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Problem Set 8 Solutions

Exercise 1. (a) Let G be a finite group, and V_1 and V_k two complex representations, $\dim V_1 = 1$, $\dim V_k = k$. Use characters to show that $V_k \otimes V_1$ is irreducible if and only if V_k is.

(b) Let V be an irreducible complex representation of G of dimension $k > 1$, and suppose that it is the only irreducible representation of G of dimension k . Show that if there is a 1-dimensional complex representation ρ_1 of G and an element $g \in G$ such that $\rho_1(g) \neq 1$, then $\chi_V(g) = 0$. This property is useful in computation of character tables.

Solution 1. (a) Let χ_k be the character of V_k , and χ_1 - the character of V_1 . Then $\chi_k \chi_1$ is the character of $V_k \otimes V_1$. Let us compute the inner product

$$(\chi_k \chi_1, \chi_k \chi_1) = \frac{1}{|G|} \sum_g \chi_k(g) \chi_1(g) \overline{\chi_k(g) \chi_1(g)}.$$

Since V_1 is one-dimensional, $\chi_1(g) = \rho_1(g)$ and is a root of unity. Therefore, $\chi_1(g) \overline{\chi_1(g)} = |\chi_1(g)|^2 = 1$ for all $g \in G$. Then we have

$$(\chi_k \chi_1, \chi_k \chi_1) = \frac{1}{|G|} \sum_g \chi_k(g) \overline{\chi_k(g)} = (\chi_k, \chi_k).$$

Therefore, $(\chi_k \chi_1, \chi_k \chi_1) = 1$ if and only if $(\chi_k, \chi_k) = 1$. Equivalently, $V_k \otimes V_1$ is irreducible if and only if V_k is.

(b) By (a) we know that $V \otimes V_1$ is irreducible, and since V is the only irreducible representation of the given dimension, we should have $V \otimes V_1 \simeq V$, and $\chi_V(g) \chi_1(g) = \chi_V(g)$ for all $g \in G$. If $\chi_1(g) \neq 1$ for some $g \in G$, this implies $\chi_V(g) = 0$.

Exercise 2. This exercise shows how to compute the symmetric and exterior powers of linear maps given by explicit matrices.

(a) Let V be a 2-dimensional vector space. Let $f : V \rightarrow V$ be given by the matrix

$$f = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Find the matrix of $S^2(f) : S^2(V) \rightarrow S^2(V)$, where $S^2(V)$ is the second symmetric power of V .

(b) Let U be a 3-dimensional vector space. Let $g : U \rightarrow U$ be given the matrix

$$g = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}.$$

Find the matrix of $\wedge^2(g) : \wedge^2(V) \rightarrow \wedge^2(V)$, where $\wedge^2(V)$ is the second exterior power of V .

Solution 2. (a) Choose a basis $\{a, b\}$ of V such that $f(a) = pa + rb$ and $f(b) = qa + sb$. Then $\{aa, ab, bb\}$ understood as commuting variables form a basis of S^2V and we have

$$\begin{aligned} S^2 f(aa) &= f(a)f(a) = (pa + rb)(pa + rb) = p^2 \cdot aa + 2pr \cdot ab + r^2 \cdot bb \\ S^2 f(ab) &= f(a)f(b) = (pa + rb)(qa + sb) = pq \cdot aa + (ps + rq) \cdot ab + rs \cdot bb \\ S^2 f(bb) &= f(b)f(b) = (qa + sb)(qa + sb) = q^2 \cdot aa + 2qs \cdot ab + s^2 \cdot bb. \end{aligned}$$

It follows that the matrix of $S^2 f$ is

$$\begin{pmatrix} p^2 & pq & q^2 \\ 2pr & ps + rq & 2qs \\ r^2 & rs & s^2 \end{pmatrix}.$$

(b) Choose a basis $\{a, b, c\}$ of V such that

$$\begin{aligned}g(a) &= ra + ub + xc, \\g(b) &= sa + vb + yc, \\g(c) &= ta + wb + zc.\end{aligned}$$

