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Problem Set 1 Solutions

Exercise 1. In class we discussed representations of associative algebras. There is a similar notion of a representation of a group. Namely, if G is a group, then a representation ρ of G over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space V together with a group homomorphism

$$\rho : G \to \mathrm{GL}(V),$$

where $\mathrm{GL}(V)$ is the group of all invertible linear transformations of the vector space V.

Show that the non-isomorphic representations of a finite group G over a field $\mathbb{K}$ are in one-to-one correspondence with the non-isomorphic representations of the algebra $\mathbb{K}[G]$.

Solution 1. Take a representation $\rho : G \to \mathrm{GL}(V)$, then for each $a = \sum_g a_g g$, where $a_g \in \mathbb{K}$, the map $\rho' : \mathbb{K}[G] \to \mathrm{End}(V)$, $\rho'(a) = \sum_g a_g \rho(g)$ is a representation of $\mathbb{K}[G]$. Indeed, this is a linear map sending $1 \cdot e$ to id and for $a = \sum_g a_g g$ and $b = \sum_h b_h g$, we have that $\rho'(ab) = \rho'(\sum_{g,h} a_g b_h gh) = \sum_{g,h} a_g b_h \rho(gh) = \sum_{g,h} a_g b_h (\rho(g) \circ \rho(h)) = (\sum_g a_g \rho(g)) \circ (\sum_h b_h \rho(g)) = \rho'(a) \circ \rho'(b)$.

Conversely, for any $\rho' : \mathbb{K}[G] \to \mathrm{End}(V)$, set $\rho(g) = \rho'(g)$. We claim that $\rho(g)$ is an automorphism of V, i.e. an isomorphism. This is clear, as $\rho(g) \circ \rho(g^{-1}) = \rho(g^{-1}) \circ \rho(g) = \rho'(e) = \mathrm{id}$. Hence, this defines a representation $\rho : G \to \mathrm{GL}(V)$.

We also need to check that $\rho_1 \simeq \rho_2$ if and only if $\rho'_1 \simeq \rho'_2$. If φ is an isomorphism of representations between ρ_1 and ρ_2, it is also an isomorphism of representations between ρ'_1 and ρ'_2, as $\varphi(\rho'_1(a)v) = \varphi(\sum_g a_g \rho_1(g)v) = \sum_g a_g \varphi(\rho_1(g)v) = \sum_g a_g \rho_2(g)\varphi(v) = \rho'_2(\sum_g a_g g)\varphi(v) = \rho'_2(a)\varphi(v)$. Conversely, if φ is an isomorphism of representations between ρ'_1 and ρ'_2, it is also an isomorphism of representations between ρ_1 and ρ_2, proving the equivalence between $\rho_1 \simeq \rho_2$ and $\rho'_1 \simeq \rho'_2$.

Exercise 2. Let (V, ρ) be a finite dimensional representation of an associative algebra A. Show that V has an irreducible subrepresentation.

Solution 2. If V is irreducible, we are done. If not, by definition there exists a nonzero subrepresentation $W_1 \subset V$ such that $W_1 \neq V$, so $\dim W_1 < \dim \mathrm{V}$. Then if W is irreducible, we are done, if not we can find $W_2 \subset W_1$ such that $\dim W_2 < \dim W_1$, and so on. The process terminates in an irreducible representation, because the dimensions are always decreasing and any 1-dimensional representation of A is irreducible by definition: the only subspaces in a 1-dimensional space is 0 and itself.

Note that this argument does not work in V is infinite dimensional.

Exercise 3. (a) Let G be a group, V a vector space and $\rho : G \to GL(V)$ be a representation of G, and W be a subrepresentation of V. Show that W is a representation of G, and that there is a basis B of V such that for all $g \in G$, the matrix of $\rho(g)$ in B has the following block form:

$$\left(\begin{array}{c|c} M & * \\ \hline 0 & * \end{array}\right),$$

where M is a matrix representing $\rho(g)|_W$.

