

September 30, 2025

## Problem Set 3

**Exercise 1.** Let  $\rho : G \rightarrow GL(1, \mathbb{C}) = \mathbb{C}^*$  be a representation of a finite group  $G$  over  $\mathbb{C}$ . Show that  $\|\rho(g)\| = 1$ ,  $\forall g \in G$ , where  $\|\cdot\|$  is the usual norm on  $\mathbb{C}$ .

**Exercise 2.** Let  $G$  be a finite group acting by permutations on the elements of a basis of a complex vector space  $V$ , thus defining a representation of  $G$  in  $V$ . Show that if  $\dim V > 1$ , then the representation is not irreducible.

**Exercise 3.** Let  $G$  be a finite group and let  $\rho : G \rightarrow GL(2, \mathbb{C})$  be a 2-dimensional representation of  $G$  over  $\mathbb{C}$ . Suppose that there are two elements  $g, h$  of  $G$  such that  $\rho(g)$  and  $\rho(h)$  do not commute. Prove that  $\rho$  is irreducible.

**Exercise 4.** Let  $\rho : S_3 \rightarrow GL(3, \mathbb{C})$  be the natural representation where the symmetric group  $S_3$  acts by permutations on an orthonormal basis in  $\mathbb{C}^3$ .

- (a) Explicitly find the elements of  $\rho(S_3)$ .
- (b) Decompose  $\rho$  as a direct sum of irreducible representations.
- (c) Is  $\rho$  completely reducible, if we replace  $\mathbb{C}$  with a finite field of two or three elements?

**Exercise 5.** Let  $G = \langle a \rangle$  be a cyclic group of prime order  $p$ . Define  $\rho : G \rightarrow GL(2, \mathbb{F}_p)$  by

$$\rho(a^r) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \forall 0 \leq r \leq p-1.$$

- (a) Show that  $\rho$  is a representation of  $G$  over  $\mathbb{F}_p$ .
- (b) Show that  $\rho$  is not irreducible.
- (c) Show that  $\rho$  cannot be decomposed as a direct sum of irreducible representations.

**Exercise 6.** Let  $G = (\mathbb{Z}, +)$ , an infinite cyclic group. Define the  $\mathbb{C}$ -representation  $\rho : G \rightarrow GL(2, \mathbb{C})$  by

$$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \forall n \in \mathbb{Z}.$$

Show that  $\rho$  is not completely reducible. (Maschke's Theorem fails for infinite groups).