Worksheet #1

Algebra V - Galois theory

September 11, 2025

You will need the following definitions to solve the exercises in this worksheet.

Definition 1. *Let $Q \neq R$ be two distinct points "drawn" in an infinite piece of paper, which we think of as the plane $\mathbb{R}^2$. A point $P \in \mathbb{R}^2$ is* ***constructible from*** *Q* ***and*** *R if $P \in \{Q, R\}$ or else P lies at the intersection of a pair of lines, of a line and a circle, or of a pair of circles that you can draw using only a straightedge and a compass following the two rules below:*

Rule 1 *you can draw the line (using the straightedge) through any two points that are already drawn,*

Rule 2 *you can open the compass to span the distance $r := d(P, \tilde{P})$ between any two points P and $\tilde{P}$ that you have already drawn, place the base at a third already drawn point, say O, and draw the circle centered at O with radius r.*

Definition 2. *Identify the field of complex numbers $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. A complex number $z = x+iy$ is* ***constructible*** *if and only if the point $P = (x, y)$ is constructible from $Q = (0, 0) \simeq 0_{\mathbb{C}}$ and $R = (1, 0) \simeq 1_{\mathbb{C}}$ as in Definition 1.*

Problem 1. *Use induction to prove that every integer is constructible.*

Problem 2. *Use the two claims below to prove that the set F of all constructible complex numbers forms a field such that $\mathbb{Q}(i) \subset F \subset \mathbb{C}$.*

Claim 1 *If L is a line and P is a point that have already been drawn, then we can draw the unique line containing P that is parallel to L.*

Claim 2 *If α and β are constructible, then so are $\alpha \cdot \beta$ and α/β (if $\beta \neq 0$).*

Problem 3. *Prove that the field F (from the previous problem) is closed under taking square roots.*

Problem 4 (a bit harder)**.** *Prove that you cannot construct the real number $\sqrt[3]{2}$ by proving that if $\alpha \in \mathbb{R}$ is constructible, then α is algebraic over $\mathbb{Q}$ and the degree of the field extension $\mathbb{Q} \subset \mathbb{Q}(\alpha)$ is a power of two*[1]*.*

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[1] Recall that this means that α is a root of a non-zero polynomial with coefficients in $\mathbb{Q}$ and the minimal such polynomial has degree a power of two.