December 16, 2025

Problem Set 13 Solutions

Exercise 1. Let V_λ denote the Specht module for S_n, where λ is a partition of n.

(a) Show that

$$\mathrm{Res}^{S_n}_{S_{n-1}} V_\lambda \simeq \bigoplus_{\mu \in R(\lambda)} V_\mu,$$

where $R(\lambda)$ is the set of Young diagrams obtained by removing one square from Y_λ.

(b) Show that

$$\mathrm{Ind}^{S_n}_{S_{n-1}} V_\mu \simeq \bigoplus_{\lambda \in A(\mu)} V_\lambda,$$

where $A(\mu)$ is the set of Young diagrams obtained by adding one square from Y_μ.

Hint: Use the formula for the character of V_μ in (a) and the Frobenius reciprocity in (b).

Solution 1. (a) Let the conjugacy class $C_{\mathbf{i}} \subset S_{n-1}$, then $C_{\mathbf{i}} \subset S_n$ leaves an element, for example n, invariant. Therefore $C_{\mathbf{i}}$ has at least one 1-cycle, $i_1 \geq 1$. Let $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_p)$ and $N \geq p$. Let $\rho = (N-1, N-2, \ldots 1, 0)$. Recall that the character $\chi_{V_\lambda}(C_{\mathbf{i}})$ of the Specht module is given by the coefficient of

$$x^{\lambda+\rho} = \prod_j x_j^{\lambda_j + N - j}$$

in the polynomial

$$\Delta(x) \prod_{m \geq 1} H_m(x)^{i_m},$$

where $\Delta(x) = \prod_{i<j}(x_i - x_j)$ and $H_m(x) = \sum_{k=1}^N x_k^m$. Then $\chi_\lambda(C_{\mathbf{i}} \cap S_{n-1})$ is the coefficient of $x^{\lambda+\rho}$ in

$$\Delta(x) \prod_{m \geq 2} H_m(x)^{i_m} H_1(x)^{i_1 - 1}(x_1 + \ldots + x_N) = \sum_i x_i \Delta(x) \prod_{m \geq 2} H_m(x)^{i_m} H_1(x)^{i_1-1}.$$

Notice that the coefficient of $x^{\lambda+\rho}$ in the first element of the sum, $x_1 \Delta(x) \prod_{m \geq 2} H_m(x)^{i_m} H_1(x)^{i_1-1}$, equals to the coefficient of $x_1^{\lambda_1 - 1 + N - 1} \prod_{j \geq 2} x_j^{\lambda_j + N - j}$ in $\Delta(x) \prod_{m \geq 2} H_m(x)^{i_m} H_1(x)^{i_1 - 1}$. Notice also that if $\lambda_1 = \lambda_2$, then we have the powers of the first two variables equal : $\lambda_1 - 1 + N - 1 = \lambda_2 + N - 2$. Then clearly this coefficient is zero, because a coefficient of a monimial $x_1^k x_2^k \prod_{j \geq 3} x_j^{k_j}$ is zero in the antisymmetric polynomial $\Delta(x) \prod_j H_m^{i_m}$ (we recall that $\Delta(x)$ is a completely antisymmetric polynomial, and each $H_m(x)$ is a symmetric polynomial). This means that the first summand of the sum

$$\sum_i x_i \Delta(x) \prod_{m \geq 2} H_m(x)^{i_m} H_1(x)^{i_1 - 1}$$

will contribute only if $\lambda_1 > \lambda_2$ and in this case its contribution is the coefficient of

$$x_1^{\lambda_1 - 1 + N - 1} \prod_{j \geq 2} x_j^{\lambda_j + N - j}$$

in

$$\Delta(x) \prod_{m \geq 2} H_m(x)^{i_m} H_1(x)^{i_1 - 1},$$

which is by definition the coefficient of $x^{\mu+\rho}$ in it, where $\mu = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_p)$. This is by definition the character of the representation V_μ of the element of the conjugacy class $C_{\mathbf{i}} \in S_{n-1}$.