(b) Let $\rho : G \to GL(V)$ be a representation of G, and W be a subrepresentation of V. Show that V/W carries a natural structure of a representation of G.

Solution 3. (a) The map $\rho_W : G \to GL(W), g \mapsto \rho(g)$ is well-defined since W is a subrepresentation of V, and is a group homomorphism since ρ is a representation of G. Therefore, W is a representation of G. Let $\iota : W \to V$ be the inclusion map of the subspace W into V. Then for al $g \in G$, we have $\rho_W(g)(\iota(w)) = \rho_W(g)(w) = \rho(g)(w) = \iota(\rho(g)(w))$, which shows that ι is a morphism of representations. It suffices to take a basis of W and complete it to a basis of V.

(b) It suffices to check that the map given by

$$\begin{array}{rccl} \rho_{V/W}: & G & \longrightarrow & GL(V/W) \\ & g & \longmapsto & \left(\begin{array}{ccc} V/W & \to & V/W \\ v+W & \mapsto & \rho(g)(v)+W \end{array}\right) \end{array}$$

is a group homomorphism.

Exercise 4. Let $\rho : G \to GL(V)$ be a representation of G, and set, for all $g \in G$,

$$\rho^*(g) = \rho(g^{-1})^{\mathrm{T}},$$

that is, $\rho^*(g)$ is the transpose of the linear map $\rho(g^{-1})$. Show that ρ^* defines a representation $G \to GL(V^*)$ of G. This is called the *dual representation*.

Solution 4. For all $g, h \in G$, we have

$$\rho^*(gh) = \rho((gh)^{-1})^{\mathrm{T}} = \rho(h^{-1}g^{-1})^{\mathrm{T}} = (\rho(h^{-1})\rho(g^{-1}))^{\mathrm{T}} = \rho(g^{-1})^{\mathrm{T}}\rho(h^{-1})^{\mathrm{T}} = \rho^*(g)\rho^*(h),$$

Also $\rho^*(e) = \rho(e^{-1})^{\mathrm{T}} = \mathrm{Id}^{\mathrm{T}} = \mathrm{Id}$, so that $\rho^* : G \to GL(V^*)$ is a group homomorphism.

Exercise 5. Consider the groups D_3 and H_3 given by generators and relations as follows:

$$D_3 = \langle r, s : r^3 = 1, s^2 = 1, srs = r^{-1}\rangle.$$

$$H_3 = \langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^3 = 1\rangle.$$

(a) Show that the two groups are isomorphic (give an explicit isomorphism)

(b) Consider the group algebra $\mathbb{C}[D_3] \simeq \mathbb{C}[H_3]$. Construct two inequivalent representations of this algebra of dimension 1 over $\mathbb{C}$ and show that there are no other inequivalent 1-dimensional representations.

(c) Consider the following maps: $\rho_1 : \mathbb{C}[D_3] \to \mathrm{End}(\mathbb{C}^2)$,

$$\rho_1(r) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} \qquad \rho_1(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\rho_2 : \mathbb{C}[H_3] \to \mathrm{End}(\mathbb{C}^2)$:

$$\rho_2(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho_2(s_2) = \begin{pmatrix} 0 & e^{-2\pi \mathrm{i}/3} \\ e^{2\pi \mathrm{i}/3} & 0 \end{pmatrix}$$

Check that ρ_1 and ρ_2 define irreducible representations of the respective algebras.

(d) Using the isomorphism of algebras $\mathbb{C}[D_3] \simeq \mathbb{C}[H_3]$, show that the representations ρ_1 and ρ_2 defined in (c) are isomorphic.

