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Problem Set 6 Solutions

Exercise 1. Recall that the character of a finite dimensional representation V of an algebra A over a field k is defined as $\chi_V(a) = \mathrm{Tr}_V \rho(a)$. Show that if V is a finite dimensional representation of A, and $W \subset V$ a subrepresentation, then the character $\chi_V = \chi_W + \chi_{V/W}$.

Solution 1. Let $\{w_1, ..., w_m\}$ be a basis of W and complete it to a basis $B = \{w_1, ..., w_m, u_1, ..., u_n\}$ of V. Similar to exercise 3 in problem set 1, the matrix of $\rho(a)$ in B is in the following block form:

$$\left(\begin{array}{c|c} M & * \\ \hline 0 & N \end{array}\right)$$

where M is the matrix representing $\rho(a)|_W = \rho_W(a)$. Moreover, the matrix N coincides with the matrix of $\rho_{V/W}(a)$ in the basis $B' = \{u_1 + W, ..., u_n + W\}$ of V/W. Indeed, let $u \in \mathrm{Span}(u_1, ..., u_n)$ and write $u + W \in V/W$ as a vector $x \in k^n$ in the basis B'. We have

$$\left(\begin{array}{c|c} M & * \\ \hline 0 & N \end{array}\right)\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} * \\ Nx \end{pmatrix}.$$

This shows that $\rho_{V/W}(a)(u + W)$ is represented by Nx in the basis B'. Since the trace of an operator in $\mathrm{End}(V)$ is independent of the choice of basis, we have

$$\chi_V(a) = \mathrm{Tr}(M) + \mathrm{Tr}(N) = \chi_W(a) + \chi_{V/W}(a)$$

for all $a \in A$.

Exercise 2. Consider the group algebra $A = \mathbb{C}[S_3]$ of the group of permutations of 3 elements.

(a) Show that $A \simeq \mathbb{C}[D_3]$, where $D_3 = \{s, r : s^2 = 1, r^3 = 1, srs = r^{-1}\}$ is the dihedral group of order 6.

(b) Classify the one-dimensional irreducible representations of A up to equivalence.

(c) Classify the two-dimensional irreducible representations of A up to equivalence.

(d) Use the obtained classifications and the theorem on the structure of finite dimensional algebras to show that A is a semisimple algebra (without use of Maschke's theorem).

Solution 2. (a) It is easy to check that the linear map $\phi : A \to \mathbb{C}[D_3]$ such that $\phi(s_1) = s, \phi(s_2) = sr$ is an algebra isomorphism.

(b) In dimension 1 we have, $\rho(s)^2 = 1$, therefore $\rho(s) = \pm 1$. Also, $\rho(r)^3 = 1$, so $\rho(r)$ is a third root of unity. However, $\rho(srs) = \rho(r^{-1}) = \rho(r)^{-1}$, therefore $\rho(r) = 1$. Then we have just two inequivalent representations: the trivial one ρ_{11}, and $\rho_{12}(s) = -1$, $\rho_{12}(r) = 1$.

(c) Because of the isomorphism with the dihedral group, we have a representation by symmetries of a regular triangle, where r acts by rotation by $2\pi/3$ and s by reflection with respect to an axis passing through the origin. It is clearly irreducible, because no linear combination of standard basis vectors is stable under both transformations. Now, by Density theorem, A surjects onto $\oplus_i \mathrm{End(V_i)}$, where V_i are the inequivalent irreducible representations. The dimension of A is 6, and the dimension of $\oplus_i \mathrm{End(V_i)}$ for the representations that we already found, is $1 + 1 + 4$. Therefore, this is the only two-dimensional irreducible representation of A up to equivalence.

(d) We have $A/\mathrm{Rad(A)} \simeq \oplus_{\mathrm{i}} \mathrm{End(V_i)}$. As $\mathrm{dim(A)} = \mathrm{dim}(\oplus_{\mathrm{i}} \mathrm{End(V_i)})$, the radical is zero, and A is semisimple.

