

Solutions to Worksheet #1

Algebra V - Galois theory

Fall 2025

Solution 1. Here is the basic idea. We first observe that 0 and 1 are given and that we can easily construct -1 as follows. Let $L(0, 1)$ be the line through 0 and 1, and let $C(0; 1)$ be the circle centered at zero with radius 1. Then -1 can be obtained as a point in the intersection $C(0; 1) \cap L(0, 1)$. Now, once you have constructed n and $-n$, $n \in \mathbb{N}$, we can construct $n + 1$ (resp. $-(n + 1)$) as a point in the intersection of the line $L(0, 1)$ and the circle $C(n; 1)$ (resp. $C(-n; 1)$) centered at n (resp. $-n$) with radius 1.

Solution 2. We first show that F is a field. Since we know how to construct the integers (problem 1) and we assume Claim 2, it suffices to show that if α and β are constructible, then $\alpha + \beta$ is also constructible. To do this, we can argue as follows. If $\alpha \cdot \beta = 0$, there is nothing to do. Otherwise, we can draw the line $L = L(0; \beta)$ through 0 and β , and the parallel line through α , say \tilde{L} , and then draw the circle $C = C(\alpha; |\beta|)$ centered at α with radius $|\beta|$. By construction, $\alpha \pm \beta \in \tilde{L} \cap C$. Note that we also already know $\mathbb{Q} \subset F$. So, to complete the exercise, we need to show that we can construct numbers of the form $a/b + c/di$, where $a, b, c, d \in \mathbb{Z}$. But this can be constructed as a point in the intersection of the line $x = a/b$ and the circle centered at a/b with radius $|c/d|$.

Solution 3. First, note that we know how to construct $\sqrt{2} = |1 + i|$ as the (positive) intersection of $L(0, 1)$ (the real axis = x -axis) and the circle $C(0; |1 + i|)$. Now, let $\alpha \in F \cap \mathbb{R}_{>0}$ ($0 = \sqrt{0}$ is given). Then $\sqrt{1 + \alpha^2} = |\alpha + i|$ and $1 + \alpha$ are also in F . Thus, the intersection points of the line $x = \sqrt{1 + \alpha^2}$ and the circle centered at 0 with radius $1 + \alpha$ are also in F . Since these points have the form

$$\sqrt{1 + \alpha^2} + i\beta,$$

where $\beta^2 = 2\alpha$, we conclude that $\sqrt{2\alpha}$ is in F . But then $\sqrt{\alpha}$ is in F .

To extend this to all $\alpha \in F$, it is handy to use polar coordinates and write $\alpha = (r, \theta)$. Then, to show you can construct $\pm\sqrt{\alpha} = \pm(\sqrt{r}, \theta/2)$, it suffices to show you can bisect the angle θ .

Solution 4. Note that if we can prove the claim, then we conclude $\alpha = \sqrt[3]{2}$ is not constructible because $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$ (the minimal polynomial of α is $x^3 - 2$). The claim is proved, for instance, in Milne's notes (Theorem 1.37 + Corollary 1.38).