Worksheets #3 & #4

Algebra V - Galois theory

October 2, 2025

Problem 1. *Let $L = \mathbb{Q}(\sqrt[4]{2})$ and $F = \mathbb{Q}(\sqrt{2})$.*

(i) Determine whether or not each of the three extensions $L/\mathbb{Q}$, $F/\mathbb{Q}$ and L/F is normal. You must justify your answer.

(ii) Find a $\mathbb{Q}$-homomorphism $\varphi : F \to F$ that does not extend to a $\mathbb{Q}$-homorphism $\tilde{\varphi} : L \to L$.

Problem 2. *For each of the following field extensions, prove that they are Galois and describe the corresponding Galois groups. Find all subgroups and the corresponding fixed fields.*

(i) $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$

(ii) $\mathbb{Q} \subset \mathbb{Q}(\alpha)$, where α is a primitive 5-th root of unity.

Problem 3. *Let $K = \mathbb{F}_2$ and $L = \mathbb{F}_2(\alpha)$, where α is a root of the polynomial $x^4 + x + 1 \in K[x]$. Prove that $\mathrm{Gal}(L/K)$ contains exactly one subgroup H of order two and find $\beta \in L$ such that $L^H = \mathbb{F}_2(\beta)$.*

Problem 4. *Give an example of fields $K \subset F \subset L$ such that the extensions L/F and F/K are normal, but the extension L/K is not.*

Problem 5. *Give three examples of a field extension L/K which is neither normal nor separable.*

Problem 6. *Let K be a field, $f \in K[x]$ an irreducible polynomial of degree four and $L = SF_K(f)$. Prove that $\mathrm{Gal}(L/K)$ cannot be isomorphic to the quaternion group of order eight.*

Problem 7. *Let L/K be a finite field extension of degree m, and let $f \in K[x]$ be irreducible (over K) of degree n. Prove that if $gcd(m, n) = 1$, then f is also irreducible over L.*

Problem 8. *Let L/K be a finite field extension and assume that $char(K)$ does not divide $[L : K]$. Prove that L is separable over K.*

Problem 9. *Let L be a field and fix $\sigma \in \mathrm{Aut}(L)$ of infinite order. Let $K \coloneqq \{a \in L \mid \sigma(a) = a\}$ be the fixed field of σ. Prove that if L/K is algebraic, then L is normal over K.*

Problem 10. *Let L/K be a finite field extension and pick $\alpha \in L$. The trace (resp. the norm) of α is defined to be the trace (resp. the determinant) of the linear map $M_\alpha : L \to L$ of K-vector spaces given by multiplication by α. Assume that $d = [L : K(\alpha)]$ and write*

$$f \coloneqq min_K(\alpha) = x^n + a_{n_1}x^{n-1} + \ldots + a_1 x + a_0 \in K[x].$$

Prove that

(i) the characteristic polynomial of M_α is f^d, and

(ii) the trace (resp. the norm) of α is the number db_{n-1} (resp. b_0^d), where $b_i \coloneqq (-1)^{n-i}a_i$.

Problem 11.

(a) Prove that any field extension of degree two is normal.

(b) Let L/K be a finite Galois extension of degree $2d$, with d odd. Prove that $\mathrm{Gal}(L/K)$ has a subgroup of index d.

Problem 12. *Let L/K be a Galois extension. Let $H \leq \mathrm{Gal}(L/K)$ and $F = L^H$ the fixed field of H. Prove that $H \trianglelefteq \mathrm{Gal}(L/K)$ if and only if F/K is normal, in which case $\mathrm{Gal}(F/K) \simeq \mathrm{Gal}(L/K)/H$.*

Problem 13. *Let $K \subset L$ be finite fields and let $|K| = q$.*

(i) Prove that the extension L/K is Galois.

(ii) Prove that $\mathrm{Gal}(L/K)$ is cyclic and generated by the Frobenius automorphism $x \mapsto x^q$.

(iii) Prove that every element of K is the norm of an element of L (see exercise 8).