

# Worksheet #6

## Algebra V - Galois theory

October 16, 2025

**Problem 1.** Let  $F$  be a finite field. Prove that  $\text{Aut}_F(\bar{F})$  is an Abelian group and that every element of this group has infinite order.

**Problem 2.** Let  $G \neq \{1\}$  be a finite Abelian group.

(i) Prove that there exists a Galois extension  $L/\mathbb{Q}$  such that  $\text{Gal}(L/\mathbb{Q}) \simeq G$ .

(ii) Let  $K/\mathbb{Q}$  be a finite extension. Prove that there exist infinitely many Galois extensions  $L/K$  such that  $\text{Gal}(L/K) \simeq G$ .

**Problem 3.** Let  $K$  be a field such that every finite extension  $L/K$  is cyclic. Show that there exists  $\sigma \in \text{Aut}_K(\bar{K})$  such that  $K = \bar{K}^\sigma = \{x \in \bar{K} ; \sigma(x) = x\}$ .

**Problem 4.** Let  $L/K$  be a Galois extension with Galois group  $G$ . Define the so-called trace map  $T : L \rightarrow L$  (cf. Problem 10, worksheet 3/4) by

$$\ell \mapsto \sum_{g \in G} g(\ell)$$

(i) Prove that  $\text{im}(T) \subset K$  and  $T$  is additive.

(ii) Assume that  $[L : K] = |G| = n$  and  $G = \langle \sigma \rangle$  is cyclic and prove that  $\ker(T) = \text{im}(\sigma - \text{id}_L)$ .

**Problem 5.** Prove that  $\mathbb{Q}(\sqrt[3]{2})$  is not a subfield of any cyclotomic extension of  $\mathbb{Q}$ .

**Problem 6.** Let  $f \in \mathbb{Q}[x]$  be a cubic polynomial such that  $\text{Gal}(SF_{\mathbb{Q}}(f)/\mathbb{Q}) = C_3$ . Prove that all the roots of  $f$  are real.

**Problem 7.** Let  $L/K$  be a cyclic extension of degree three of fields of characteristic not equal to two. Write  $G = \text{Gal}(L/K) = \{1, \sigma, \sigma^2\}$  and for  $\beta \in L$  define (cf. Problem 10, worksheet 3/4)

$$N_{L/K}(\beta) = \prod_{g \in G} g(\beta).$$

Fix  $\beta \in L^\times \setminus K$  not a square with  $N_{L/K}(\beta) = 1$  and let  $\alpha$  be a root of  $x^2 - \beta$ .

(i) Prove that the normal closure of  $L(\alpha)$  over  $K$  is a Galois extension of  $K$  with Galois group  $A_4$ .

(ii) Let now  $L = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$  and  $K = \mathbb{Q}$ , where  $\zeta_7$  is a primitive 7-th root of unity. Then,  $L/K$  is a cubic cyclic extension. Check that  $\beta = \zeta_7 + \zeta_7^{-1}$  has the properties outlined above.