Exercise 3. Consider the group $D_4 = \langle r, s \mid r^4 = 1, s^2 = 1, srs = r^{-1}\rangle$. Recall the irreducible complex representations of D_4 that we have constructed earlier in class (see Section 4.1 of RT-1.pdf). Use the structure theorem for finite dimensional semisimple algebras to show that this is a complete list of irreducible representations of D_4. Find the conjugacy classes of D_4 and compute the characters $\chi_V(g)$ for each irreducible representation V and each conjugacy class of D_4.

Solution 3. See Example 11.7 in RT-1.pdf.

Exercise 4. (a) Let V_1 and V_2 be two-dimensional complex vector spaces with bases $\{x_1, x_2\}$ and $\{y_1, y_2\}$ respectively. Let $A : V_1 \to V_1$ be the linear map given in the basis $\{x_1, x_2\}$ by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $B : V_2 \to V_2$ the linear map given in the basis $\{y_1, y_2\}$ by the matrix

$$B = \begin{pmatrix} s & t \\ u & v \end{pmatrix}.$$

The linear map $A \otimes B$ is defined as follows: $(A \otimes B)(v_1 \otimes v_2) = A(v_1) \otimes B(v_2)$. Compute the matrix $A \otimes B$ in the basis $\{x_1 \otimes y_1, x_1 \otimes y_2, x_2 \otimes y_1, x_2 \otimes y_2\}$.

(b) Apply the above to find the matrices of the representation $\rho \otimes \rho$ of the group $D_4 = \langle s, r \mid s^2 = 1, r^4 = 1, srs = r^{-1}\rangle$, where ρ is the unique irreducible 2-dimensional representation:

$$\rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(r) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Derive the decomposition of $\rho \otimes \rho$ into a direct sum of irreducible components.

Solution 4. (a) We will write $(t, p)^T$ for a vector with components t and p in the given basis in V_1 or V_2. We have

$$(A \otimes B)(x_1 \otimes y_1) = (A \otimes B)((1,0)^T \otimes (1,0)^T) = A(1,0)^T \otimes B(1,0)^T = (a,c)^T \otimes (s,u)^T =$$

$$= (ax_1 + cx_2) \otimes (sy_1 + uy_2) = as(x_1 \otimes y_1) + au(x_1 \otimes y_2) + cs(x_2 \otimes y_1) + cu(x_2 \otimes y_2) = (as, au, cs, cu)^T.$$

This is the expression of $(A \otimes B)(x_1 \otimes y_1)$ in the basis $\{x_1 \otimes y_1, x_1 \otimes y_2, x_2 \otimes y_1, x_2 \otimes y_2\}$. Similarly,

$$(A \otimes B)(x_1 \otimes y_2) = (A \otimes B)((1,0)^T \otimes (0,1)^T) = A(1,0)^T \otimes B(0,1)^T = (a,c)^T \otimes (t,v)^T = (at, av, ct, cv)^T.$$

$$(A \otimes B)(x_2 \otimes y_1) = (A \otimes B)((0,1)^T \otimes (1,0)^T) = A(0,1)^T \otimes B(1,0)^T = (b,d)^T \otimes (s,u)^T = (bs, bu, ds, du)^T.$$

$$(A \otimes B)(x_1 \otimes y_2) = (A \otimes B)((0,1)^T \otimes (0,1)^T) = A(0,1)^T \otimes B(0,1)^T = (b,d)^T \otimes (t,v)^T = (bt, bv, dt, dv)^T.$$

So finally the matrix of $A \otimes B$ in the basis $\{x_1 \otimes y_1, x_1 \otimes y_2, x_2 \otimes y_1, x_2 \otimes y_2\}$ is given by

$$A \otimes B = \begin{pmatrix} as & at & bs & bt \\ au & av & bu & bv \\ cs & ct & ds & dt \\ cu & cv & du & dv \end{pmatrix} = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix}.$$

The same computation generalizes to higher dimensional vector spaces.

