September 16, 2025

Problem Set 1

Exercise 1. In class we discussed representations of associative algebras. There is a similar notion of a representation of a group. Namely, if G is a group, then a representation ρ of G over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space V together with a group homomorphism

$$\rho : G \to \mathrm{GL}(V),$$

where $\mathrm{GL}(V)$ is the group of all invertible linear transformations of the vector space V.

Show that the non-isomorphic representations of a finite group G over a field $\mathbb{K}$ are in one-to-one correspondence with the non-isomorphic representations of the algebra $\mathbb{K}[G]$.

Exercise 2. Let (V, ρ) be a finite dimensional representation of an associative algebra A. Show that V has an irreducible subrepresentation.

Exercise 3. (a) Let G be a group, V a vector space and $\rho : G \to GL(V)$ be a representation of G, and W be a subrepresentation of V. Show that W is a representation of G, and that there is a basis B of V such that for all $g \in G$, the matrix of $\rho(g)$ in B has the following block form:

$$\left(\begin{array}{c|c} M & * \\ \hline 0 & * \end{array}\right),$$

where M is a matrix representing $\rho(g)|_W$.

(b) Let $\rho : G \to GL(V)$ be a representation of G, and W be a subrepresentation of V. Show that the quotient space V/W carries a natural structure of a representation of G.

Exercise 4. Let $\rho : G \to GL(V)$ be a representation of G, and set, for all $g \in G$,

$$\rho^*(g) = \rho(g^{-1})^{\mathrm{T}},$$

that is, $\rho^*(g)$ is the transpose of the linear map $\rho(g^{-1})$. Show that ρ^* defines a representation $G \to GL(V^*)$ of G. This is called the *dual representation*.

Exercise 5. Consider the groups D_3 and H_3 given by generators and relations as follows:

$$D_3 = \langle r, s : r^3 = 1, s^2 = 1, srs = r^{-1} \rangle.$$

$$H_3 = \langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^3 = 1 \rangle.$$

(a) Show that the two groups are isomorphic (give an explicit isomorphism)

(b) Consider the group algebra $\mathbb{C}[D_3] \simeq \mathbb{C}[H_3]$. Construct two inequivalent representations of this algebra of dimension 1 over $\mathbb{C}$ and show that there are no other inequivalent 1-dimensional representations.

(c) Consider the following maps: $\rho_1 : \mathbb{C}[D_3] \to \mathrm{End}(\mathbb{C}^2)$,

$$\rho_1(r) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} \qquad \rho_1(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\rho_2 : \mathbb{C}[H_3] \to \mathrm{End}(\mathbb{C}^2)$:

$$\rho_2(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho_2(s_2) = \begin{pmatrix} 0 & e^{-2\pi \mathrm{i}/3} \\ e^{2\pi \mathrm{i}/3} & 0 \end{pmatrix}$$

Check that ρ_1 and ρ_2 define irreducible representations of the respective algebras.

(d) Using the isomorphism of algebras $\mathbb{C}[D_3] \simeq \mathbb{C}[H_3]$, show that the representations ρ_1 and ρ_2 defined in (c) are isomorphic.

Exercise 6. Consider the $\mathbb{C}$-algebra $U(sl_2)$ generated over $\mathbb{C}$ by $\{e, f, h\}$ with the relations

$$he - eh = 2e, \qquad hf - fh = -2f, \qquad ef - fe = h.$$

(a) Show that the assignment

$$\rho(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \qquad \rho(e) = x\frac{\partial}{\partial y}, \qquad \rho(f) = y\frac{\partial}{\partial x}$$

defines a representation of $U(sl_2)$ on the vector space $\mathbb{C}_2[x, y]$ of homogeneous polynomials of degree 2.

(b) Consider the 3-dimensional vector space V_3 with basis $\{e, f, h\}$, and define a map $\vartheta : U(sl_2) \to \mathrm{End}(V_3)$ by

$$\vartheta(e)(t) = et - te, \quad \vartheta(f)(t) = ft - tf, \quad \vartheta(h)(t) = ht - th$$

for any $t \in V_3$. Show that ϑ defines a representation of $U(sl_2)$ in V_3 and that this representation is isomorphic to the representation $(\rho, \mathbb{C}_2[x, y])$ constructed in (a).