n-form on M, where $n \geq 1$. Suppose that ω is compactly supported in the domain of a single smooth chart (U, φ) for M that is either positively or negatively oriented. We define *the integral of ω over M* to be

$$\int_M \omega = \pm \int_{\varphi(U)} \left(\varphi^{-1}\right)^* \omega \tag{11.1}$$

with the positive sign for a positively oriented chart, and the negative sign otherwise. (See Figure 11.1.) Since $\left(\varphi^{-1}\right)^*\omega$ is a compactly supported n-form on the open subset $\varphi(U) \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, its integral is defined as in Definition 11.12.

Figure 11.1: The integral of a form over a manifold

Proposition 11.17. *If M and ω are as above, then $\int_M \omega$ does not depend on the choice of smooth chart whose domain contains* $\operatorname{supp}\omega$.

Proof. Let (U, φ) and $(\widetilde{U}, \widetilde{\varphi})$ be two smooth charts such that $\operatorname{supp}\omega \subseteq U \cap \widetilde{U}$. If both charts are similarly oriented, then $\widetilde{\varphi} \circ \varphi^{-1}\colon \varphi(U \cap \widetilde{U}) \to \widetilde{\varphi}(U \cap \widetilde{U})$ is an orientation-preserving diffeomorphism (see the proof of Proposition 10.18 and Lemma 10.27), so

$$\begin{aligned} \int_{\widetilde{\varphi}(\widetilde{U})} \left(\widetilde{\varphi}^{-1}\right)^* \omega &= \int_{\widetilde{\varphi}(\widetilde{U} \cap U)} \left(\widetilde{\varphi}^{-1}\right)^* \omega \overset{\textit{Proposition 11.15}}{=\!=\!=\!=} \int_{\varphi(U \cap \widetilde{U})} \left(\widetilde{\varphi} \circ \varphi^{-1}\right)^* \left(\widetilde{\varphi}^{-1}\right)^* \omega \\ &= \int_{\varphi(U \cap \widetilde{U})} \left(\varphi^{-1}\right)^* \underbrace{\widetilde{\varphi}^* \left(\widetilde{\varphi}^{-1}\right)^*}_{=\,\mathrm{Id}^*} \omega = \int_{\varphi(U)} \left(\varphi^{-1}\right)^* \omega. \end{aligned}$$

If the charts are oppositely oriented, then the two definitions given by (11.1) have opposite signs, but this is compensated by the fact that $\widetilde{\varphi} \circ \varphi^{-1}$ is orientation-reversing, so Proposition 11.15 introduces an extra negative sign into the above computation. Hence, in either case, the two definitions of $\int_M \omega$ agree. □

To integrate over an entire manifold, we combine this definition with a partition of unity.

Definition 11.18. Let M be an oriented smooth n-manifold with or without boundary and let ω be a compactly supported n-form on M, where $n \geq 1$. Let $\{U_i\}$ be a finite open cover of $\operatorname{supp} \omega$ by domains of positively or negatively oriented smooth charts[2], and let $\{\psi_i\}$ be a smooth partition of unity subordinate to this cover. We define *the integral of ω over M* to be

$$\int_M \omega = \sum_i \int_M \psi_i \, \omega \,. \tag{11.2}$$

Since for each i the n-form $\psi_i \, \omega$ is compactly supported in U_i, each of the terms in this (finite) sum is well defined according to our previous discussion.

The following proposition shows that the integral is well defined.

Proposition 11.19. *The definition (11.2) does not depend on the choice of open cover or partition of unity.*

Proof. Let $\{\widetilde{U}_j\}$ be another open cover of $\operatorname{supp} \omega$ by domains of positively or negatively oriented smooth charts, and let $\{\widetilde{\psi}_j\}$ be a subordinate smooth partition of unity. Since

$$\int_M \psi_i \, \omega = \int_M \Big(\sum_j \widetilde{\psi}_j\Big) \psi_i \, \omega = \sum_j \int_M \widetilde{\psi}_j \, \psi_i \, \omega \quad \text{for every } i,$$

we obtain

$$\sum_i \int_M \psi_i \, \omega = \sum_{i,j} \int_M \widetilde{\psi}_j \, \psi_i \, \omega.$$

Each term in this last sum is the integral of a form that is compactly supported in the domain of a single smooth chart (e.g., in U_i), so by Proposition 11.17 each term is well defined, regardless of which coordinate map we use to compute it. The same argument, starting with $\int_M \widetilde{\psi}_j \, \omega$ instead, shows that

$$\sum_j \int_M \widetilde{\psi}_j \, \omega = \sum_{i,j} \int_M \widetilde{\psi}_j \, \psi_i \, \omega.$$

Thus, both definitions yield the same value for $\int_M \omega$. □

We have a special definition in the zero-dimensional case. The integral of a compactly supported 0-form (i.e., a real-valued function) f over an oriented 0-manifold M is defined to be the sum

$$\int_M f \coloneqq \sum_{p \in M} \pm f(p), \tag{11.3}$$

---

[2] The reason we allow for negatively oriented charts is that it may not be possible to find positively oriented boundary charts on a 1-manifold with boundary, as noted in the proof of Proposition 10.18.

where we take the positive sign at points where the orientation is positive and the negative sign otherwise. The assumption that f is compactly supported implies that there are only finitely many non-zero terms in this sum.

If $S \subseteq M$ is an oriented immersed k-dimensional manifold (with or without boundary) and if ω is a k-form on M whose restriction to S is compactly supported, then we interpret $\int_S \omega$ as $\int_S \iota^*\omega$, where $\iota\colon S \hookrightarrow M$ is the inclusion map. In particular, if M is a compact, oriented, smooth n-manifold with boundary and if ω is an $(n-1)$-form on M, then we can interpret $\int_{\partial M} \omega$ unambiguously as the integral of $\iota^*\omega$ over ∂M, where ∂M is always understood to have the induced (Stokes) orientation; see Proposition 10.23.

Proposition 11.20 (Properties of integrals)**.** *Let M and N be nonempty oriented smooth n-manifolds with or without boundary, and let ω and η be compactly supported n-forms on M.*

(a) Linearity*: If $a, b \in \mathbb{R}$, then*

$$\int_M a\,\omega + b\,\eta = a\int_M \omega + b\int_M \eta\,.$$

(b) Orientation reversal*: If $-M$ denotes M with opposite orientation, then*

$$\int_{-M} \omega = -\int_M \omega\,.$$

(c) Positivity*: If ω is a positively oriented orientation form, then*

$$\int_M \omega > 0\,.$$

(d) Diffeomorphism invariance*: If $F\colon N \to M$ is an orientation-preserving or an orientation-reversing diffeomorphism, then*

$$\int_N F^*\omega = \begin{cases} \displaystyle\int_M \omega, & \text{if } F \text{ is orientation-preserving,} \\[2ex] \displaystyle -\int_M \omega, & \text{if } F \text{ is orientation-reversing.} \end{cases}$$

Proof.

(a) Exercise.

(b) Exercise (follows from the usual change of variables formula).

(c) Since ω is a positively oriented orientation form on M, if $\big(U, \varphi = (x^i)\big)$ is a positively oriented smooth chart (with connected domain U), then $(\varphi^{-1})^*\omega$ is a positive function times $\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ (while if (U, φ) is negatively oriented, then it is a negative function times the same form); see the proof of Proposition 10.14. Therefore, each term in (11.2) defining $\int_M \omega$ is nonnegative, with at least one strictly positive term, proving thus (c).

(d) It suffices to treat the case when ω is compactly supported in a single positively or negatively oriented smooth chart. If (U, φ) is a positively oriented such chart and if F is orientation-preserving, then it is easy to check that $\big(F^{-1}(U),\ \varphi \circ F\big)$ is a positively-oriented smooth chart on N whose domain contains $\operatorname{supp}(F^*\omega)$, so the claim follows from Proposition 11.15. The remaining cases follow from this result and (b). ☐

11.2.3 Stokes' theorem

We now state the cental result in theory of integration on manifolds, *Stokes' theorem.* However, we do not provide its complete proof; instead, we refer to [Lee13, Theorem 16.11] for the details.

Theorem 11.21 (Stokes' theorem)**.** *Let M be an oriented smooth n-manifold with boundary and let ω be a compactly supported smooth $(n-1)$-form on M. Then*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega\,.$$

Here, ∂M is understood to have the induced (Stokes) orientation, and the ω on the right-hand side is to be interpreted as $\iota_{\partial M}^*\omega$, where $\iota_{\partial M}\colon \partial M \hookrightarrow M$ is the canonical inclusion. If $\partial M = \varnothing$, then the right-hand side is to be interpreted as 0. When M is 1-dimensional, the right-hand integral is just a finite sum, see (11.3). Note finally that $\mathrm{d}\omega$ is smooth n-form on M which is also compactly supported.