Then $\{a \wedge b, a \wedge c, b \wedge c\}$ is a basis of $\wedge^2 V$ and we have

$$\begin{aligned}\wedge^2 g(a \wedge b) &= g(a) \wedge g(b) = (ra + ub + xc) \wedge (sa + vb + yc) \\&= (rv - us) \cdot a \wedge b + (ry - xs) \cdot a \wedge c + (uy - xv) \cdot b \wedge c, \\ \wedge^2 g(a \wedge c) &= g(a) \wedge g(c) = (ra + ub + xc) \wedge (ta + wb + zc) \\&= (rw - ut) \cdot a \wedge b + (rz - xt) \cdot a \wedge c + (uz - xw) \cdot b \wedge c, \\ \wedge^2 g(b \wedge c) &= g(b) \wedge g(c) = (sa + vb + yc) \wedge (ta + wb + zc) \\&= (sw - vt) \cdot a \wedge b + (sz - yt) \cdot a \wedge c + (vz - yw) \cdot b \wedge c.\end{aligned}$$

It follows that the matrix of $\wedge^2 g$ is

$$\begin{pmatrix}rv - us & rw - ut & sw - vt \\ry - xs & rz - xt & sz - yt \\uy - xv & uz - xw & vz - yw\end{pmatrix}.$$

Exercise 3. Let $V \simeq \mathbb{C}^n$ and $A : V \rightarrow V$ be a linear map with eigenvalues $\{\lambda_i\}_{i=1}^n$. Consider the linear maps $S^2(A) : S^2(V) \rightarrow S^2(V)$ and $\wedge^2 A : \wedge^2 V \rightarrow \wedge^2 V$. This exercise expresses the trace of a symmetric and exterior square of a linear map in terms of traces in V .

- (a) Express the trace $\text{tr}(S^2(A))$ in terms of $\text{tr}(A)$ and $\text{tr}(A^2)$.
(b) Express the trace $\text{tr}(\wedge^2(A))$ in terms of $\text{tr}(A)$ and $\text{tr}(A^2)$.
(c) Let V be a representation of a finite group G , $\dim(V) \geq 2$ and let $g \in G$. Use (a) and (b) to express the characters of the representations S^2V and \wedge^2V in terms of $\chi_V(g)$ and $\chi_V(g^2)$.

Solution 3. Let $\{v_1, \dots, v_n\}$ be a basis of V such that v_i is a generalized eigenvector associated to λ_i for all $i = 1, \dots, n$ and A is in Jordan normal form with respect to this basis, i.e. $Av_i = \lambda_i v_i + \mu_i v_{i-1}$ with $\mu_i \in \{0, 1\}$.

- (a) The vectors $\{v_i v_j \mid 1 \leq i \leq j \leq n\}$ form a basis of S^2V and we have $S^2A(v_i v_j) = (Av_i)(Av_j) = \lambda_i \lambda_j \cdot v_i v_j + \mu_i \lambda_j \cdot v_{i-1} v_j + \lambda_i \mu_j \cdot v_i v_{j-1} + \mu_i \mu_j \cdot v_{i-1} v_{j-1}$, so $\{\lambda_i \lambda_j \mid 1 \leq i \leq j \leq n\}$ is the (multi)set of eigenvalues of S^2A . As the trace of S^2A is the sum of all eigenvalues of S^2A , we obtain

$$\text{tr}(S^2A) = \sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j = \frac{1}{2} \cdot \left(\sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} \lambda_i \lambda_j + \sum_{1 \leq i \leq n} \lambda_i^2 \right) = \frac{1}{2} \cdot (\text{tr}(A)^2 + \text{tr}(A^2)),$$

where we also use the fact that the eigenvalues of A^2 are $\{\lambda_i^2 \mid i = 1, \dots, n\}$.

- (b) The vectors $\{v_i \wedge v_j \mid 1 \leq i < j \leq n\}$ form a basis of $\wedge^2 V$ and we have $\wedge^2 A(v_i \wedge v_j) = (Av_i) \wedge (Av_j) = \lambda_i \lambda_j \cdot v_i \wedge v_j + \mu_i \lambda_j \cdot v_{i-1} \wedge v_j + \lambda_i \mu_j \cdot v_i \wedge v_{j-1} + \mu_i \mu_j \cdot v_{i-1} \wedge v_{j-1}$, so $\{\lambda_i \lambda_j \mid 1 \leq i < j \leq n\}$ is the (multi)set of eigenvalues of $\wedge^2 A$. As the trace of $\wedge^2 A$ is the sum of all eigenvalues of $\wedge^2 A$, we obtain

$$\text{tr}(\wedge^2 A) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = \frac{1}{2} \cdot \left(\sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i \leq n} \lambda_i^2 \right) = \frac{1}{2} \cdot (\text{tr}(A)^2 - \text{tr}(A^2)),$$

as in part (a).

