November 11, 2025

Problem Set 8

Exercise 1. (a) Let G be a finite group, and V_1 and V_k two complex representations, $\mathrm{dim V_1 = 1}$, $\mathrm{dim V_k = k}$. Use characters to show that $V_k \otimes V_1$ is irreducible if and only if V_k is.

(b) Let V be an irreducible complex representation of G of dimension $k > 1$, and suppose that it is the only irreducible representation of G of dimension k. Show that if there is a 1-dimensional complex representation ρ_1 of G and an element $g \in G$ such that $\rho_1(g) \neq 1$, then $\chi_V(g) = 0$. This property is useful in computation of character tables.

Exercise 2. This exercise shows how to compute the symmetric and exterior powers of linear maps given by explicit matrices.

(a) Let V be a 2-dimensional vector space. Let $f : V \to V$ be given by the matrix

$$f = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Find the matrix of $S^2(f) : S^2(V) \to S^2(V)$, where $S^2(V)$ is the second symmetric power of V.

(b) Let U be a 3-dimensional vector space. Let $g : U \to U$ be given the matrix

$$g = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}.$$

Find the matrix of $\wedge^2(g) : \wedge^2(V) \to \wedge^2(V)$, where $\wedge^2(V)$ is the second exterior power of V.

Exercise 3. Let $V \simeq \mathbb{C}^n$ and $A : V \to V$ be a linear map with eigenvalues $\{\lambda_i\}_{i=1}^n$. Consider the linear maps $S^2(A) : S^2(V) \to S^2(V)$ and $\wedge^2 A : \wedge^2 V \to \wedge^2 V$. This exercise expresses the trace of a symmetric and exterior square of a linear map in terms of traces in V.

(a) Express the trace $\mathrm{tr}(S^2(A))$ in terms of $\mathrm{tr}(A)$ and $\mathrm{tr}(A^2)$.

(b) Express the trace $\mathrm{tr}(\wedge^2(A))$ in terms of $\mathrm{tr}(A)$ and $\mathrm{tr}(A^2)$.

(c) Let V be a representation of a finite group G, $\dim(V) \geq 2$ and let $g \in G$. Use (a) and (b) to express the characters of the representations S^2V and $\wedge^2 V$ in terms of $\chi_V(g)$ and $\chi_V(g^2)$.

Exercise 4. Let V be an n-dimensional complex vector space. Then the group of invertible linear transformations $GL(V)$ acts in the space $\wedge^m(V)$ by $g \cdot (v_1 \wedge v_2 \wedge \ldots \wedge v_m) = gv_1 \wedge gv_2 \wedge \ldots gv_m$, where $m \leq n$.

Show that $\wedge^m(V)$ is an irreducible representation of $GL(V)$, $m \leq n$. *Hint:* Let $\{v_i\}_{i=1}^n$ be a basis in V. Find an element $H \in GL(V)$ such that $\wedge^m H$ is a diagonal operator with all distinct eigenvalues in $\wedge^m(V)$. Then any subrepresentation $W \subset \wedge^m(V)$ should contain a subset of eigenvectors of H. Use an element $P \in GL(V)$ that permutes the basis $\{v_i\}$ of V to conclude that $W = \wedge^m(V)$.