

## Lecture # 13 (11/12/2025)

Today: Infinite Galois Extensions

(Reference: J.S. Milne. Fields and Galois Theory)

Recall: A field extension  $L/K$  is

- normal if it is a splitting field of a collection of polynomials in  $K[x]$
- separable if it is algebraic and  $\min_K \alpha$  has no repeated roots  $\forall \alpha \in L$
- Galois if it is both normal and separable

Notation: If  $L/K$  is Galois, we write

$$\text{Gal}(L/K) = \text{Aut}_K(L)$$

! When  $L/K$  is not finite, not all subgroups of  $\text{Gal}(L/K)$  correspond to subextensions of  $L/K$ . We want to consider a topology on  $\text{Gal}(L/K)$  that captures those that do.

Here, is a Key observation:

**Lemma** A field extension  $L/K$  is Galois iff it is a union of finite Galois extensions

So, we want to consider all finite Galois subextensions in some sense.

## A digression on inverse limits

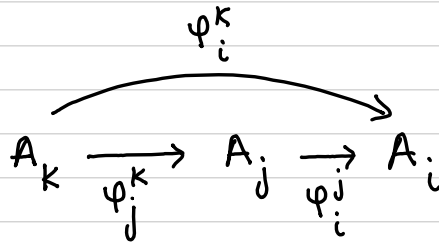
### Definitions

① A **directed set** is a partially ordered set  $(I, \leq)$  such that for all  $i, j \in I$  we can find  $K \in I$  s.t.  $i \leq K$  and  $j \leq K$ .

$$\begin{cases} i \leq i \\ i \leq j \text{ \& } j \leq i \Rightarrow i = j \\ i \leq j \text{ \& } j \leq k \Rightarrow i \leq k \end{cases}$$

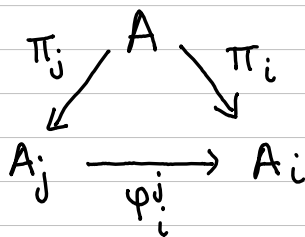
② Given a category  $\mathcal{C}$  and a directed set  $(I, \leq)$  an **inverse system** is a collection of objects  $(A_i)_{i \in I}$  together with

morphisms  $(\varphi_i^j: A_j \rightarrow A_i)_{i \leq j}$  s.t.  
 $\varphi_i^i = \text{id}_{A_i} \forall i$  and  $\varphi_i^j \circ \varphi_j^k = \varphi_i^k \forall i \leq j \leq k$ .

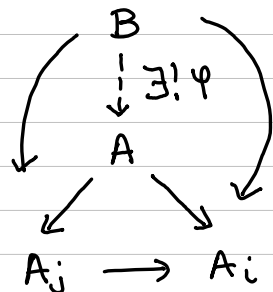


③ Given an inverse system as in ②, an inverse limit is an object  $A$  (in  $\mathcal{C}$ )

plus morphisms  $\pi_i: A \rightarrow A_i$  satisfying  
 $\varphi_i^j \circ \pi_j = \pi_i \forall i \leq j$



+ universal property.



**FACT** In the category of groups inverse limits always exist.

Given an inverse system of groups

$$(G_i)_{i \in I} + (\varphi_i^j : G_j \rightarrow G_i)_{i \leq j}$$

$$\varprojlim G_i := \left\{ (g_i) \in \prod G_i ; \varphi_i^j(g_j) = g_i \forall i \leq j \right\}$$

this is always a subgroup of  $\prod G_i$

**Definition** Groups which are inverse limits of finite groups are called **profinite**.

One can prove that every profinite group is the Galois group of some Galois extension.

**FACT** If we have an inverse system in the category of groups, the inverse limit is also an inverse limit in the category of topological groups if

- $\prod G_i$  has the product topology &
- $\varprojlim G_i$  has the subspace topology.

- ! Every finite group is a topological group with the discrete topology. So, every profinite group has a "natural" topology.

In fact, we obtain Hausdorff, compact and totally disconnected spaces.

↳ only subspaces which are connected are the singletons.

## Back to Galois extensions

Given  $L/K$  Galois, we can consider

- $I =$  set of intermediate fields  $K < F < L$  s.t.  $F/K$  is finite Galois.
- $\leq$  = inclusion (given  $K < F < L$  and  $K < F' < L$ ,  $F, F' < (FF')$ ) → composite
- For each  $K < F < L$ , and each  $F < F'$ , we have  $\varphi_F^{F'} : \text{Gal}(F'/K) \rightarrow \text{Gal}(F/K)$   
 $\sigma \mapsto \sigma|_F$

## Example 1

- Let  $K = \mathbb{Q}$  and for each  $n \in \mathbb{N}$  let  $F_n = \text{SF}_{\mathbb{Q}}(x^{n-1})$ .