Proof of the case $M = \mathbb{R}^2$. We have to show that

$$\int_{\mathbb{R}^2} \mathrm{d}\omega = 0, \quad \text{where } \omega = f\,\mathrm{d}x + g\,\mathrm{d}y \in \Omega^1_c(\mathbb{R}^2).$$

Since f and g have compact support, we may pick $r > 0$ such that both $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are contained in the interior of the square $[-r,r]\times[-r,r]$. Then

$$\begin{aligned}\int_{\mathbb{R}^2} \mathrm{d}\omega = \int_{\mathbb{R}^2} \left(\frac{\partial f}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial g}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y\right) &\overset{\text{Fubini}}{=\!=\!=} \\ &= -\int_{-r}^{r}\int_{-r}^{r} \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{-r}^{r}\int_{-r}^{r} \frac{\partial g}{\partial x}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_{-r}^{r} \underbrace{\big[f(x,y)\big]_{y=-r}^{y=r}}_{=0}\,\mathrm{d}x + \int_{-r}^{r} \underbrace{\big[g(x,y)\big]_{x=-r}^{x=r}}_{=0}\,\mathrm{d}y \\ &= 0\end{aligned}$$

□

11.2.4 Applications of Stokes' theorem

Example 11.22. Let M be a smooth manifold. Let $\gamma\colon [a,b] \hookrightarrow M$ be a smooth embedding, so that $S := \gamma\big([a,b]\big)$ is an embedded 1-submanifold with 0-dimensional boundary $\partial S = \big\{\gamma(a),\gamma(b)\big\}$ in M (and γ is a diffeomorphism onto its image S). If we give S the orientation (via the differential $\mathrm{d}\gamma_p$, $p \in [a,b]$) such that γ is orientation-preserving, then for any $f \in C^\infty(M)$ we obtain

$$\begin{aligned}\int_\gamma \mathrm{d}f &\overset{\textit{Definition } 11.2}{=\!=\!=\!=} \int_{[a,b]} \gamma^*(\mathrm{d}f) \underset{\gamma^{-1}\text{: chart}}{\overset{\textit{Definition } 11.18}{=\!=\!=\!=}} \int_S \mathrm{d}f \\ &\overset{\textit{Theorem } 11.21}{=\!=\!=\!=} \int_{\partial S} f \overset{(11.3)}{=\!=} f\big(\gamma(b)\big) - f\big(\gamma(a)\big),\end{aligned}$$

because the boundary orientation at $\gamma(a)$ is -1, while at $\gamma(b)$ is $+1$. Thus, Stokes' theorem reduces to the *fundamental theorem for line integrals* (Theorem 11.9) in this case. In particular, when $\gamma\colon [a,b] \hookrightarrow \mathbb{R}$ is the inclusion map, then Stokes' theorem is just the *fundamental theorem of calculus.*

Theorem 11.23 (Green's theorem)**.** *Let $D \subseteq \mathbb{R}^2$ be a compact regular domain (i.e., properly embedded codimension-0 submanifold with boundary), and let P, Q be smooth real-valued functions on D. Then*

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y = \int_{\partial D} P \, \mathrm{d}x + Q \, \mathrm{d}y.$$

Proof. Apply Stokes' theorem to the 1-form $P \, \mathrm{d}x + Q \, \mathrm{d}y$. ☐

In particular, with $P(x,y) = -y$ and $Q(x,y) = x$, we compute the *area* of D:

$$A(D) = \frac{1}{2} \int_{\partial D} (x \, \mathrm{d}y - y \, \mathrm{d}x) \, .$$

Exercise 11.24: Using Green's theorem, compute the area of the disc

$$D_r \coloneqq \left\{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le r^2 \right\} \subset \mathbb{R}^2,$$

where $r > 0$.

Corollary 11.25 (Integrals of exact forms)**.** *If M is a compact, oriented, smooth n-manifold without boundary, then the integral of any exact n-form over M is zero:*

$$\int_M \mathrm{d}\omega = 0 \quad \textit{if} \ \ \partial M = \varnothing.$$

Corollary 11.26 (Integrals of closed forms over boundaries)**.** *If M is a compact, oriented, smooth n-manifold with boundary, then the integral over ∂M of any closed $(n-1)$-form on M is zero:*

$$\int_{\partial M} \omega = 0 \quad \textit{if} \ \ \mathrm{d}\omega = 0 \ \ \textit{on} \ \ M.$$

Corollary 11.27. *Let M be a smooth n-manifold with or without boundary, let $S \subseteq M$ be an oriented, compact, smooth k-dimensional submanifold without boundary, and let ω be a closed k-form on M. If $\int_S \omega \neq 0$, then both of the following are true:*

(a) *ω is not exact on M.*

(b) *S is not the boundary of an oriented, compact, smooth submanifold with boundary in M.*

Proof.

(a) If ω were exact on M, then $\omega = \mathrm{d}\eta$ for some $(k-1)$-form η on M, so we would have

$$0 \neq \int_S \omega \overset{\text{dfn}}{=\!=} \int_S \iota_S^* \omega = \int_S \iota_S^* (\mathrm{d}\eta) \xlongequal{\textit{Proposition 8.28}} \int_S \mathrm{d}(\iota_S^* \eta) \xlongequal{\textit{Corollary 11.25}} 0,$$

which is a contradiction.

(b) Argue again by contradiction and invoke Corollary 11.26. ☐

Example 11.28. Consider the smooth covector field ω on $M = \mathbb{R}^2 \backslash \{0\}$ discussed in Example 11.4, namely

$$\omega = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}.$$

We will show that ω is closed, but not exact. Indeed:

- *Closedness*: Setting

$$f(x,y) = \frac{x}{x^2+y^2} \quad \text{and} \quad g(x,y) = -\frac{y}{x^2+y^2},$$

so that $\omega = f\,\mathrm{d}y + g\,\mathrm{d}x$, we compute that

$$\begin{aligned} \mathrm{d}\omega = \mathrm{d}f \wedge \mathrm{d}y + \mathrm{d}g \wedge \mathrm{d}x &= \frac{\partial f}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial g}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}x \\ &= \frac{y^2 - x^2}{(x^2+y^2)^2}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{y^2 - x^2}{(x^2+y^2)^2}\,\mathrm{d}y \wedge \mathrm{d}x \\ &= 0. \end{aligned}$$

Therefore, ω is closed.

- *Non-exactness*: If ω were exact, then by Theorem 11.9 it would be *conservative* (i.e., its line integral over every piecewise smooth closed curve segment is zero), but this would contradict our earlier computation in Example 11.10 that $\int_\gamma \omega = 2\pi \neq 0$. Therefore, ω is not exact.

Finally, if (r,θ) are polar coordinates on the right half-plane $H = \big\{(x,y) \mid x > 0\big\} \subseteq M$, then we may compute the polar coordinate expression for $\omega \in \mathfrak{X}^*(M)$ as follows: Since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned} \omega &= \frac{r\cos\theta}{r^2}\,\mathrm{d}(r\sin\theta) - \frac{r\sin\theta}{r^2}\,\mathrm{d}(r\cos\theta) \\ &= \frac{\cos\theta}{r}\,(\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta) - \frac{\sin\theta}{r}\,(\cos\theta\,\mathrm{d}r - \sin\theta\,\mathrm{d}\theta) \\ &= \mathrm{d}\theta. \end{aligned}$$

APPENDIX A

THE REAL PROJECTIVE SPACE

Most of the smooth manifolds that we encountered in this course were intrinsically subspaces of some Euclidean space $\mathbb{R}^n$. However, the set-up of the general theory (that is, endowing topological manifolds with a smooth structure) is designed precisely so as to allow our objects of study to come along as abstract spaces, rather than requiring them to be subsets of some $\mathbb{R}^n$. Hence, it would be nice to see an example of a smooth manifold which takes advantage of this abstract set-up. An elementary, yet important, example is the real projective space $\mathbb{RP}^n$, which will be described in this appendix.

The underlying set of $\mathbb{RP}^n$:

Let $n \in \mathbb{N}^*$. There is a natural group action of $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$ on $\mathbb{R}^{n+1} \setminus \{0\}$ given by

$$\begin{aligned} \mathbb{R}^\times \times (\mathbb{R}^{n+1} \setminus \{0\}) &\to \mathbb{R}^{n+1} \setminus \{0\} \\ (\lambda, x) &\mapsto \lambda x. \end{aligned}$$

As with any group action, we can form the quotient set, whose points are the orbits of the action. Concretely, we define the *real projective space of dimension n*, denoted by $\mathbb{RP}^n$, to be the quotient of the above action, i.e.,

$$\mathbb{RP}^n := \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \mathbb{R}^\times.$$

Note that $\mathbb{RP}^n$ comes equipped with a natural surjection

$$\begin{aligned} \pi \colon \mathbb{R}^{n+1} \setminus \{0\} &\to \mathbb{RP}^n \\ x &\mapsto [x] := \mathbb{R}^\times \cdot x. \end{aligned}$$

In particular, notice that points of $\mathbb{RP}^n$ are in one-to-one correspondence with one-dimensional subspaces of $\mathbb{R}^{n+1}$: if $[x] \in \mathbb{RP}^n$, then $[x] \cup \{0\} = \mathbb{R} \cdot x$ is the one-dimensional subspace of $\mathbb{R}^{n+1}$ generated by x, while if L is any one-dimensional subspace of $\mathbb{R}^{n+1}$, then $L \setminus \{0\} = [x]$ for any $x \in L \setminus \{0\}$. (This is the geometric picture you should have in mind when thinking about $\mathbb{RP}^n$.) If

$$x = (x_0, \dots, x_n)$$

is a point of $\mathbb{R}^{n+1} \setminus \{0\}$, then we denote by

$$\pi(x) = [x] = [x_0 : \ldots : x_n]$$

the corresponding point of $\mathbb{RP}^n$. Note that $[x_0 : \ldots : x_n] = [y_0 : \ldots : y_n]$ if and only if there exists $\lambda \neq 0$ such that $\lambda x_i = y_i$ for all i.