- (c) From (a) and (b) we have

$$\begin{aligned}\chi_{S^2V}(g) &= \frac{1}{2} ((\chi_V(g))^2 + \chi_V(g^2)), \\ \chi_{\wedge^2V}(g) &= \frac{1}{2} ((\chi_V(g))^2 - \chi_V(g^2)).\end{aligned}$$

In particular, we have $\chi_V(g^2) = \chi_{S^2V}(g) - \chi_{\wedge^2V}(g)$.

Exercise 4. Let V be an n -dimensional complex vector space. Then $GL(V)$ acts in the space $\wedge^m(V)$ by $g \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_m) = gv_1 \wedge gv_2 \wedge \dots \wedge gv_m$, where $m \leq n$. Show that $\wedge^m(V)$ is an irreducible representation of $GL(V)$, $m \leq n$. *Hint:* Let $\{v_i\}_{i=1}^n$ be a basis in V . Find an element $H \in GL(V)$ such that $\wedge^m H$ is a diagonal operator with all distinct eigenvalues in $\wedge^m(V)$. Then any subrepresentation $W \subset \wedge^m(V)$ should contain a subset of eigenvectors of H . Use an element $P \in GL(V)$ that permutes the basis $\{v_i\}$ of V to conclude that $W = \wedge^m(V)$.

Solution 4. Let H be the diagonal matrix in the basis $\{v_i\}_{i=1}^n$ with the diagonal elements given by distinct prime numbers p_1, \dots, p_n . A basis in $\wedge^m(V)$ is given by the set $\{v_{i_1} \wedge \dots \wedge v_{i_m}\}$ where $1 \leq i_1 < i_2 < \dots < i_m \leq n$. The operator $\wedge^m H$ is diagonal in this basis with the eigenvalues given by the products of m distinct primes:

$$\wedge^m H(v_{i_1} \wedge \dots \wedge v_{i_m}) = (p_{i_1} p_{i_2} \dots p_{i_m}) v_{i_1} \wedge \dots \wedge v_{i_m}.$$

Since p_i are all distinct primes, these eigenvalues are all distinct. Suppose $W \subset \wedge^m(V)$ is a subrepresentation of the group $GL(V)$. Then $HW \subset W$, which means that W is spanned by a subset of the eigenvectors of H . Suppose that $v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_m} \in W$, and consider another element of the basis of $\wedge^m(V)$, say $v_{j_1} \wedge v_{j_2} \wedge \dots \wedge v_{j_m}$. Let σ_m be the permutation that sends $v_{i_k} \rightarrow v_{j_k}$ for all $k = 1, \dots, m \leq n$. Extend it in an arbitrary way to a permutation $\sigma \in S_n$. Viewed as a permutation of the basis $\{v_i\}_{i=1}^n$ in V , σ is a linear operator given by an invertible matrix $P_\sigma \in GL(V)$. Since W is invariant under $P \in GL(V)$, we have

$$P(v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_m}) = v_{j_1} \wedge v_{j_2} \wedge \dots \wedge v_{j_m} \in W.$$

Therefore, all elements of the basis of $\wedge^m(V)$ are in W , and thus $\wedge^m(V)$ is an irreducible representation of $GL(V)$.

Remark: In fact $GL(V)$ also acts in the space $S^k(V)$ by $g \cdot (u_1 u_2 \dots u_k) = (gu_1)(gu_2) \dots (gu_k)$ and $S^k(V)$ is an irreducible representation of $GL(V)$ for any k . The proof of this fact uses a similar method but requires a more careful construction of operators in $GL(V)$ in order to obtain all elements of the standard basis starting from one of them. The case of $\wedge^m(V)$ is easier because every element of the basis of V can occur at most once as a factor in a basis element of $\wedge^m(V)$.