# Week #7

Algebra V - Galois theory

Oct 30, 2025

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**Problem 1.** *Let $n$ be any positive integer and let $\Phi_n(x) = \prod_{i=1}^{\phi(n)}(x - \zeta_i)$ denote the $n$-th cyclotomic polynomial. That is, $\Phi_n(x) \in \mathbb{C}[x]$ is a monic polynomial, and its roots are exactly the primitive $n$-th roots of unity in $\mathbb{C}$. Prove that $\Phi_n(x) \in \mathbb{Z}[x]$ and that $\Phi_n(x)$ is irreducible over $\mathbb{Q}$.*

**Problem 2.** *Let $n$ be any positive integer and let $\mathbb{Q}_n \coloneqq SF_{\mathbb{Q}}(x^n - 1)$.*

*(i) Prove that $[\mathbb{Q}_n : \mathbb{Q}] = \phi(n)$ and $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times$.*

*(ii) If $m$ is an odd (positive) integer, prove that $\mathbb{Q}_{2m} = \mathbb{Q}_m$.*

*(iii) Find $n$ such that $\mathbb{Q}_n$ contains a subfield which is not a cyclotomic extension of $\mathbb{Q}$.*

*(iv) Find all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}_8$, and between $\mathbb{Q}$ and $\mathbb{Q}_{12}$.*

**Problem 3.** *Prove that $\mathbb{Q}(\cos(2\pi/n))/\mathbb{Q}$ is a Galois extension for every $n \in \mathbb{N}$.*

**Problem 4.** *Consider a radical extension $K = K_0 \subset K_1 \subset \ldots \subset K_n = L$ as defined in the lecture, where each $K_{j+1}/K_j$ is obtained by extracting an $m_j$-th root. Let $m = \prod_{j=1}^{n} m_j$ (assume that char$(K)$ does not divide the $m_j$s) and consider the field $F = K(\mu_m)$. Prove that the tower of composite fields*

$$K \subset F = FK_0 \subset FK_1 \subset \ldots \subset FK_n = FL$$

*is a radical tower and that each $FK_{j+1}/FK_j$ is Galois with cyclic Galois group.*

**Problem 5.** *Let $L/K$ and $M/K$ be two finite field extensions such that $L, M \subset \bar{K}$ and $M/K$ and $LM/M$ are solvable. Prove that $L/K$ is solvable as well.*

**Problem 6.** *Let $L/K$ and $M/K$ be two finite Galois extensions. Prove that $LM/M$ and $L/L \cap M$ are also Galois and that*

$$\mathrm{Gal}(LM/M) \simeq \mathrm{Gal}(L/L \cap M)$$

**Problem 7.** *Let $L/K$ be a finite separable extension of prime degree $p$. Let $\alpha \in L$ be such that $L = K(\alpha)$ and let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_p$ be the $K$-conjugates of $\alpha$ in $\bar{K}$. Prove that if $\alpha_2 \in L$, then $L/K$ is Galois with cyclic Galois group.*