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## Problem Set 6

**Exercise 1.** Recall that the character of a finite dimensional representation $V$ of an algebra $A$ over a field $k$ is defined as $\chi_V(a) = \mathrm{Tr}_V \rho(a)$. Show that if $V$ is a finite dimensional representation of $A$, and $W \subset V$ a subrepresentation, then the character $\chi_V = \chi_W + \chi_{V/W}$.

**Exercise 2.** Consider the group algebra $A = \mathbb{C}[S_3]$ of the group of permutations of 3 elements.

(a) Show that $A \simeq \mathbb{C}[D_3]$, where $D_3 = \{s, r : s^2 = 1, r^3 = 1, srs = r^{-1}\}$ is the dihedral group of order 6.

(b) Classify the one-dimensional irreducible representations of $A$ up to equivalence.

(c) Classify the two-dimensional irreducible representations of $A$ up to equivalence.

(d) Use the obtained classifications and the theorem on the structure of finite dimensional algebras to show that $A$ is a semisimple algebra (without use of Maschke's theorem).

**Exercise 3.** Consider the group $D_4 = \langle r, s \mid r^4 = 1, s^2 = 1, srs = r^{-1}\rangle$. Recall the irreducible complex representations of $D_4$ that we have constructed earlier in class (see Section 4.1 of RT-1.pdf). Use the structure theorem for finite dimensional semisimple algebras to show that this is a complete list of irreducible representations of $D_4$. Find the conjugacy classes of $D_4$ and compute the characters $\chi_V(g)$ for each irreducible representation $V$ and each conjugacy class of $D_4$.

**Exercise 4.** (a) Let $V_1$ and $V_2$ be two-dimensional complex vector spaces with bases $\{x_1, x_2\}$ and $\{y_1, y_2\}$ respectively. Let $A : V_1 \to V_1$ be the linear map given in the basis $\{x_1, x_2\}$ by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $B : V_2 \to V_2$ the linear map given in the basis $\{y_1, y_2\}$ by the matrix

$$B = \begin{pmatrix} s & t \\ u & v \end{pmatrix}.$$

The linear map $A \otimes B$ is defined as follows: $(A \otimes B)(v_1 \otimes v_2) = A(v_1) \otimes B(v_2)$. Compute the matrix $A \otimes B$ in the basis $\{x_1 \otimes y_1, x_1 \otimes y_2, x_2 \otimes y_1, x_2 \otimes y_2\}$.

(b) Apply the above to find the matrices of the representation $\rho \otimes \rho$ of the group $D_4 = \langle s, r \mid s^2 = 1, r^4 = 1, srs = r^{-1}\rangle$, where $\rho$ is the unique irreducible 2-dimensional representation:

$$\rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(r) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Derive the decomposition of $\rho \otimes \rho$ into a direct sum of irreducible components.

**Exercise 5.** Let $A$, $B$ be finite dimensional algebras. Then $A \otimes B$ is also an algebra, with the multiplication given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

(a) Show that $\mathrm{Mat_n}(\mathbb{K}) \otimes \mathrm{Mat_m}(\mathbb{K}) \simeq \mathrm{Mat_{nm}}(\mathbb{K})$ as associative algebras.

(b) Let $V$ and $W$ be irreducible finite dimensional representations of $A$ and $B$, respectively. Show that $V \otimes W$ with the action $\rho(a \otimes b)(v \otimes w) = \rho(a)v \otimes \rho(b)w$, is a finite dimensional irreducible representation of $A \otimes B$. *Hint:* To show irreducibility, use the density theorem and (a).