Exercise Sheet n°1

Exercise 1:

In the first order language of set theory,

1. write a formula $\mathrm{P}(z,x,y)$ which defines $z = \{x,y\}$, that is, z is the pair set of x and y,
2. write a formula $\mathrm{PO}(z,x,y)$ which defines $z = \langle x,y \rangle$, that is, z is the ordered pair of x and y,
3. write a formula which is equivalent to the following proposition, you have seen in class:

$$\forall x \forall y \forall x' \forall y' \big((\langle x,y \rangle = \langle x',y' \rangle) \leftrightarrow (x = x' \wedge y = y') \big).$$

Prove the proposition, by distinguishing the cases $x = y$ and $x \neq y$.

Exercise 2:

Show that the theory ZF is equivalent to the theory ZF^* obtained from ZF by substituting the axiom schema of replacement with the following axiom schema:

$$\mathrm{Repl}_F \ : \ \forall \vec{c} (\forall x \forall y \forall z ((F(x,y,\vec{c}) \wedge F(x,z,\vec{c})) \rightarrow y = z) \rightarrow \forall A \exists B \forall y (\exists x \in A \ F(x,y,\vec{c}) \rightarrow y \in B))$$

for all formulae $F(x,y,\vec{c})$ of set theory.

Exercise 3:

Denote by ZFC_{fin} the axiomatic system obtained from ZFC by removing the axiom of infinity. The goal of this exercise is to exhibit a model of ZFC_{fin}.

Definition. *For all natural numbers $q \in \mathbb{N}$, write $[q]$ for the unique set of natural numbers $\{p_1, \ldots, p_n\}$ such that $q = \sum_{i=1}^{n} 2^{p_i}$ (i.e. the elements of $[q]$ are the positions of the binary expansion of q, counted right to left, where a 1 appears). Let E be the binary relation on $\mathbb{N}$ defined by pEq iff $p \in [q]$ for all $p,q \in \mathbb{N}$.*

Prove the following:

1. The structure $(\mathcal{P}_{\text{fin}}(\mathbb{N}), \epsilon)$, where $\mathcal{P}_{\text{fin}}(\mathbb{N})$ is the set of finite subsets of $\mathbb{N}$ and ϵ is defined as $F \epsilon G$ iff $\sum_{n \in F} 2^n \in G$ for $F,G \in \mathcal{P}_{\text{fin}}(\mathbb{N})$, is isomorphic to $(\mathbb{N},E)$.
2. The structure $(\mathbb{N}, E)$ satisfies the axiom of extensionality.
3. The structure $(\mathbb{N}, E)$ is a model of ZFC_{fin} and the axiom of infinity does not hold in this structure.