Solution 5. (a) In order to define a homomorphism from a group (ar algebra) defined by generators and relations to another group (or algebra), it is sufficient to define the images of the generators and verify that these satisfy the same relations. Here, we can define a homomorphism $\varphi\colon D_3 \to H_3$ with $s \mapsto s_1$ and $r \mapsto s_1 s_2$ since

$$(s_1 s_2)^3 = 1, \qquad s_1^2 = 1 \qquad \text{and} \qquad s_1(s_1 s_2)s_1 = s_2 s_1 = (s_1 s_2)^{-1},$$

matching the relations in D_3. Similarly, we can define a homomorphism $\psi\colon H_3 \to D_3$ with $s_1 \mapsto s$ and $s_2 \mapsto sr$ since

$$s^2 = 1, \qquad (sr)^2 = srsr = r^{-1}r = 1 \qquad \text{and} \qquad (ssr)^3 = r^3 = 1,$$

matching the relations in H_3. Now we have

$$\psi \circ \varphi(s) = \psi(s_1) = s \qquad \text{and} \qquad \psi \circ \varphi(r) = \psi(s_1 s_2) = ssr = r,$$

so $\psi \circ \varphi$ is the identity on D_3 (as s and r generate D_3). Analogously, we see that $\varphi \circ \psi$ is the identity on H_3, so φ and ψ are isomorphisms and $D_3 \cong H_3$.

(b) The elements of the group $D_3 \cong H_3$ form a basis of the group algebra $\mathbb{C}[D_3] \cong \mathbb{C}[H_3]$, so we only need to specify their images in $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. We define group homomorphisms $\lambda_1 \colon H_3 \to \mathbb{C}^\times$ by $g \mapsto 1$ for all $g \in D_3$ and $\lambda_2 \colon H_3 \to \mathbb{C}^\times$ with $s_1 \mapsto -1$ and $s_2 \mapsto -1$. Note that λ_2 is well defined since

$$(-1)^2 = 1 = (-1)^2 = 1 \qquad \text{and} \qquad \big((-1)\cdot(-1)\big)^3 = 1.$$

Now extending λ_1 and λ_2 to linear maps $\mathbb{C}[H_3] \to \mathbb{C} \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C})$ gives rise to algebra homomorphisms $\hat{\lambda}_1$ and $\hat{\lambda}_2$, respectively, (by definition of the multiplicative structure on $\mathbb{C}[H_3]$) and therefore to one-dimensional representations of $\mathbb{C}[H_3]$. As $\mathrm{End}_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}$ is a commutative algebra, any two equivalent one-dimensional representations must be equal, and as $\hat{\lambda}_1 \neq \hat{\lambda}_2$, we conclude that $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are non-equivalent.

Now let $\hat{\lambda}$ be an arbitrary one-dimensional representation of $\mathbb{C}[H_3]$. Then

$$\hat{\lambda}(s_1)^2 = \hat{\lambda}(s_1^2) = 1 \qquad \text{and} \qquad \hat{\lambda}(s_2)^2 = \hat{\lambda}(s_2^2) = 1$$

and therefore $\hat{\lambda}(s_1), \hat{\lambda}(s_2) \in \{\pm 1\}$. If $\hat{\lambda}(s_1) \neq \hat{\lambda}(s_2)$ then $\hat{\lambda}(s_1 s_2) = \hat{\lambda}(s_1)\cdot\hat{\lambda}(s_2) = -1$ and

$$1 = \hat{\lambda}\big((s_1 s_2)^3\big) = \hat{\lambda}(s_1 s_2)^3 = -1,$$

a contradiction. We conclude that $\hat{\lambda}(s_1) = \hat{\lambda}(s_2) \in \{\pm 1\}$ and therefore $\hat{\lambda} = \hat{\lambda}_1$ or $\hat{\lambda} = \hat{\lambda}_2$.

(c) In order to show that ρ_1 and ρ_2 define representations, it suffices to prove that the images of the generators in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$ satisfy the relations defining the groups. For ρ_1, it is straightforward to see that

$$\begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}^3 = 1, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = 1$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}^{-1},$$

matching the relations of D_3. Now denote by $\zeta = e^{\frac{2\pi}{3}\cdot \mathrm{i}}$ a primitive third root of unity. For ρ_2, we obtain

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & \zeta^{-1} \\ \zeta & 0 \end{pmatrix}^2 = 1 \qquad \text{and} \qquad \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \zeta^{-1} \\ \zeta & 0 \end{pmatrix}\right)^3 = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}^3 = 1$$

matching the relations in H_3.

