# Worksheet #5

Algebra V - Galois theory

October 9, 2025

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**Problem 1.** *In what follows, let* $L = SF_K(f)$ *for some* $f \in K[x]$*. Determine* $\mathrm{Gal}(L/K)$ *and find all intermediate subfields of* $L/k$*.*

*(i)* $K = \mathbb{Q}$ *and* $f(x) = x^4 - 7$

*(ii)* $K = \mathbb{F}_5$ *and* $f(x) = x^4 - 7$

*(iii)* $K = \mathbb{F}_2$ *and* $f(x) = x^6 + 1$

*(iv)* $K = \mathbb{Q}$ *and* $f(x) = x^8 - 1$

**Problem 2.** *Let* $L/K$ *be a (finite) Galois extension such that* $\mathrm{Gal}(L/K) = A_4$*. Prove that there is no intermediate field* $K \subset F \subset L$ *satisfying that* $[F : K] = 2$*.*

**Problem 3.** *Give examples of finite field extensions* $L/K$ *with*

*(i)* $L/K$ *normal but not separable,*

*(ii)* $L/K$ *separable but not normal,*

*(iii)* $[L : K] = 4$ *and there is no intermediate field* $K \subset F \subset L$ *with* $[F : K] = 2$*.*

**Problem 4.** *Let* $f(x) \in \mathbb{Q}[x]$ *be irreducible of prime degree* $p$ *and assume that* $f$ *has exactly two non-real roots (in* $\mathbb{C}$*). Prove that* $\mathrm{Gal}(SF_{\mathbb{Q}}(f)/\mathbb{Q}) = S_p$ *(the symmetric group on* $p$ *letters).*

**Problem 5.** *Let* $f(x) = x^4 + ax^2 + b$ *be an irreducible quartic polynomial in* $\mathbb{Q}[x]$*, with roots* $\pm\alpha, \pm\beta$ *and let* $L = SF_{\mathbb{Q}}(f)$*.*

*(i) Prove that* $\mathrm{Gal}(L/\mathbb{Q})$ *is isomorphic to a subgroup of* $D_8$*.*

*(ii) Prove that this subgroup is isomorphic to* $C_4$ *if and only if* $\alpha/\beta - \beta/\alpha \in \mathbb{Q}$*.*