The topology of $\mathbb{RP}^n$:

By definition, $\mathbb{RP}^n$ is a quotient of $\mathbb{R}^{n+1} \setminus \{0\}$, and the latter can be equipped with its natural Euclidean topology. Recall that, in general, there is a procedure with which the quotient of some topological space can be equipped with a natural topology. Concretely, one can easily show that the collection

$$\mathcal{T}_{\mathbb{RP}^n} := \big\{ U \subseteq \mathbb{RP}^n \mid \pi^{-1}(U) \subseteq \mathbb{R}^{n+1} \setminus \{0\} \text{ is open} \big\}$$

is a topology on $\mathbb{RP}^n$. Moreover, if we endow $\mathbb{RP}^n$ with this topology, then the quotient map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is continuous, and a map $f\colon \mathbb{RP}^n \to X$ from $\mathbb{RP}^n$ to some topological space X is continuous if and only if so is the composite map $f \circ \pi$. The same is true for any subset $A \subseteq \mathbb{RP}^n$ endowed with the subspace topology.

At this point, there are several things that need to be checked about the topological space $\mathbb{RP}^n$.

Exercise A.1: Show that $\mathbb{RP}^n$ is *Hausdorff* by going through the following steps:

(i) Show that the quotient map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is open.

(ii) Show that the set

$$\widetilde{\Delta} := \big\{ (x,y) \in \big(\mathbb{R}^{n+1} \setminus \{0\}\big) \times \big(\mathbb{R}^{n+1} \setminus \{0\}\big) \mid [x] = [y] \big\}$$

is closed in $\big(\mathbb{R}^{n+1} \setminus \{0\}\big) \times \big(\mathbb{R}^{n+1} \setminus \{0\}\big)$.

(iii) Show that the set

$$\Delta := \big\{ \big([x],[x]\big) \in \mathbb{RP}^n \times \mathbb{RP}^n \mid [x] \in \mathbb{RP}^n \big\}$$

is closed in $\mathbb{RP}^n \times \mathbb{RP}^n$.

(iv) Conclude that $\mathbb{RP}^n$ is Hausdorff.

[Hint: Use (iii) and that the collection

$$\big\{ U \times V \mid U, V \in \mathcal{T}_{\mathbb{RP}^n} \big\}$$

is a basis for the topology of $\mathbb{RP}^n \times \mathbb{RP}^n$ by definition of the product topology.]

Solution:

(i) Note that we have

$$\pi^{-1}\big(\pi(U)\big) = \bigcup_{\lambda \in \mathbb{R}^\times} \lambda \cdot U.$$

As multiplication by a scalar $\lambda \in \mathbb{R}^\times$ is a homeomorphism, the sets $\lambda \cdot U$ are open, and thus $\pi^{-1}(\pi(U))$ is open as well. By definition of the quotient topology, we conclude that $\pi(U)$ is open.

(ii) Notice that $[x] = [y]$ if and only if the $2 \times (n+1)$ matrix with lines x and y has submaximal rank. By the solution to part (c) of [*Exercise Sheet* 2, *Exercise* 4], the set $\widetilde{\Delta}$ of such matrices is closed.

(iii) As π is open, the map

$$\pi \times \pi \colon (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}) \to \mathbb{RP}^n \times \mathbb{RP}^n$$

is open as well (it suffices to check that it maps basis elements $U \times V$ to open subsets). It is straightforward to see that

$$\pi \times \pi(\widetilde{\Delta}^c) = \Delta^c$$

where $\bullet^c$ denotes the complement. Hence, Δ^c is open, and thus Δ is closed.

(iv) Let $[x] \neq [y]$ be two distinct points of $\mathbb{RP}^n$. Then $([x],[y]) \in \Delta^c$, and as Δ^c is open, there exist open subset U, V of $\mathbb{RP}^n$ such that

$$([x],[y]) \in U \times V \subseteq \Delta^c.$$

Now notice that $U \cap V = \emptyset$, because otherwise $U \times V$ would contain a point of the diagonal. Hence $[x]$ and $[y]$ can be separated by open subsets, i.e. $\mathbb{RP}^n$ is Hausdorff.

Exercise A.2: Show that $\mathbb{RP}^n$ is *second-countable.*

[Hint: Use *Exercise* 1(i).]

Solution: Let $\mathcal{B}$ be a countable basis for $\mathbb{R}^{n+1} \setminus \{0\}$. Set

$$\mathcal{B}' := \big\{\pi(B) \mid B \in \mathcal{B}\big\}$$

and notice that as π is open, this is a collection of open subsets of $\mathbb{RP}^n$. Let us show that $\mathcal{B}'$ is a basis for the topology of $\mathbb{RP}^n$. To this end, let $U \subseteq \mathbb{RP}^n$ be open and $[x] \in U$ a point. Then $x \in \pi^{-1}(U)$, and thus there exists $B \in \mathcal{B}$ such that $x \in B \subseteq \pi^{-1}(U)$. But then $[x] \in \pi(B) \subseteq U$. Hence, $\mathcal{B}'$ is a countable basis for $\mathbb{RP}^n$.

Exercise A.3: Show that $\mathbb{RP}^n$ is *locally Euclidean of dimension* n as follows.

(i) For each $0 \leq i \leq n$, set

$$U_i := \big\{[x_0 : \ldots : x_n] \mid x_i \neq 0\big\} \subseteq \mathbb{RP}^n.$$

Show that U_i is open, and that

$$\mathbb{RP}^n = \bigcup_{i=0}^{n} U_i.$$

(ii) For each $0 \leq i \leq n$, consider the map

$$\begin{aligned} \varphi_i \colon U_i &\to \mathbb{R}^n \\ [x_0 : \ldots : x_n] &\mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right). \end{aligned}$$

Show first that φ_i is well-defined, and subsequently that it is a homeomorphism. Conclude that $\mathbb{RP}^n$ is locally Euclidean of dimension n.

Solution:

(i) Notice that

$$\pi^{-1}(U_i) = \{(x_0, \ldots, x_n) \mid x_i \neq 0\} \subseteq \mathbb{R}^{n+1}$$

is open, and thus U_i is open by definition of the quotient topology.

Now, note that for any $[x] \in \mathbb{RP}^n$ we have $x \in \mathbb{R}^{n+1} \setminus \{0\}$, and thus there exists an index $i \in \{0, \ldots, n\}$ such that $x_i \neq 0$. Hence $[x] \in U_i$, and as $[x]$ was arbitrary, we infer that

$$\mathbb{RP}^n = \bigcup_{i=0}^{n} U_i.$$

(ii) The ratio x_j/x_i is invariant under scaling x, and thus φ_i is well-defined. To check that it is continuous, it suffices to check that $\varphi_i \circ \pi$ is a continuous map from $\pi^{-1}(U_i) = \mathbb{R}^{n+1}_{x_i \neq 0}$ to $\mathbb{R}^n$, but this is straightforward by the defining formula. Finally, to show that φ_i is a homeomorphism, we construct a continuous inverse. Consider the map

$$\begin{aligned} \Psi_i \colon \mathbb{R}^n &\to \pi^{-1}(U_i) \\ (y_1, \ldots, y_n) &\mapsto (y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n), \end{aligned}$$

which is clearly continuous, and set $\psi_i = \pi \circ \Psi_i$. This is continuous, since it is a composition of continuous maps, and it is straightforward to see that φ_i and ψ_i are mutually inverse. Hence, φ_i is a homeomorphism with inverse $\varphi_i^{-1} = \psi_i$.

Exercise A.4:

(i) Show that $\mathbb{RP}^n$ is connected.

(ii) Show that the restriction of π to $\mathbb{S}^n \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ is still surjective. Conclude that $\mathbb{RP}^n$ is compact.

Solution: Recall that the continuous image of a connected, respectively compact, space is connected, respectively compact.

(i) For every $n > 0$, the space $\mathbb{R}^{n+1} \setminus \{0\}$ is connected. Hence, $\mathbb{RP}^n = \pi\big(\mathbb{R}^{n+1} \setminus \{0\}\big)$ is also connected.

(ii) If $[x] \in \mathbb{RP}^n$, then $x/\|x\| \in \mathbb{S}^n$ and $\pi\big(x/\|x\|\big) = [x]$. Hence, $\mathbb{RP}^n = \pi(\mathbb{S}^n)$, and since $\mathbb{S}^n$ is compact, $\mathbb{RP}^n$ is also compact.

$\rightsquigarrow$ By the above exercises we infer that $\mathbb{RP}^n$ is an *n-dimensional topological manifold*, which is additionally compact and connected.

Before continuing the study of $\mathbb{RP}^n$, a few words about the open subsets U_i defined in *Exercise* 3(i) are in order. The open cover $\mathbb{RP}^n = \bigcup_{i=0}^n U_i$ is called the *standard open cover* of $\mathbb{RP}^n$. The equality, for example, $\varphi_n\big([x]\big) = y$, means that the line corresponding to $[x]$ meets the plane $\mathbb{R}^n \times \{1\}$ at the point $(y,1)$. The complement of U_n consists of those lines which do not intersect the plane $\mathbb{R}^n \times \{1\}$, which (as you may convince yourself) are precisely the lines contained in $\mathbb{R}^n \times \{0\}$. Hence, we may somewhat suggestively write

$$\mathbb{RP}^n = U_n \sqcup \mathbb{P}\big(\mathbb{R}^n \times \{0\}\big) \cong \mathbb{R}^n \sqcup \mathbb{RP}^{n-1}.$$

We may thus regard $\mathbb{RP}^n$ as a compactification of $\mathbb{R}^n$ by adding the points of $\mathbb{RP}^{n-1}$, which from this point of view are often called *points at infinity.* In particular, the real projective line $\mathbb{RP}^1$ ($n = 1$) may be regarded a one-point compactification of the real line $\mathbb{R}^1$, obtained by adding to it a "point at infinity", and the real projective plane $\mathbb{RP}^2$ ($n = 2$) may be viewed as a compactification of the real plane $\mathbb{R}^2$ by adding to it a "line at infinity".