If either of the two-dimensional representations ρ_1 and ρ_2 is not irreducible then it must admit a one-dimensional subrepresentation, so there exists $0 \neq v \in \mathbb{C}^2$ such that $\rho_1(a)\cdot v \in \mathbb{C}\cdot v$ for all $a \in \mathbb{C}[D_3]$ (or similarly for ρ_2 and H_3). Then v is an eigenvector of $\rho_1(a)$ for all $a \in \mathbb{C}[D_3]$. Now it is straightforward to see that the eigenspaces of the matrix $\rho_1(s)$ (for the eigenvalues 1 and -1, respectively) are spanned by the standard basis vectors e_1 and e_2. However, neither of e_1 and e_2 is an eigenvector of $\rho_1(r)$ and it follows that ρ_1 is irreducible. Analogously, we find that the eigenspaces of $\rho_2(s_1)$ are spanned by $e_1 + e_2$ and $e_1 - e_2$, but neither of these vectors is an eigenvector of the matrix $\rho_2(s_2)$, so ρ_2 is irreducible.

(d) In order to show that ρ_1 and ρ_2 define equivalent representations of $\mathbb{C}[D_3] \cong \mathbb{C}[H_3]$, we need to find $T \in \mathrm{GL}_2(\mathbb{C})$ such that

$$T \cdot \rho_1\big(\psi(g)\big) = \rho_2(g)\cdot T$$

for all $g \in H_3$, where $\psi \colon H_3 \to D_3$ is the isomorphism from part (a). With $T = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}$, we have

$$T \cdot \rho_1\big(\psi(s_1)\big) = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix} = \rho_2(s_1)\cdot T$$

and (writing $a = \cos(2\pi/3)$ and $b = \sin(2\pi/3)$ so that $\zeta = a + \mathrm{i}b$)

$$\begin{aligned} T \cdot \rho_1\big(\psi(s_2)\big) = T\cdot\rho_1(sr) = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} &= \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} a & -b \\ -b & -a \end{pmatrix} = \begin{pmatrix} a - \mathrm{i}b & -b - \mathrm{i}a \\ a + \mathrm{i}b & -b + \mathrm{i}a \end{pmatrix} \\ &= \begin{pmatrix} \zeta^{-1} & -\mathrm{i}\zeta^{-1} \\ \zeta & \mathrm{i}\zeta \end{pmatrix} = \begin{pmatrix} 0 & \zeta^{-1} \\ \zeta & 0 \end{pmatrix}\begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix} = \rho_2(s_2)\cdot T \end{aligned}$$

and as s_1 and s_2 generate H_3, we conclude that ρ_1 and ρ_2 are equivalent.

Exercise 6. Consider the $\mathbb{C}$-algebra $U(sl_2)$ generated over $\mathbb{C}$ by $\{e, f, h\}$ with the relations

$$he - eh = 2e, \qquad hf - fh = -2f, \qquad ef - fe = h.$$

(a) Show that the assignment

$$\rho(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \qquad \rho(e) = x\frac{\partial}{\partial y}, \qquad \rho(f) = y\frac{\partial}{\partial x}$$

defines an irreducible representation of $U(sl_2)$ on the vector space $\mathbb{C}_2[x, y]$ of homogeneous polynomials of degree 2.

(b) Consider the 3-dimensional vector space V_3 with basis $\{e, f, h\}$, and define a map $\vartheta : U(sl_2) \to \mathrm{End}(V_3)$ by

$$\vartheta(e)(t) = et - te, \quad \vartheta(f)(t) = ft - tf, \quad \vartheta(h)(t) = ht - th$$

for any $t \in V_3$. Show that ϑ defines a representation of $U(sl_2)$ in V_3 and that this representation is isomorphic to the representation $(\rho, \mathbb{C}_2[x, y])$ constructed in (a).