Similarly, the other nonzero contributions will come from the coefficient of $x^{\lambda+\rho}$ in

$$x_k\Delta(x)\prod_{m\geq 2}H_m(x)^{i_m}H_1(x)^{i_1-1},$$

where $\lambda_k>\lambda_{k+1}$, which means exactly that we can remove a square from the row λ_k and still obtain a valid Yound diagram of size $n-1$. Each time this contribution will be equal to the coefficient of $x^{\mu+\rho}$, where $\mu_k=\lambda_k-1$, and $\mu_i=\lambda_i$ for $i\neq k$, in the polynomial

$$\Delta(x)\prod_{m\geq 2}H_m(x)^{i_m}H_1(x)^{i_1-1},$$

which corresponds to computing the character of V_μ on the conjugacy class $C_{\mathbf{i}}\in S_{n-1}$, with one fewer 1-cycles than $C_{\mathbf{i}}\in S_n$. Finally we have that the character of the restriction of the irreducible representation V_λ to S_{n-1} equals to the sum of characters of the Specht modules V_μ for S_{n-1}, where μ is a partition of $(n-1)$ obtained from the partition λ of n by shortening one row, so that the resulting Young diagram still satisfies the non-increasing condition. This means that $\mu\in R(\lambda)$. We have

$$\chi_{\mathrm{Res}^{S_n}_{S_{n-1}}V_\lambda=\sum_{\mu\in R(\lambda)}}\chi_{V_\mu},$$

and

$$\mathrm{Res}^{S_n}_{S_{n-1}}V_\lambda\simeq\bigoplus_{\mu\in R(\lambda)}V_\mu.$$

(b) We will use (a) and the Frobenius reciprocity result. Suppose that

$$\mathrm{Ind}^{S_n}_{S_{n-1}}V_\nu\simeq\bigoplus_\lambda m_\lambda V_\lambda,$$

where

$$m_\lambda=\dim\mathrm{Hom}_{S_n}(\mathrm{Ind}^{S_n}_{S_{n-1}}V_\nu,V_\lambda)=\dim\mathrm{Hom}_{S_{n-1}}(V_\nu,\mathrm{Res}^{S_n}_{S_{n-1}}V_\lambda)=$$

$$=\dim\mathrm{Hom}_{S_{n-1}}(V_\nu,\bigoplus_{\mu\in R(\lambda)}V_\mu)=\left[\begin{array}{ll}1, & \nu\in R(\lambda),\\ 0, & \nu\notin R(\lambda).\end{array}\right.$$

We have $\nu\in R(\lambda)$ when Y_ν is obtained from Y_λ by removing a square, which is equivalent to saying that Y_λ is obtained from Y_ν by adding a square, or $\lambda\in A(\nu)$. Therefore,

$$m_\lambda=\left[\begin{array}{ll}1, & \lambda\in A(\nu),\\ 0, & \lambda\notin A(\nu).\end{array}\right.$$

Finally we have

$$\mathrm{Ind}^{S_n}_{S_{n-1}}V_\nu\simeq\bigoplus_{\lambda\in A(\nu)}V_\lambda,$$

as required.

Exercise 2. (Transitivity of the induction) Let $K\subset H\subset G$ be subgroups of a finite group G and V a complex representation of K. Show that

$$\mathrm{Ind}^G_H\,\mathrm{Ind}^H_K V\simeq\mathrm{Ind}^G_K V.$$

Hint: Use the tensor product form of the induced representations.

Solution 2. We have $\mathrm{Ind}^H_K V\simeq\mathbb{C}[H]\otimes_{\mathbb{C}[K]}V$ and $\mathrm{Ind}^G_H W\simeq\mathbb{C}[G]\otimes_{\mathbb{C}[H]}W$. Then

$$\mathrm{Ind}^G_H\,(\mathrm{Ind}^H_K V)\simeq\mathrm{Ind}^G_H\,(\mathbb{C}[H]\otimes_{\mathbb{C}[K]}V)$$

$$\simeq\mathbb{C}[G]\otimes_{\mathbb{C}[H]}(\mathbb{C}[H]\otimes_{\mathbb{C}[K]}V)\simeq(\mathbb{C}[G]\otimes_{\mathbb{C}[H]}\mathbb{C}[H])\otimes_{\mathbb{C}[K]}V)\simeq\mathbb{C}[G]\otimes_{\mathbb{C}[K]}V\simeq\mathrm{Ind}^G_K V.$$

Here we used the isomorphism

$$\mathbb{C}[G]\otimes_{\mathbb{C}[H]}\mathbb{C}[H]\simeq\mathbb{C}[G]$$

given by the map $f:\mathbb{C}[G]\otimes_{\mathbb{C}[H]}\mathbb{C}[H]\to\mathbb{C}[G]$, where $f:\ g\otimes h=gh\otimes 1\to gh$.