The smooth structure of $\mathbb{RP}^n$:

The standard open cover

$$\mathbb{RP}^n = \bigcup_{i=0}^{n} U_i,$$

together with the homeomorphisms

$$\varphi_i \colon U_i \to \mathbb{R}^n,\ [x_0 : \ldots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right),\ 0 \le i \le n,$$

determine an atlas of $\mathbb{RP}^n$. According to Proposition 1.8(a), to obtain a smooth structure on $\mathbb{RP}^n$, it only remains to check that the charts $\big\{(U_i,\varphi_i)\big\}_{0\le i\le n}$ are smoothly compatible.

Exercise A.5: Let $0 \le i < j \le n$. Show that the transition map from (U_i, φ_i) to (U_j, φ_j) is a diffeomorphism by computing that

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1} \colon \mathbb{R}^n_{x_j \neq 0} &\to \mathbb{R}^n_{x_{i+1}\neq 0} \\ (x_1, \ldots, x_n) &\mapsto \frac{1}{x_j}\left(x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\right), \end{aligned}$$

and

$$\begin{aligned} \varphi_i \circ \varphi_j^{-1} \colon \mathbb{R}^n_{x_{i+1} \neq 0} &\to \mathbb{R}^n_{x_j\neq 0} \\ (x_1, \ldots, x_n) &\mapsto \frac{1}{x_{i+1}}\left(x_1, \ldots, x_i, x_{i+2}, \ldots, x_j, 1, x_{j+1}, \ldots, x_n\right). \end{aligned}$$

Solution: It is a straightforward albeit a tedious calculation to verify the formulas. Once verified, it is immediate that the transition functions are smooth.

It follows from *Exercise* 5 that

$$\mathcal{A}_{\mathbb{RP}^n} := \big\{(U_i, \varphi_i)\big\}_{i=0}^{n}$$

is a smooth atlas for $\mathbb{RP}^n$, and the induced by Proposition 1.8(a) smooth structure on $\mathbb{RP}^n$ is referred to as the *standard* one. Thus, we now have a smooth manifold, namely $\mathbb{RP}^n$, which is not intrinsically defined as a subset of $\mathbb{R}^n$!

Comment: A posteriori, *Whitney's embedding theorem* (see Appendix B) asserts that there is a smooth embedding $\mathbb{RP}^n \hookrightarrow \mathbb{R}^{2n}$ (and the exponent $2n$ is in fact minimal if n is a power of 2), so we can realize the smooth manifold $\mathbb{RP}^n$ as a submanifold of $\mathbb{R}^{2n}$. However, it would be very awkward if we were only able to speak about $\mathbb{RP}^n$ as a smooth manifold once we find such an embedding, so the flexibility of defining it abstractly is certainly very helpful.

Further exercises about $\mathbb{RP}^n$:

Exercise A.6: Prove the following assertions:

(i) The quotient map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is smooth.

(ii) A map $F\colon \mathbb{RP}^n \to M$ to a smooth manifold M is smooth if and only if the composite map $F \circ \pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to M$ is smooth.

(iii) For any point $p \in \mathbb{R}^{n+1} \setminus \{0\}$, the differential $d\pi_p\colon T_p\big(\mathbb{R}^{n+1} \setminus \{0\}\big) \to T_{[p]}\mathbb{RP}^n$ is surjective (i.e., $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is a smooth submersion) and its kernel is the subspace generated by p.

Solution: See [*Exercise Sheet* 5, *Exercise* 2].

Exercise A.7: Show that the map

$$F\colon \mathbb{R}^n \to \mathbb{RP}^n,\ (x^1, \dots, x^n) \mapsto [x^1 : \dots : x^n : 1]$$

is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$.

Exercise A.8: Let

$$P\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$$

be a smooth map, and suppose that for some $d \in \mathbb{Z}$ we have $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R}^\times$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Show that the map

$$\widetilde{P}\colon \mathbb{RP}^n \to \mathbb{RP}^k,\ \widetilde{P}\big([x]\big) := [P(x)]$$

given by is well-defined and smooth.

Exercise A.9: Consider the smooth map

$$F\colon \mathbb{R}^2 \to \mathbb{RP}^2,\ (x, y) \mapsto [x : y : 1]$$

and the smooth vector field X on $\mathbb{R}^2$ defined by

$$X = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}.$$

Show that there is a smooth vector field Y on $\mathbb{RP}^2$ that is F-related to X, and compute its coordinate representation in terms of each of the charts defined in *Exercise* 3(ii).

APPENDIX B

SARD'S THEOREM AND WHITNEY'S THEOREMS

B.1 Sard's Theorem and Applications

Theorem B.1 (Sard's theorem)**.** *If $F\colon M \to N$ is a smooth map between smooth manifolds, then the set of critical values of F has measure zero in N.*

$\rightsquigarrow$ "almost all" $c \in N$ are regular values of F

$\Rightarrow$ "almost all" level sets $F^{-1}(c)$ of F are properly embedded submanifolds of M of dimension $\dim M - \dim N$ by Corollary 5.13.

Corollary B.2. *Let $F\colon M \to N$ be a smooth map. If $\dim M < \dim N$, then $F(M)$ has measure zero in N.*

Proof. Follows immediately from Theorem B.1, because in this case each point of M is a critical point of F; see Remark 5.11(1). ☐

Corollary B.3. *If M is a smooth manifold and if S is an immersed submanifold of M such that $\dim S < \dim M$, then S has measure zero in M.*

Proof. Apply Corollary B.2 to the inclusion map $\iota\colon S \hookrightarrow M$. ☐

B.2 Whitney's Theorems

Theorem B.4 (Whitney's embedding theorem)**.** *Every smooth n-manifold admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.*

$\rightsquigarrow$ Every smooth n-manifold is diffeomorphic to a properly embedded submanifold of $\mathbb{R}^{2n+1}$.

(Hint: Use Whitney's embedding theorem, Proposition 5.3, Claim 3 from the proof of Proposition 4.6 and [*Exercise Sheet* 8, *Exercise* 1].)

Theorem B.5 (Whitney's immersion theorem)**.** *Every smooth n-manifold admits a smooth immersion into $\mathbb{R}^{2n}$.*

The above two theorems are sometimes referred to as the easy or *weak* Whitney embedding and immersion theorems, because Whitney obtained later the following improvements.

Theorem B.6 (Strong Whitney's embedding theorem)**.** *Given $n \geq 1$, every smooth n-manifold admits a proper smooth embedding into $\mathbb{R}^{2n}$.*

Theorem B.7 (Strong Whitney's immersion theorem)**.** *Given $n \geq 2$, every smooth n-manifold admits a smooth immersion into $\mathbb{R}^{2n-1}$.*

For the proofs of all the above results, as well as a discussion of sets of measure zero (in $\mathbb{R}^n$ or in smooth manifolds) we refer to [Lee13, Chapter 6 and Appendix C].

APPENDIX C

MULTILINEAR ALGEBRA

C.1 The Dual of a Vector Space

Definition C.1. Let V be a finite-dimensional real vector space.

(a) A *covector* on V is a real-valued linear functional on V, i.e., a linear map $\omega\colon V \to \mathbb{R}$.

(b) The set of all covectors on V is a real vector space under the obvious operations of pointwise addition and scalar multiplication. It is denoted by V^* and called *the dual space of* V.

The next proposition expresses the most important fact about V^* (in the finite-dimensional case).

Proposition C.2. *Let V be a real vector space of dimension n. Given any basis $(E_1, \dots, E_n)$ for V, consider the covectors $\varepsilon^1, \dots, \varepsilon^n \in V^*$ defined by*

$$\varepsilon^i(E_j) = \delta^i_j.$$

Then $(\varepsilon^1, \dots, \varepsilon^n)$ is a basis for V^*, called the dual basis to (E_j). *In particular,*

$$\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} V^*.$$

Proof. Exercise! □

In general, if (E_j) is a basis for V and if (ε^i) is its dual basis, then for any vector $v = v^j E_j \in V$ we have

$$\varepsilon^i(v) = v^j \varepsilon^i(E_j) = v^j \delta^i_j = v^i.$$

Thus, the i-th basis covector ε^i picks out the i-th component of a vector with respect to the basis (E_j).

More generally, we can express an arbitrary covector $\omega \in V^*$ in terms of the dual basis as

$$\omega = \omega_i \, \varepsilon^i,$$

where the i-th component is determined by $\omega_i = \omega(E_i)$. Thus, the action of the given covector $\omega \in V^*$ on a vector $v = v^j E_j \in V$ is

$$\omega(v) = \omega_i \, v^j \, \varepsilon^i(E_j) = \omega_i \, v^i.$$

Definition C.3. Let V and W be real vector spaces and let $A\colon V \to W$ be a linear map. *The dual map of* A is the linear map $A^*\colon W^* \to V^*$ defined by

$$(A^*\omega)(v) := \omega(Av), \;\; \omega \in W^*, \;\; v \in V.$$

It is straightforward to check that the dual map satisfies the following properties:

(a) $(A \circ B)^* = B^* \circ A^*$.

