November 4, 2025

Problem Set 7

Exercise 1. Let D_{2n} be the dihedral group, the group of symmetries of a regular n-gon. This group has $2n$ elements.

(a) Describe all irreducible complex representations of D_n. Start with the 1-dimensional representations, then consider representations coming from the symmetries of a regular n-gon, and use the sum of squares formula to complete the classification. Consider cases of odd and even n.

(b) Use the character table to find the decompositions of the tensor products $V_i \otimes V_j$ into a direct sum of irreducible representations. (It is enough to consider the case when $\mathrm{dimV_i} > 1, \mathrm{dimV_j} > 1$).

Exercise 2. (a) Suppose $H \subset G$ is a normal subgroup of a finite group, and $\rho : G/H \to \mathrm{Aut(V)}$ is a representation of G/H. Let $\phi : G \to G/H$ be the natural surjective homomorphism. Check that $\tilde{\rho} = \rho \circ \phi$ defines a representation of G in V. If ρ is irreducible, show that $\tilde{\rho}$ is irreducible as well. Show that inequivalent representations of G/H lift to inequivalent representations of G.

(b) Let Q_8 denote the group of quaternions, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with the defining relations

$$i = jk = -kj, \quad j = ki = -ik, \quad k = ij = -ji, \quad -1 = i^2 = j^2 = k^2.$$

Find the center $Z(Q_8)$, and describe the structure of $Q_8/Z(Q_8)$. Use (a) to lift the irreducible representations of $Q_8/Z(Q_8)$ to Q_8.

(c) Use the structure theorem of semisimple finite dimensional algebras to find the dimensions of the remaining irreducible representations of Q_8. Use the orthogonality relations to determine their characters.

(d) Use characters to decompose the tensor products of the irreducible representations of Q_8 of dimension > 1 into a direct sum.

Exercise 3. Let G be a group of invertible upper triangular 2×2 matrices with coefficients in $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$.

(a) Find the conjugacy classes of G.

(b) Find a normal subgroup $H \subset G$ such that G/H is abelian.

(c) Use (a), (b) and the "sum of squares" formula to find the dimensions of the irreducible complex representations of G.

(d) Use the orthogonality relations to compute the table of characters of G.