Consider  $L = F_1 F_2 F_3 \dots$

- $L/\mathbb{Q}$  is Galois (char  $\mathbb{Q} = 0$  & composite of normal extensions is normal)

- Note that  $F_m \subset F_n \Leftrightarrow m|n$  and we

have  $\text{Gal}(F_n/\mathbb{Q}) \rightarrow \text{Gal}(F_m/\mathbb{Q})$

$$(\mathbb{Z}/n\mathbb{Z})^{\times} \rightarrow (\mathbb{Z}/m\mathbb{Z})^{\times}$$

$$a \mapsto a \pmod{m}$$

- $\text{Gal}(L/\mathbb{Q}) \cong \varprojlim (\mathbb{Z}/n\mathbb{Z})^{\times} = \left\{ (a_n) \in \prod_{n=1}^{\infty} (\mathbb{Z}/n\mathbb{Z})^{\times} ; \right.$

$$a_n \equiv a_m \pmod{m}$$

$$\left. \forall m|n \right\}$$

## Example 2 (Rings & fields)

$K$  a field,  $K^{\text{sep}} = \{ \alpha \in \bar{K} \mid \alpha \text{ separable over } K \}$

- $K^{\text{sep}}/K$  is Galois and  $\bar{K}/K^{\text{sep}}$  is purely inseparable

\* if  $\text{char } K = 0$ , then  $\bar{K} = K^{\text{sep}}$

- If  $K = \mathbb{F}_p$ , then we have (for  $\forall m | n$ )

$$\begin{array}{ccc} \text{Gal}(\mathbb{F}_{p^n} / \mathbb{F}_p) & \longrightarrow & \text{Gal}(\mathbb{F}_{p^m} / \mathbb{F}_p) \\ \cong & & \cong \\ \mathbb{Z}/n\mathbb{Z} & & \mathbb{Z}/m\mathbb{Z} \end{array}$$

$$a \mapsto a \pmod{m}$$

$$\begin{array}{c} \text{Gal}(\mathbb{F}_p^{\text{sep}} / \mathbb{F}_p) = \left\{ (a_n) \in \prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}; a_n \equiv a_m \pmod{m} \right. \\ \left. \forall m | n \right\} \\ \parallel \\ \hat{\mathbb{Z}} \end{array}$$

**Proposition** If  $L/K$  is a Galois extension, then

$$\text{Gal}(L/K) = \text{Aut}_K(L) \xrightarrow{\cong} \varprojlim \text{Gal}(F/K) \subset \prod_{\substack{F/K \\ \text{finite} \\ \text{Galois}}} \text{Gal}(F/K)$$

$$\sigma \mapsto (\sigma|_F)$$

is a group isomorphism.

**Definition** If  $L/K$  is Galois, then the Krull topology on  $\text{Gal}(L/K)$  is the unique topology on  $\text{Gal}(L/K)$  making the above group isomorphism a homeomorphism.

\* This topology has a basis consisting of all  $\text{Gal}(L/F)$  such that  $F/K$  is finite and normal.

\* Note that if  $G = \varprojlim \text{Gal}(F/K)$ , then

$\text{Ker}(\pi_F : G \rightarrow \text{Gal}(F/K))$  are open normal subgroups of finite index

$\rightarrow$  we get a collection of opens around  $\downarrow G$ .

**Proposition** Let  $L/K$  be a Galois extension.

(i) If  $K \subset F \subset L$  is an intermediate field, then  $\text{Gal}(L/F)$  is closed in  $\text{Gal}(L/K)$ , and  $L^{\text{Gal}(L/F)} = F$

(ii) If  $H \leq \text{Gal}(L/F)$  is a subgroup, then  $\text{Gal}(L/L^H)$  is the closure of  $H$  in  $\text{Gal}(L/F)$ .

**idea of (i)**

$$\text{Gal}(L/F) = \bigcap_{\substack{S \subset F \\ \text{finite}}} \underbrace{\{ \sigma \in \text{Gal}(L/K); \sigma s = s \ \forall s \in S \}}_{G(S)}$$

- Each  $G(S)$  is open subgroup (of finite index).
- On a topological group, every open subgroup is also closed.

**The fundamental theorem**

Let  $L/K$  be Galois with Galois group  $G$ . The

maps  $H \mapsto L^H$ ,  $F \mapsto \text{Gal}(L/F)$

define

inverse bijections between

$$\{ \text{closed subgroups } H \leq G \} \xleftrightarrow{1:1} \{ \text{intermediate } K \subset F \subset L \}$$

Moreover,

(i) the corresp. is order reversing

(ii)  $H \leq G$  closed is open  $\Leftrightarrow [L^H : K] < \infty$ ,

in which case  $[L^H : K] = [G : H]$

(iii)  $\sigma H \sigma^{-1} \leftrightarrow \sigma F$

(iv)  $H \leq G$  closed is normal  $\Leftrightarrow L^H/K$  is

Galois, in which case  $\text{Gal}(L^H/K) \cong G/H$

Some last remarks

Ⓐ Not all subgroups of  $\text{Gal}(L/K)$  are closed.

(Krull) Any infinite Galois extension includes non-closed subgroups.

③ In the Krull topology,

open + normal  $\Rightarrow$  finite index

but  $\exists$  normal of finite index which are not open

$\leadsto$  Krull topology  $\subset$  profinite topology