(b) $(\mathrm{Id}_V)^* = \mathrm{Id}_{V^*}$.

Proposition C.4. *The assignment that sends a vector space to its dual space and a linear map to its dual linear map is a contravariant functor from the category of real vector spaces to itself.*

Another important fact about the dual of a finite-dimensional vector space is the following.

Proposition C.5. *Let V be a finite-dimensional real vector space. For any given $v \in V$, define a linear functional $\xi(v)$ by*

$$\begin{aligned} \xi(v)\colon V^* &\to \mathbb{R} \\ \omega &\mapsto \xi(v)(\omega) := \omega(v). \end{aligned}$$

Then $\xi(v) \in (V^*)^*$; that is, $\xi(v)$ is a linear functional on V^*. Moreover, the map

$$\begin{aligned} \xi\colon V &\to (V^*)^* \\ v &\mapsto \xi(v) \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism, which is canonical *(it is defined without reference to any basis).*

Proof. The proof that both $\xi(v)$ and ξ are $\mathbb{R}$-linear maps are left as exercises. Since by Proposition C.2 we have

$$\dim V = \dim V^* = \dim (V^*)^*\,,$$

it suffices to prove that ξ is injective. To this end, let $v \in V$ be non-zero, complete it to a basis $\{v = E_1, E_2, \ldots, E_n\}$ of V, and let (ε^i) be its dual basis. Then

$$\xi(v)(\varepsilon^1) = \varepsilon^1(v) = \varepsilon^1(E_1) = 1,$$

so $\xi(v) \neq 0$. Therefore, $\ker \xi = 0$; in other words, ξ is injective, as desired. □

Due to Proposition C.5, the real number $\omega(v)$ obtained by applying a covector ω to a vector v is sometimes denoted by either of the more symmetric-looking notations $\langle \omega, v\rangle$ or $\langle v, \omega\rangle$; both expressions can be thought of either as the action of the covector $\omega \in V^*$ on the vector $v \in V$, or as the action of the linear functional $\xi(v) \in V^{**}$ on the element $\omega \in V^*$. There should be no cause for confusion with the use of the same angle bracket notation for inner products: whenever one of the arguments is a vector and the other a covector, the notation $\langle \omega, v\rangle$ is always to be interpreted as the natural pairing between vectors and covectors, not as an inner product.

There is also a symmetry between bases and dual bases for a finite-dimensional vector space V: any basis for V determines a dual basis for V^*, and conversely, any basis for V^* determines a dual basis for $V^{**} \cong V$. If (ε^i) is the basis for V^* dual to a basis (E_j) for V, then (E_j) is the basis dual to (ε^i), because both statements are equivalent to the relation $\langle \varepsilon^i, E_j\rangle = \delta^i_j$.

C.2 Multilinear Maps and Tensors

In the preceding section, we defined and briefly examined the dual of a vector space (in the finite-dimensional case), which is the space of real-valued linear functions on the given vector space. A natural, and from the point of view of (differential) geometry very important, generalization is to consider functions with several arguments, which are linear in each individual argument. These are called *multilinear* functions.

Definition C.6. Let $V_1, \dots, V_k$ and W be real vector spaces. A map $F\colon V_1 \times \cdots \times V_k \to W$ is called *multilinear* if it is linear as a function of each variable separately when the others are held fixed; that is, if $1 \leq i \leq k$ is arbitrary, and if we are given elements $v_i, v_i' \in V_i$ and real numbers $a, a' \in \mathbb{R}$, then

$$F(v_1, \dots, av_i + a'v_i', \dots, v_k) = aF(v_1, \dots, v_i, \dots, v_k) + a'F(v_1, \dots, v_i', \dots, v_k).$$

Denote by $L(V_1, \dots, V_k; W)$ the set of multilinear maps from $V_1 \times \cdots \times V_k$ to W, and note that $L(V_1, \dots, V_k; W)$ has the structure of a real vector space. In the special case when $V_1 = \ldots = V_k = V$ and $W = \mathbb{R}$, we often call an element of the space $L(V, \dots, V; \mathbb{R})$ a *k-multilinear function* on V; see Definition C.11.

Now, if the target space is $W = \mathbb{R}$, then there is a simple operation with which one can succesively build multilinear maps.

Definition C.7. Let $V_1, \dots, V_k$ and $W_1, \dots, W_l$ be real vector spaces, and consider $F \in L(V_1, \dots, V_k; \mathbb{R})$ and $G \in L(W_1, \dots, W_l; \mathbb{R})$. The function

$$\begin{aligned} F \otimes G\colon V_1 \times \cdots \times V_k \times W_1 \times \cdots \times W_l &\to \mathbb{R} \\ (v_1, \dots, v_k, w_1, \dots, w_l) &\mapsto F(v_1, \dots, v_k)\, G(w_1, \dots, w_l) \end{aligned}$$

is called the *tensor product of F and G*.

Exercise C.8:

(a) Show that, given F and G as above, the function $F \otimes G$ is multilinear, that is,

$$F \otimes G \in L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R}).$$

(b) Show that the tensor product operation

$$\begin{aligned} - \otimes -\colon L(V_1, \dots, V_k; \mathbb{R}) \times L(W_1, \dots, W_l; \mathbb{R}) &\to L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R}) \\ (F, G) &\mapsto F \otimes G \end{aligned}$$

is *bilinear*, i.e., multilinear with two variables, and *associative*, i.e., for any multilinear real-valued functions F, G, H, we have $F \otimes (G \otimes H) = (F \otimes G) \otimes H$.

Given a finite-dimensional real vector space V, we described in Section C.1 how to obtain a basis for the dual space $V^* = L(V; \mathbb{R})$ from a basis for V. With the above operation at hand, we may now generalize this to the space $L(V_1, \dots, V_k; \mathbb{R})$.

Proposition C.9. *Let* $V_1, \dots, V_k$ *be* $\mathbb{R}$*-vector spaces of dimensions* $n_1, \dots n_k$*, respectively. For each* $1 \le j \le k$*, let* $\left(E_1^{(j)}, \dots, E_{n_j}^{(j)}\right)$ *be a basis of* V_j*, and denote by* $\left(\varepsilon_{(j)}^1, \dots, \varepsilon_{(j)}^{n_j}\right)$ *the corresponding dual basis of* V_j^**. Then the set*

$$\mathcal{B} := \left\{ \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k} \mid 1 \le i_1 \le n_1, \dots, 1 \le i_k \le n_k \right\}$$

is a basis for $L(V_1, \dots, V_k; \mathbb{R})$*, which therefore has dimension* $n_1 \dots n_k$*.*

Proof. First, given $F \in L(V_1, \dots, V_k; \mathbb{R})$, define for each multi-index $I = (i_1, \dots, i_k)$ with $1 \le i_j \le n_j$ for all $1 \le j \le k$, a number $F_I \in \mathbb{R}$ by

$$F_I := F\left(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)}\right).$$

Also, use the short-hand notation

$$\varepsilon^{\otimes I} := \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k}.$$

We will show that

$$F = \sum_I F_I \, \varepsilon^{\otimes I},$$

where the sum is taken over all multi-indices as above, and thereby show that $\mathcal{B}$ spans $L(V_1, \dots, V_k; \mathbb{R})$. To this end, take $(v_1, \dots, v_k) \in V_1 \times \cdots \times V_k$. For integers i_j between 1 and n_j, let $v_j^{i_j} \in \mathbb{R}$ be the coefficient of v_j with respect to the basis $\left(E_1^{(j)}, \dots, E_{n_j}^{(j)}\right)$, i.e.,

$$v_j^{i_j} = \varepsilon_{(j)}^{i_j}(v_j).$$

Then by the multilinearity of F we have

$$F(v_1, \dots, v_k) = \sum_I v_1^{i_1} \cdots v_k^{i_k} F\left(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)}\right) = \sum_I v_1^{i_1} \cdots v_k^{i_k} F_I.$$

On the other hand, we have

$$\left[\sum_I F_I \, \varepsilon^{\otimes I}\right](v_1, \dots, v_k) = \sum_I F_I \, \varepsilon^{\otimes I}(v_1, \dots, v_k) = \sum_I v_1^{i_1} \cdots v_k^{i_k} F_I.$$

Hence F and $\sum_I F_I \varepsilon^{\otimes I}$ agree at any k-tuple and thus are equal, so $\mathcal{B}$ indeed spans $L(V_1, \dots, V_k; \mathbb{R})$.

Finally, in order to see that $\mathcal{B}$ is linearly independent, suppose that we have

$$\sum_I \lambda_I \, \varepsilon^{\otimes I} = 0$$

for some real numbers $\lambda_I \in \mathbb{R}$ indexed by multi-indices I. Evaluating both sides at $\left(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)}\right)$ for some fixed multi-index $I = (i_1, \dots, i_k)$, we obtain by the same computation as above that $\lambda_I = 0$. Hence, $\mathcal{B}$ is linearly independent. □

The proof of Proposition C.9 shows also that the components $F_{i_1\dots i_k}$ of a multilinear function F in terms of the basis elements in $\mathcal{B}$ are given by

$$F_{i_1\dots i_k} = F\left(E^{(1)}_{i_1}, \dots, E^{(k)}_{i_k}\right).$$

Thus, F is completely determined by its action on all possible sequences of basis vectors.

