SOLUTIONS TO WORKSHEET #12

In this worksheet, if K is a field and $f \in K[x]$ is an irreducible and separable polynomial over K, we will call the group $\mathrm{Gal}(\mathrm{SF}_K(f)/K)$ the Galois group of the polynomial f.

Problem 1. — *For each of the following polynomials, compute its Galois group.*

(i) $f(x) = x^3 + (2t+3)x - 1 \in \mathbf{Q}(t)[x]$.
(ii) $f(x) = x^4 - 7x^2 + 1 \in \mathbf{Q}[x]$.
(iii) $f(x) = x^4 - 5x + 2 \in \mathbf{Q}[x]$.
(iv) $f(x) = x^5 - x - 1 \in \mathbf{Q}[x]$.
(v) $f(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 \in \mathbf{Q}[x]$.

Solution. For (i), observe that its discriminant is

$$\Delta(f) = -(4(2t+3)^3 + 27) = -(32t^3 + 144t^2 + 216t + 135)$$

which, upon reduction modulo 7, may be seen to be an irreducible polynomial in t. In particular, this is not a square in $\mathbf{Q}(t)$. Since $f(x)$ is irreducible—as can be seen by specializing $t = 0$ and then reducing the resulting polynomial $x^3 + 3x - 1 \in \mathbf{Q}[x]$ modulo 2—this implies that the Galois group of $f(x)$ is S_3.

For (ii), observe that there is a factorization $f(x) = x^4 - 7x^2 + 1 = (x^2 - 3x + 1)(x^2 + 3x + 1)$ and so the roots of $f(x)$ may be easily determined with the quadratic formula:

$$\alpha_1 = \frac{1}{2}(3+\sqrt{5}),\ \ \alpha_2 = \frac{1}{2}(3-\sqrt{5}),\ \ \alpha_3 = \frac{1}{2}(-3+\sqrt{5}),\ \ \alpha_4 = \frac{1}{2}(-3-\sqrt{5}).$$

Thus the roots of the associated resolvent cubic are

$$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4 = 2,\ \ \beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4 = -2,\ \ \beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3 = -7.$$

Thus the resolvent is totally split over $\mathbf{Q}$, implying that the Galois group of $f(x)$ is the Klein group V.

For (iii), we use the methods from last week to determine the Galois group—see the solution of the next problem for more. In short, for the quartic $f(x) = x^4 - 5x + 2$, its resolvent cubic is $r(x) = x^3 - 8x - 25$ and its discriminant is $\Delta = -14827$. Reduction modulo 5 shows that both $f(x)$ and $r(x)$ are irreducible. Since $\Delta < 0$, it certainly is not a square in $\mathbf{Q}$, so this implies that the Galois group of $f(x)$ is S_4.

For (iv), the discriminant of $f(x) = x^5 - x - 1$ is

$$\Delta = 2^8 \cdot (-1)^5 + 5^5 \cdot (-1)^4 = 2869$$

which is not a square. Since $f(x)$ is irreducible—as can be seen reduction modulo 2—and 5 is prime, this implies that the Galois group is S_5.

For (v), we pull a rabbit out of a hat and observe that

$$x^{14} - 1 = (x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)(x-1)(x+1).$$

Since $x^7 - 1 = (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)(x-1)$, this means that $f(x) = \Phi_{14}(x)$ is the 14-th cyclotomic polynomial. Therefore the splitting field of $f(x)$ is that obtained by adjoining a primitive 14-th root of unity, so its Galois group is $(\mathbf{Z}/14\mathbf{Z})^\times$. ■

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Problem 2. — *In each of the following cases, explain why G is the Galois group of the given polynomial $f(x) = x^4 + ax + b \in \mathbf{Q}[x]$.*

(i) $G = S_4$ *and* $(a, b) = (1, 1)$.

(ii) $G = A_4$ *and* $(a, b) = (8, 12)$.

(iii) $G = D_4$ *and* $(a, b) = (3, 3)$.

(iv) $G = C_4$ *and* $(a, b) = (5, 5)$.

(v) $G = V$ *and* $(a, b) = (0, 1)$.

Solution. As thoroughly discussed on the previous worksheet, the resolvent cubic and the discriminant associated with the quartic polynomial $f(x) = x^4 + ax + b$ are given by

$$r(x) = x^3 - 4bx - a^2 \text{ and } \Delta := \Delta(f) = \Delta(r) = 256b^3 - 27a^4.$$

The Galois group of $f(x)$ for each individual case may now be determined via the arithmetic properties of $f(x)$, $r(x)$, and Δ:

For $f(x) = x^4 + x + 1$ as in (i), so that $r(x) = x^3 - 4x - 1$ and $\Delta = 229$. Considering $r(x)$ modulo 3 shows that it is irreducible. Since 229 is not a square, this implies that $G = S_4$.

For $f(x) = x^4 + 8x + 12$ as in (ii), so that $r(x) = x^3 - 48x - 64$ and $\Delta = 331776$. Considering $r(x)$ modulo 5 shows that it is irreducible. Since $331776 = 576^2$, this implies that $G = A_4$.

For $f(x) = x^4 + 3x + 3$ as in (iii), so that $r(x) = x^3 - 12x - 9$ and $\Delta = 4725$. Reducing $f(x)$ modulo 2 shows it is irreducible. Since $r(x) = (x+3)(x^2 - 3x - 3)$, this implies that $G = D_4$.

For $f(x) = x^4 + 5x + 5$ as in (iv), so that $r(x) = x^3 - 20x - 25$ and $\Delta = 15125$. Again, reducing $f(x)$ modulo 2 shows that it is irreducible. Since $r(x) = (x-5)(x^2 + 5x + 5)$, actually, $G = D_4$ again.

For $f(x) = x^4 + 1$ as in (v), so that $r(x) = x^3 - 4x$ and $\Delta = 256$. Reducing $f(x)$ modulo 3 shows that it is irreducible, whereas $r(x) = x(x+2)(x-2)$. This implies that $G = V$. ■