Exercise 3. (a) Let G be a finite group and V_R an irreducible representation of G defined over the real numbers. Show that its complexification $V = \mathbb{C} \otimes_{\mathbb{R}} V_R$ is a representation of real type.

(b) Show that all Specht modules V_λ for S_n are of real type.

(c) Use the Frobenius-Schur indicator to find the sum of dimensions of all irreducible representations of S_n.

Solution 3. (a) Suppose that the representation V_R has real entries. Choose $\langle\ ,\ \rangle_0$ to be a positive definite symmetric bilinear form on V_R. Let $\langle\ ,\ \rangle$ be the G-invariant nondegenerate symmetric form on V_R (constructed by the Weyl's unitary trick, see Lecture 3). More explictly, we may define

$$\langle v, w\rangle := \sum_{g\in G} \frac{1}{G} \langle g\cdot v, g\cdot w\rangle_0.$$

Let $V = \mathbb{C} \otimes_{\mathbb{R}} V_R$ be the complexification of V_R. Then we can define

$$\langle v + iw, u + iz\rangle = \langle v, u\rangle - \langle w, z\rangle + i\langle w, u\rangle + i\langle v, z\rangle.$$

This is a symmetric G-invariant $\mathbb{C}$-bilinear form on V. It is also nondegenerate: If $(v, w) \neq (0, 0)$ we can find (u, z) such that $\langle v + iw, u + iz\rangle \neq 0$. Indeed, if $v \neq 0$, we can choose u so that $\langle v, u\rangle \neq 0$ and set $z = 0$, and similarly in case $w \neq 0$. Therefore, we have a symmetric G-invariant $\mathbb{C}$-bilinear form on V. This implies that the space S^2V^* of symmetric bilinear forms on V has a G-invariant element, or that $V_0 \subset S^2V$ as a G-representation, which by definition means that V^* is a representation of real type. Since V^* is of real type if and only if V is, we conclude that V is a representation of real type.

(b) We have the Specht module V_λ defined as $V_\lambda = \mathbb{C}[S_n]c_\lambda$, where $c_\lambda = a_\lambda b_\lambda$ is a rational linear combination of the elements of S_n. Therefore the action of any $g \in S_n$ in V_λ is given by a matrix with rational coefficients, in particular all matrices $\rho_\lambda(g)$ have real values and thus we may see V_λ as the complexification of representation $\mathbb{R}[S_n]c_\lambda$. Thus by part (a), the representation V_λ is of real type.

(c) The theorem on the Frobenius-Schur indicator claims that the number of all involutions in G (elements of order ≤ 2) equals to $\sum_V FS(V) \cdot \dim(V)$ over the irreducible representations, where $FS(V) = 1$ if V is of real type, 0 if it is of complex type and -1 if it is of quaternionic type. In case of S_n we have by (b) that all irreducible representations of the symmetric group S_n are of real type, therefore the sum of the dimensions of the irreducible representations of S_n equals the number of involutions of S_n. The involutions of S_n are the permutations of cycle type $(2, 2, 2, ...2)$ or the identity element. We can present these elements as products of (possibly zero) disjoint transpositions. Therefore the number of involutions can be computed by summing over all possible choices of disjoint transpositions. For example, we have $\binom{n}{2}$ transpositions in S_n. To find the number of different products of k disjoint transpositions, we have to choose $2k$ numbers out of n and then pair them up. In a set of $2k$ elements, we have $(2k-1)!! = (2k-1)(2k-3)\dots 1$ possible pairings, they are called perfect matchings of the complete graph of $2k$ vertices. Indeed, we have $(2k - 1)$ ways to find a pair for the first element, and once this choice is done, we have $(2k - 3)$ ways to pair off the second element, and so on till we exhaust the available elements. Therefore, the total number N_{inv} of involutions in S_n equals to

$$N_{inv} = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k - 1)!!$$

This also equals to the sum of the dimensions of all irreducible representations of S_n:

$$\sum_{V\in \mathrm{Irr}(S_n)} \dim V = N_{inv} = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k - 1)!!$$