Remark C.10. You might have already encountered the abstract construction of the *tensor product of vector spaces.* If so, then regarding the above discussion (which shows that the real vector space $L(V_1, \dots, V_k; \mathbb{R})$ can be viewed as the set of all linear combinations of objects of the form $\omega^1 \otimes \dots \otimes \omega^k$, where $\omega^i \in V_i^*$ are covectors), one should remark the following: given finite-dimensional real vector spaces $V_1, \dots, V_k$, there is a canonical isomorphism

$$V_1^* \otimes \dots \otimes V_k^* \cong L(V_1, \dots, V_k; \mathbb{R}),$$

which is induced by the multilinear map

$$\begin{aligned} \Phi\colon V_1^* \times \ldots \times V_k^* &\to L(V_1, \dots, V_k; \mathbb{R}) \\ \Phi\left(\omega^1, \dots, \omega^k\right)(v_1, \dots, v_k) &:= \left(\omega^1 \otimes \dots \otimes \omega^k\right)(v_1, \dots, v_k) \\ &= \omega^1(v_1)\cdots\omega^k(v_k). \end{aligned}$$

Under this canonical isomorphism, abstract tensors correspond to the concrete tensor product of multilinear functions defined above. As it is a natural isomorphism, we may use the expression $V_1^* \otimes \dots \otimes V_k^*$ as a notation for $L(V_1, \dots, V_k; \mathbb{R})$ (this is a typical example of slight abuse of notation, where one identifies naturally isomorphic objects). Finally, using Proposition C.5, we also obtain a canonical identification

$$V_1 \otimes \dots \otimes V_k \cong L(V_1^*, \dots, V_k^*; \mathbb{R}).$$

Therefore, we may view the above construction as a concrete construction of the abstract tensor product.

Let us now turn our attention to various spaces of multilinear functions on a finite-dimensional real vector space that naturally appear in (differential) geometry.

Definition C.11. Let V be a finite-dimensional real vector space. For any integer $k \geq 1$, we denote by $T^k(V^*)$ the space of k-multilinear functions on V, i.e.,

$$T^k(V^*) := L(\underbrace{V, \dots, V}_{k \text{ times}}; \mathbb{R}) \cong \underbrace{V^* \otimes \ldots \otimes V^*}_{k \text{ copies}}.$$

By convention, we also define $T^0(V^*) := \mathbb{R}$. The elements of $T^k(V^*)$ are often referred to as *covariant k-tensors on V.*

Observe that every linear functional $\omega\colon V \to \mathbb{R}$ is (trivially) multilinear, so a covariant 1-tensor is just a covector on V. Thus,

$$T^1(V^*) = V^*.$$

According to Proposition C.9, we obtain a basis for $T^k(V^*)$ as follows. Assume that V has dimension n, let $(E_1, \dots, E_n)$ be a basis for V and denote by $(\varepsilon^1, \dots, \varepsilon^n)$ the dual basis for V^*. For a multi-index $I = (i_1, \dots, i_k)$, where $1 \leq i_j \leq n$ for all j, define the *elementary covariant k-tensor* $\varepsilon^{\otimes I}$ by the formula

$$\varepsilon^{\otimes I} := \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$$

(see the proof of Proposition C.9) and for an integer $m \in \mathbb{Z}_{\geq 1}$, denote by $[m]$ the set $\{1, \dots, m\}$. Then the set

$$\left\{\varepsilon^{\otimes I} \mid I \in [n]^{[k]}\right\}$$

is a basis for $T^k(V^*)$; in particular, we have

$$\dim_{\mathbb{R}} T^k(V^*) = n^k.$$

Therefore, every covariant k-tensor $\alpha \in T^k(V^*)$ can be written uniquely in the form

$$\alpha = \alpha_I \, \varepsilon^{\otimes I} = \alpha_{i_1 \dots i_k} \, \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k},$$

where the n^k coefficients $\alpha_I = \alpha_{i_1 \dots i_k}$ are determined by

$$\alpha_{i_1 \dots i_k} = \alpha(E_{i_1}, \dots, E_{i_k}).$$

For example, $T^2(V^*)$ is the space of *bilinear forms* on V – note that a covariant 2-tensor on V is simply a real-valued bilinear function of two vectors – and every bilinear form on V can be written as $\beta = \beta_{ij} \, \varepsilon^i \otimes \varepsilon^j$ for some uniquely determined $n \times n$ matrix (β_{ij}).

Definition C.12. For a covariant k-tensor $\alpha \in T^k(V^*)$ and a permutation $\sigma \in S_k$, denote by ${}^{\sigma}\alpha$ the covariant k-tensor given by

$$\begin{aligned} {}^{\sigma}\alpha \colon V \times \cdots \times V &\to \mathbb{R} \\ (v_1, \dots, v_k) &\mapsto \alpha\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big). \end{aligned}$$

In the following two sections we will discuss two important subspaces of $T^k(V^*)$, namely the subspaces of *symmetric* resp. *alternating* covariant k-tensors. Both notions are described by the way that a permutation of the arguments of the given covariant k-tensor changes its value. A significant application of symmetric tensors in the theory of smooth manifolds is in the form of *Riemannian metrics*. Loosely speaking, a Riemannian metric is a choice of an inner product on each tangent space of the given manifold, varying smoothly from point to point, and allows one to define geometric concepts such as lenghts, angles and distances on the manifold. Riemannian metrics will not be discussed in this course, and this is the main reason why the discussion about symmetric tensors in Section C.3 will be kept to a minimum. On the other hand, *differential forms* will be discussed thoroughly in this course. They constitute a significant application of alternating tensors in smooth manifold theory, and they will be presented in Section C.4.

C.3 Symmetric Tensors

In all probability, you have already encountered the concept of *inner product* on a finite-dimensional real vector space V. It is a bilinear map $\langle\cdot,\cdot\rangle\colon V\times V\to\mathbb{R}$ which is symmetric and positive definite; in particular, $\langle\cdot,\cdot\rangle$ is a covariant 2-tensor on V, having the additional property that its value is unchanged when the two input arguments are exchanged; namely, we have $\langle v_1,v_2\rangle=\langle v_2,v_1\rangle$ for any $v_1,v_2\in V$. We now generalize this notion to any covariant k-tensor on V.

Definition C.13. Let V be a finite-dimensional real vector space.

(a) A covariant k-tensor $\alpha\in T^k(V^*)$ on V is said to be *symmetric* if its value is unchanged by interchanging any pair of its arguments; namely, for all $v_1,\dots,v_k\in V$ and all $1\le i<j\le k$, we have

$$\alpha(v_1,\dots,v_i,\dots,v_j,\dots,v_k)=\alpha(v_1,\dots,v_j,\dots,v_i,\dots,v_k).$$

(b) The set of symmetric covariant k-tensors on V is denoted by $\Sigma^k(V^*)$. It is clearly a linear subspace of $T^k(V^*)$. By convention, we define $\Sigma^0(V^*):=\mathbb{R}$, and we also note that $\Sigma^1(V^*)=T^1(V^*)=V^*$.

Exercise C.14: We define a projection $\operatorname{Sym}\colon T^k(V^*)\to\Sigma^k(V^*)$, called *symmetrization*, by the formula

$$\operatorname{Sym}(\alpha):=\frac{1}{k!}\sum_{\sigma\in S_k}{}^{\sigma}\alpha,$$

where ${}^{\sigma}\alpha$ was defined in Definition C.12. Show that Sym is well-defined and linear, and that the following are equivalent:

(a) α is symmetric,

(b) $\alpha={}^{\sigma}\alpha$ for all $\sigma\in S_k$,

(c) $\alpha=\operatorname{Sym}(\alpha)$.

C.4 Alternating Tensors

Recall that the determinant may be regarded as a function $\det\colon\mathbb{R}^n\times\cdots\times\mathbb{R}^n\to\mathbb{R}$, taking as input n column vectors with n entries each, and having as output the determinant of the $n\times n$ matrix formed by these n column vectors. This map is multilinear, so det is a covariant n-tensor on $\mathbb{R}^n$. Moreover, it has the property that its value changes sign whenever two of its input entries are interchanged; in other words, det is an *alternating* n-tensor. We now generalize this notion to arbitrary covariant k-tensors.

Definition C.15. Let V be a finite-dimensional real vector space.

(a) A covariant k-tensor $\alpha\in T^k(V^*)$ on V is said to be *alternating* (or *anti-symmetric* or *skew-symmetric*) if its value changes sign whenever any two of its arguments are interchanged; namely, for all $v_1,\dots,v_k\in V$ and $1\le i<j\le k$, we have

$$\alpha(v_1,\dots,v_i,\dots,v_j,\dots,v_k)=-\alpha(v_1,\dots,v_j,\dots,v_i,\dots,v_k).$$

(b) The set of alternating covariant k-tensors on V is denoted by $\Lambda^k(V^*)$. It is clearly a linear subspace of $T^k(V^*)$ and its elements of $\Lambda^k(V^*)$ are also called *exterior forms*, *multicovectors* or *k-covectors*. By convention, we define $\Lambda^0(V^*) := \mathbb{R}$, and we also note that $\Lambda^1(V^*) = T^1(V^*) = V^*$.

