Worksheet #2

Algebra V - Galois theory

September 18, 2025

Problem 1. *Let L/K be a finite field extension of degree two. Prove that the following statements are true.*

(i) If char(k) $\neq 2$, then $L = K(x)$ for some $x \in L$ such that $x^2 \in K$.

(ii) if char(k) $= 2$, then either $L = K(x)$ with $x^2 \in K$, or $L = K(x)$ with $x^2 + x \in K$.

Problem 2. *Let L/K be a finite field extension whose degree is a prime number. Prove that there are no intermediate fields between K and L.*

Problem 3. *Let $K(\alpha)/K$ be a finite field extension of odd degree.*

(i) Prove that $K(\alpha) = K(\alpha^2)$.

(ii) Give an example to show that the previous statement (i.e., (i)) can be false if $[K(\alpha) : K]$ is even.

Problem 4. *Let L/K be an algebraic field extension, denote the group of $K-$automorphisms of L by $Aut(L/K)$ and assume that for every $\alpha \in L \backslash K$ we can find $\varphi \in Aut(L/K)$ such that $\varphi(\alpha) \neq \alpha$. Pick $\beta \in L$ and let $f \in K[x]$ be the corresponding minimal polynomial. Prove that the group $Aut(L/K)$ acts transitively on the set of roots of f.*

Problem 5. *Let $K = \mathbb{F}_2 = \{0, 1\}$ and $L = K(\alpha)$, where α is a root of $1 + x + x^2 \in K[x]$. Prove that $\varphi : L \to L$ given by $a + b\alpha \mapsto a + b + b\alpha$ is an element of $Aut(L/K)$, where $a, b \in K$.*

Problem 6. *Prove that the set $\mathbb{A}$ of complex numbers that are algebraic over $\mathbb{Q}$ form a subfield of $\mathbb{C}$ that is algebraically closed. This gives an example of an algebraic field extension which is not finite.*

Problem 7. *Let L/K be a field extension such that one can pick $x \neq y \in L$ transcendental over K. Prove that x is algebraic over $K(y)$ if and only if y is algebraic over $K(x)$.*