(b) Using the given matrices of the two-dimensional irreducible representation $\rho : V \to V$ of D_4 and (a), we easily compute

$$\rho^{\otimes 2}(s) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho^{\otimes 2}(r) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic equation for both matrices is $\lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2 = 0$. So the only eigenvalues of both matrices are ± 1, which means that the two-dimensional irreducible representation where r acts by rotation by $\pi/2$ does not occur in $\rho^{\otimes 2}$. The matrices are easily diagonalizable and we obtain that $V \otimes V \simeq V_0 \oplus V_1 \oplus V_2 \oplus V_3$.

Remark. Of course this decomposition can be easily obtained by computing the character. You can use the character table for D_4, Example 11.7 in the course notes, RT-1.pdf available on Moodle. In particular, $\chi_\rho(1) = 2, \chi_\rho(r^2) = -2$, other values of ρ are zeros. Then $\chi_{\rho^{\otimes 2}}(1) = \chi_{\rho^{\otimes 2}}(r^2) = 4$, other values are 0. The character table gives the unique decomposition

$$\chi_{\rho^{\otimes 2}} = \chi_0 + \chi_1 + \chi_2 + \chi_3.$$

Exercise 5. Let A, B be finite dimensional algebras. Then $A \otimes B$ is also an algebra, with the multiplication given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

(a) Show that $\mathrm{Mat_n}(\mathbb{K}) \otimes \mathrm{Mat_m}(\mathbb{K}) \simeq \mathrm{Mat_{nm}}(\mathbb{K})$ as associative algebras.

(b) Let V and W be irreducible finite dimensional representations of A and B, respectively. Show that $V \otimes W$ with the action $\rho(a \otimes b)(v \otimes w) = \rho(a)v \otimes \rho(b)w$, is a finite dimensional irreducible representation of $A \otimes B$. *Hint:* To show irreducibility, use the density theorem and (a).

Solution 5. (a) Direct computation. Let E^n_{ij} denote a square $n \times n$ matrix with the only nonzero entry, equal to 1, at the position (i, j), and zeros everywhere else. Then $\phi : \mathrm{Mat_n}(\mathbb{K}) \otimes \mathrm{Mat_m}(\mathbb{K}) \to \mathrm{Mat_{nm}}(\mathbb{K})$, $\phi(E^n_{ij} \otimes E^m_{lk}) = E^{nm}_{mi+l,mj+k}$ respects matrix multiplication:

$$\phi(E^n_{ij} \otimes E^m_{lk}) \cdot \phi(E^n_{st} \otimes E^m_{pq}) = E^{nm}_{mi+l,mj+k} \cdot E^{nm}_{ms+p,mt+q} =$$

$$= \delta_{js}\delta_{kp} E^{nm}_{mi+l,mt+q} = \phi(\delta_{js}\delta_{kp}(E^n_{it} \otimes E^m_{lq})) = \phi((E^n_{ij} \otimes E^m_{lk}) \cdot (E^n_{st} \otimes E^m_{pq})).$$

Extending by bilinearity to $\mathrm{Mat_n}(\mathbb{K}) \otimes \mathrm{Mat_m}(\mathbb{K})$ and noticing that $\{E^n_{ij}\}$ form a basis in $\mathrm{Mat_n}(\mathbb{K})$, completes the proof.

(b) The map $\rho(a \otimes b)(v \otimes w) = \rho(a)v \otimes \rho(b)w$ indeed defines a representation of $A \otimes B$: $\rho((a_1 \otimes b_1) \cdot (a_2 \otimes b_2))(v \otimes w) = \rho(a_1 a_2)v \otimes \rho(b_1 b_2)w = \rho(a_1)\rho(a_2)v \otimes \rho(b_1)\rho(b_2)w = \rho(a_1 \otimes b_1)\rho(a_2 \otimes b_2)(v \otimes w)$. SInce V and W are irreducible, by density theorem, the algebra A surjects onto End(V) and the algebra B surjects onto End(W), so $A \otimes B$ surjects onto End(V) $\otimes$ End(W). This space is isomorphic to End(V $\otimes$ W) by (a). Thus, $V \otimes W$ is an irreducible representation of $A \otimes B$.