Note that every covariant 2-tensor β can be expressed as a sum of an alternating and a symmetric tensor, because

$$\begin{aligned}\beta(v,w) &= \frac{1}{2}\big(\beta(v,w)-\beta(w,v)\big)+\frac{1}{2}\big(\beta(v,w)+\beta(w,v)\big)\\ &= \alpha(v,w)+\sigma(v,w),\end{aligned}$$

where

$$\alpha(v,w) := \frac{1}{2}\big(\beta(v,w)-\beta(w,v)\big) \in \Lambda^2(V^*)$$

is an alternating 2-tensor on V and

$$\sigma(v,w) := \frac{1}{2}\big(\beta(v,w)+\beta(w,v)\big) \in \Sigma^2(V^*)$$

is a symmetric 2-tensor on V. However, this is not true for tensors of higher rank, as the following exercise demonstrates.

Exercise C.16: Let (e^1, e^2, e^3) be the standard dual basis for $(\mathbb{R}^3)^*$. Show that $e^1 \otimes e^2 \otimes e^3$ is not equal to a sum of an alternating tensor and a symmetric tensor.

Recall that there is a group homomorphism $\operatorname{sgn}\colon S_k \to \{\pm 1\}$, which maps a permutation $\sigma \in S_k$ to 1 if it is a product of an even number of transpositions (even permutation), and to -1 otherwise (odd permutation). We may use it to describe alternating tensors as follows.

Exercise C.17: We define a projection $\operatorname{Alt}\colon T^k(V^*) \to \Lambda^k(V^*)$, called *alternation*, by the formula

$$\operatorname{Alt}(\alpha) := \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\,{}^{\sigma}\alpha,$$

where ${}^{\sigma}\alpha$ was defined in Definition C.12. Show that Alt is well-defined and linear, and that the following are equivalent:

(a) α is alternating,

(b) $\alpha = (\operatorname{sgn}\sigma)\,{}^{\sigma}\alpha$ for all $\sigma \in S_k$,

(c) $\alpha = \operatorname{Alt}(\alpha)$,

(d) $\alpha(v_1, \dots, v_k) = 0$ whenever $v_1, \dots, v_k \in V$ are linearly dependent,

(e) $\alpha(v_1, \dots, v_k) = 0$ whenever there are $i \neq j$ such that $v_i = v_j$.

Example C.18. Let us explicitly compute Alt for 1-, 2- and 3-tensors.

- If α is a 1-tensor, then $\operatorname{Alt}(\alpha) = \alpha$.

- If β is a 2-tensor, then

$$\operatorname{Alt}(\beta)(u,v) = \frac{1}{2}\big(\beta(u,v) - \beta(v,u)\big).$$

- If γ is a 3-tensor, then

$$\begin{aligned}\operatorname{Alt}(\gamma)(u,v,w) = \frac{1}{6}\big(&\gamma(u,v,w) + \gamma(v,w,u) + \gamma(w,u,v)\\ &-\gamma(v,u,w) - \gamma(u,w,v) - \gamma(w,v,u)\big).\end{aligned}$$

C.4.1 Elementary Alternating Tensors

Recall that for any basis of V, we described an induced basis of $T^k(V^*)$ in terms of tensor products of elements of the dual basis; cf. Proposition C.9. We obtain here a similar description for a basis of $\Lambda^k(V^*)$.

Let V be a real vector space of dimension n, let $(E_1, \dots, E_n)$ be a basis for V, and denote by $(\varepsilon^1, \dots, \varepsilon^n)$ the corresponding dual basis for V^*. For a multi-index $I = (i_1, \dots, i_k) \in [n]^{[k]}$, define the *elementary alternating k-tensor* (or *elementary k-covector*) ε^I by the formula

$$\varepsilon^I := k! \operatorname{Alt}\left(\varepsilon^{\otimes I}\right),$$

where

$$\varepsilon^{\otimes I} = \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} \in T^k(V^*)$$

is the elementary k-tensor. Therefore, if $v_1, \dots, v_k \in V$, then the value of ε^I at the k-tuple $(v_1, \dots, v_k)$ is given by the formula

$$\begin{aligned}\varepsilon^I(v_1, \dots, v_k) &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \varepsilon^{\otimes I}\left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right)\\ &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \prod_{1 \le j \le k} \varepsilon^{i_j}\left(v_{\sigma(j)}\right)\\ &= \det\begin{pmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k)\\ \vdots & \ddots & \vdots\\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k)\end{pmatrix}.\end{aligned}$$

In other words, to compute $\varepsilon^I(v_1, \dots, v_k)$, we write the coefficients of $(v_1, \dots, v_k)$ with respect to the basis $(E_1, \dots, E_n)$ of V in the form of a $n \times k$-matrix, we consider the $k \times k$ submatrix formed by the lines $i_1, \dots, i_k$, and then we compute its determinant.

Example C.19. In terms of the standard dual basis (e^1, e^2, e^3) for $(\mathbb{R}^3)^*$, we have

$$e^{13}(v,w) = \det\begin{pmatrix} v^1 & w^1\\ v^3 & w^3\end{pmatrix} = v^1 w^3 - v^3 w^1,$$

since $v = v^1 e_1 + v^2 e_2 + v^3 e_3$ and $w = w^1 e_1 + w^2 e_2 + w^3 e_3$, and

$$e^{123}(v,w,z) = \det(v,w,z).$$

Since $\mathrm{Alt}\colon T^k(V^*) \to \Lambda^k(V^*)$ is surjective, we know that $\left\{\varepsilon^I \mid I \in [n]^{[k]}\right\}$ is a generating set of $\Lambda^k(V^*)$. To extract from it a basis of $\Lambda^k(V^*)$, we need the following lemma, which describes the redundancy of $\left\{\varepsilon^I \mid I \in [n]^{[k]}\right\}$. In order to state it nicely, we need to introduce the following notation: for a multi-index $I \in [n]^{[k]}$ and a permutation $\sigma \in S_k$, denote by I_σ the multi-index

$$I_\sigma = \left(i_{\sigma(1)}, \dots, i_{\sigma(k)}\right).$$

Also, denote by δ^I_J the following generalization of the Kronecker-delta to multi-indices $I, J \in [n]^{[k]}$:

$$\delta^I_J := \begin{cases} \operatorname{sgn}\sigma & \text{if neither } I \text{ nor } J \text{ have repeated entries and } J = I_\sigma \text{ for some } \sigma \in S_k, \\ 0 & \text{if } I \text{ or } J \text{ have repeated entries or } J \text{ is not a permutation of } I. \end{cases}$$

and observe that

$$\delta^I_J = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \dots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \dots & \delta^{i_k}_{j_k} \end{pmatrix}.$$

Lemma C.20. *With the same notation as in the preceeding paragraph, the following statements hold:*

(a) *If I has a repeated index, then $\varepsilon^I = 0$.*

(b) *If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^J = (\operatorname{sgn}\sigma)\, \varepsilon^I$.*

(c) *For $I, J \in [n]^{[k]}$ we have*

$$\varepsilon^I\left(E_{j_1}, \dots, E_{j_k}\right) = \delta^I_J.$$

Proof. Exercise! ☐

Lemma C.20 tells us that from the generating set $\left\{\varepsilon^I \mid I \in [n]^{[k]}\right\}$ of $\Lambda^k(V^*)$, we may discard all those ε^I's for which I has a repeated index, and for any I having no repeated index, we need only take one element from the set $\{\varepsilon^{I_\sigma} \mid \sigma \in S_k\}$ and discard the rest. A nice choice is thus the following: notice that for any multi-index I having no repeated indices, there exists a unique permutation $\sigma \in S_k$ such that I_σ is *strictly increasing*, i.e., $i_{\sigma(1)} < \dots < i_{\sigma(k)}$. Therefore, according to Lemma C.20, the set $\left\{\varepsilon^I \mid I \in [n]^{[k]} \text{ is strictly increasing}\right\}$ still generates $\Lambda^k(V^*)$, and there is no obvious redundancy in it. Essentially due to Lemma C.20(c), this set is linearily independent, and thus we obtain the following result:

Proposition C.21. *With the same notation as above, the set*

$$\left\{\varepsilon^I \mid I \in [n]^{[k]} \textit{ is a strictly increasing multi-index}\right\}$$

is a basis for $\Lambda^k(V^*)$. In particular, we have

$$\dim_{\mathbb{R}} \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and

$$\Lambda^k(V^*) = \{0\} \quad \textit{for } k > n.$$

Proof. Assume first that $k > n$. Since then every k-tuple of vectors is linearly dependent, it follows from Exercise C.17(d) that $\Lambda^k(V^*) = \{0\}$.

Assume now that $k \leq n$. We need to show that

$$\mathcal{E} := \left\{ \varepsilon^I \mid I \in [n]^{[k]} \text{ is a strictly increasing multi-index} \right\}$$

is linearly independent and spans $\Lambda^k(V^*)$. The fact that $\mathcal{E}$ generates $\Lambda^k(V^*)$ was already discussed above. Suppose now that we have some linear relation

$$\sum_{I \in [n]^{[k]} \text{ strictly increasing}} \lambda_I \, \varepsilon^I = 0$$

for some $\lambda_I \in \mathbb{R}$. If we fix a strictly increasing multi-index $J \in [n]^{[k]}$, then evaluating the above relation at $(E_{j_1}, \ldots, E_{j_k})$ gives $\lambda_J = 0$ according to Lemma C.20(c). Thus, $\mathcal{E}$ is linearly independent. In conclusion, $\mathcal{E}$ is a basis of $\Lambda^k(V^*)$, as desired. □

In particular, if V is a real vector space of dimension n, then the above proposition implies that $\Lambda^n(V^*)$ is 1-dimensional, spanned by the elementary n-covector $\varepsilon^{(1,\ldots,n)}$. As discussed in the beginning of this subsection, $\varepsilon^{(1,\ldots,n)}$ sends an n-tuple $(v_1, \ldots, v_n)$ to the determinant of the matrix $(v^i_j)_{1 \leq i,j, \leq n}$, where $v^i_j = \varepsilon^i(v_j)$ is the i-th component of v_j with respect to the chosen basis of V. Note that when $V = \mathbb{R}^n$ with the standard basis, the covector $\varepsilon^{(1,\ldots,n)}$ (which by definition is a function from $(\mathbb{R}^n)^n = \mathbb{R}^{n^2}$ to $\mathbb{R}$) is precisely the usual determinant function.