Solution 6. (a) It is sufficient to verify that the endomorphisms $\rho(h)$, $\rho(e)$ and $\rho(f)$ satisfy the relations of $U(sl_2)$. By the product rule and symmetry of second derivatives, we have

$$x\frac{\partial}{\partial y} \circ y\frac{\partial}{\partial x} - y\frac{\partial}{\partial x} \circ x\frac{\partial}{\partial y} = \left(x\frac{\partial}{\partial x} + xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right) - \left(y\frac{\partial}{\partial y} + xy\frac{\partial}{\partial y}\frac{\partial}{\partial x}\right) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},$$

so $\rho(e) \circ \rho(f) - \rho(f) \circ \rho(e) = \rho(h)$. Analogously, we compute that

$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) \circ x\frac{\partial}{\partial y} - x\frac{\partial}{\partial y} \circ \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$$
$$= \left(x\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial x}\frac{\partial}{\partial y} - xy\frac{\partial}{\partial y}\frac{\partial}{\partial y}\right) - \left(x^2\frac{\partial}{\partial y}\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - xy\frac{\partial}{\partial y}\frac{\partial}{\partial y}\right) = 2 \cdot x\frac{\partial}{\partial y},$$

so $\rho(h) \circ \rho(e) - \rho(e) \circ \rho(h) = 2 \cdot \rho(e)$, and the third relation $\rho(h) \circ \rho(f) - \rho(f) \circ \rho(h) = -2 \cdot \rho(f)$ can be checked similarly.

To show that V_3 is irreducible, let $v = ax^2 + bxy + cy^2 \in V_3$ be an arbitrary vector. Then action by $\rho(e)$, if needed twice, maps v to the vector λx^2 with $\lambda \in \mathbb{C}^*$. Further acting by $\rho(f)$ on λx^2 we can obtain the whole space V_3, since $\{x^2, xy, y^2\}$ form a basis in V_3. Therefore the representation V_3 does not have nontrivial subrepresentations, and is irreducible.

(b) Again, we need to verify that the endomorphisms $\vartheta(h)$, $\vartheta(e)$ and $\vartheta(f)$ satisfy the relations defining $U(sl_2)$. For $t \in V_3$, we have

$$\big(\vartheta(h) \circ \vartheta(e)\big)(t) - \big(\vartheta(h) \circ \vartheta(e)\big)(t) = \vartheta(h)\big(et - te\big) - \vartheta(e)(ht - th) = het - hte - eth + teh - eht + eth + hte - the$$
$$= (he - eh) \cdot t - t \cdot (he - eh) = 2et - 2te = 2 \cdot \vartheta(e)(t),$$

so $\vartheta(h) \circ \vartheta(e) - \vartheta(e) \circ \vartheta(h) = 2 \cdot \vartheta(e)$. The two remaining relations can be checked analogously. Now consider the linear isomorphism $\varphi\colon V_3 \to \mathbb{C}[x, y]_2$ with $\varphi(e) = -x^2/2$, $\varphi(f) = y^2/2$ and $\varphi(h) = xy$. We claim that φ is an isomorphism of representations of $U(sl_2)$. To that end, we need to prove that $\varphi\big(\vartheta(e)(t)\big) = \rho(e)\big(\varphi(t)\big)$ for all $t \in V_3$ and similarly for f and h. By linearity, it suffices to check this equality for $t \in \{e, f, h\}$. We have

$$\varphi\big(\vartheta(e)(e)\big) = \varphi(e^2 - e^2) = 0 = x\frac{\partial}{\partial y}(-x^2/2) = \rho(e)\big(\varphi(e)\big),$$

$$\varphi\big(\vartheta(e)(f)\big) = \varphi(h) = xy = x\frac{\partial}{\partial y}(y^2/2) = \rho(e)\big(\varphi(f)\big)$$

and

$$\varphi\big(\vartheta(e)(h)\big) = \varphi(-2e) = -x^2 = x\frac{\partial}{\partial y}(xy) = \rho(e)\big(\varphi(h)\big).$$

The computations for f and h are similar and the claim follows.