One consequence of this observation is the following useful description of the behavior of an n-covector on an n-dimensional vector space under linear maps. Recall that if $T \colon V \to V$ is a linear map, then the *determinant* of T is defined to be the determinant of the matrix representation of T with respect to any basis (recall that any two such matrix representation are conjugations of each other and hence have the same determinant, so this is well-defined).

Proposition C.22. *Let V be an n-dimensional real vector space and let $\omega \in \Lambda^n(V^*)$. If $T \colon V \to V$ is any linear map and if $v_1, \ldots, v_n \in V$ are arbitrary vectors, then*

$$\omega(Tv_1, \ldots, Tv_n) = (\det T)\, \omega(v_1, \ldots, v_n). \tag{$\bullet$}$$

Proof. Let (E_i) be any basis for V, and let (ε^i) be the dual basis. Denote by $(T^j_i)_{1 \leq i,j \leq n}$ the matrix of T with respect to this basis, and set $T_i = TEi = \sum_j T^j_i E_j$. By Proposition C.21, we can write $\omega = c\varepsilon^{(1,\ldots,n)}$ for some $c \in \mathbb{R}$. Since both sides of ($\bullet$) are multilinear functions of $(v_1, \ldots, v_n)$, it suffices to verify the identity when the v_i's are basis vectors. Furthermore, since both sides are alternating, by Lemma C.20 we only need to check the case $(v_1, \ldots, v_n) = (E_1, \ldots, E_n)$. In this case, the right-hand side of ($\bullet$) is

$$(\det T)\, c\, \varepsilon^{(1,\ldots,n)}(E_1, \ldots, E_n) = c \det T.$$

On the other hand, the left-hand side of ($\bullet$) reduces to

$$\omega(TE_1, \ldots, TE_n) = c\, \varepsilon^{(1,\ldots,n)}(T_1, \ldots, T_n) = c \det \big((\varepsilon^j(T_i))_{1 \leq i,j \leq n} \big) = c \det \big((T^j_i)_{1 \leq i,j \leq n} \big).$$

which is thus equal to the right-hand side. □

C.4.2 The Wedge Product

Recall that for any covariant tensors $\alpha \in T^k(V^*)$ and $\beta \in T^l(V^*)$ we defined the covariant $(k+l)$-tensor $\alpha \otimes \beta$; see Definition C.7. This allowed us to build 'higher' covariant tensors out of lower ones, and also to describe a basis for $T^k(V^*)$ in terms of tensor products of elements of a dual basis. We now describe a similar construction for alternating tensors.

Definition C.23. Let V be a finite-dimensional real vector space, and let $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$ be alternating tensors on V. The *wedge product* (or *exterior product*) of ω and η is denoted by $\omega \wedge \eta$ and is defined to be the $(k+l)$-covector given by the formula

$$\omega \wedge \eta \coloneqq \frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta).$$

As $\otimes$ is bilinear and Alt is linear, the map $- \wedge -\colon \Lambda^k(V^*) \times \Lambda^l(V^*) \to \Lambda^{k+l}(V^*)$ is bilinear. It is therefore natural to examine what the wedge product looks like on basis vectors. This also motivates the somewhat mysterious normalization factor $(k+l)!/(k!l!)$, because we have the following result.

Lemma C.24. *Let V be a finite-dimensional real vector space, and let $(\varepsilon^1, \dots, \varepsilon^n)$ be a basis for V^*. For any multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$ we have the formula*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{I \frown J},$$

where $I \frown J = (i_1, \dots, i_k, j_1, \dots, j_l)$ is the $(k+l)$-multi-index obtained by concatenating I and J.

Proof. By multilinearity, as in the proof of Proposition C.9, it suffices to show that

$$\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) = \varepsilon^{I \frown J}(E_{p_1}, \dots, E_{p_{k+l}}) \tag{$\star$}$$

for any sequence of basis vectors $(E_{p_1}, \dots, E_{p_{k+l}})$. We do this by considering several cases.

Case 1: *The multi-index $P = (p_1, \dots, p_{k+l})$ has a repeated index.* Then by part (e) of Exercise C.17, both sides of $(\star)$ evaluate to 0.

Case 2: *P contains an index that does not appear in either I or J.* In this case, the right-hand side of $(\star)$ is zero by part (c) of Lemma C.20. Similarly, each term in the expansion of the left-hand side of $(\star)$ involves either I or J evaluated on a sequence of basis vectors that is not a permutation of I or J, respectively, so the left-hand side is also zero.

Case 3: *$P = I \frown J$ and P has no repeated indices.* In this case, the right-hand side of $(\star)$ is equal to 1, again by part (c) of Lemma C.20, so we need to show that the left-hand side is also equal to 1. By definition,

$$\begin{aligned} &\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) = \\ &\quad = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\varepsilon^I \otimes \varepsilon^J) \\ &\quad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \varepsilon^I(E_{p_{\sigma(1)}}, \dots, E_{p_{\sigma(k)}}) \varepsilon^J(E_{p_{\sigma(k+1)}}, \dots, E_{p_{\sigma(k+l)}}). \end{aligned}$$

By Lemma C.20 again, the only terms in the sum above that give nonzero values are those in which σ permutes the first k indices and the last l indices of P separately. In other words, σ must be of the form $\sigma = \tau\eta$, where $\tau \in S_k$ acts by permuting $\{1, \ldots, k\}$ and $\eta \in S_l$ acts by permuting $\{k+1, \ldots, k+l\}$. Since then $\operatorname{sgn}\sigma = (\operatorname{sgn}\tau)(\operatorname{sgn}\eta)$, we have

$$\begin{aligned}
&\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \ldots, E_{p_{k+l}}) = \\
&= \frac{1}{k!l!} \sum_{\substack{\tau \in S_k \\ \eta \in S_l}} (\operatorname{sgn}\tau)(\operatorname{sgn}\eta)\, \varepsilon^I(E_{p_{\tau(1)}}, \ldots, E_{p_{\tau(k)}})\, \varepsilon^J(E_{p_{k+\eta(1)}}, \ldots, E_{p_{k+\eta(l)}}) \\
&= \left(\frac{1}{k!} \sum_{\tau \in S_k} (\operatorname{sgn}\tau)\, \varepsilon^I(E_{p_{\tau(1)}}, \ldots, E_{p_{\tau(k)}})\right) \left(\frac{1}{l!} \sum_{\eta \in S_l} (\operatorname{sgn}\eta)\, \varepsilon^J(E_{p_{k+\eta(1)}}, \ldots, E_{p_{k+\eta(l)}})\right) \\
&= \big(\operatorname{Alt}(\varepsilon^I)(E_{p_1}, \ldots, E_{p_k})\big)\, \big(\operatorname{Alt}(\varepsilon^J)(E_{p_{k+1}}, \ldots, E_{p_{k+l}})\big) \\
&= \varepsilon^I(E_{p_1}, \ldots, E_{p_k})\, \varepsilon^J(E_{p_{k+1}}, \ldots, E_{p_{k+l}}) \\
&= 1
\end{aligned}$$

where we used that Alt fixes alternating tensors by Exercise C.17, and again used part (c) of Lemma C.20 (recall that we are in the case $P = I \frown J$).

Case 4: *P is a permutation of $I \frown J$ and has no repeated indices.* In this case, applying a permutation to P brings us back to Case 3. As both sides of ($\star$) are alternating, the effect of this permutation is to multiply both sides by the same sign. Hence the result holds in this final case as well.

This completes the proof of the lemma. ☐

Together with the bilinearity of $- \wedge -$, this gives the following properties of the wedge product.

Proposition C.25. *Let ω, η, ξ be multicovectors on a finite-dimensional real vector space V. Then we have the following properties:*

(a) Associativity:

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi.$$

(b) Anticommutativity: *if $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, then*

$$\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega.$$

(c) *If $(\varepsilon^1, \ldots, \varepsilon^n)$ is a basis of V^* and $I = (i_1, \ldots, i_k)$ a multi-index, then*

$$\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k} = \varepsilon^I.$$

(d) *For any $\omega^1, \ldots, \omega^k \in V^*$ and $v_1, \ldots, v_k \in V$ we have*

$$\omega^1 \wedge \ldots \wedge \omega^k(v_1, \ldots, v_k) = \det\left(\left(\omega^j(v_i)\right)_{1 \le i,j \le k}\right).$$

Proof. Exercise! ☐

Due to Proposition C.25(c), we generally use the notations ε^I and $\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k}$ interchangably.

An element $\eta \in \Lambda^k(V^*)$ is said to be *decomposable* if it can be expressed in the form $\eta = \omega^1 \wedge \ldots \wedge \omega^k$ for some covectors $\omega^1, \ldots, \omega^k \in V^*$. Note that not every k-covector is decomposable when $k > 1$; however, it follows from Proposition C.21 and Proposition C.25(c) that every k-covector can be written as a linear combination of decomposable ones.

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