December 9, 2025

Problem Set 12 Solutions

Exercise 1. Recall that for a Young tableau $\mathcal{T}_\lambda$ of size n that corresponds to a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_p)$, such that $\sum_{i=1}^{p} \lambda_i = n$, we can define two subgroups $P_\lambda \in S_n$ and $Q_\lambda \in S_n$ that are the stabilizers respectively of rows and columns of a Young tableau $\mathcal{T}_\lambda$.

Consider two Young tableaux of size $n = 7$ and $\lambda = (3, 2, 2)$.

$\mathcal{T}_\lambda =$

1	2	3
4	5	
6	7	

$\mathcal{T}'_\lambda =$

5	2	1
3	7	
6	4	

We showed in class that if there is no couple (i, j) of numbers in the same row of $\mathcal{T}_\lambda$ and in the same column of $\mathcal{T}'_\lambda = g\mathcal{T}_\lambda$, then $g = pq$, where $p \in P_\lambda$ and $q \in Q_\lambda$. Find the elements p and q in this case.

Exercise 2. Recall that an irreducible Specht module V_λ for S_n is determined by a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_p)$, such that $\sum_{i=1}^{p} \lambda_i = n$. It can be defined by $V(\lambda) = \mathbb{C}[S_n]c_\lambda$, where $c_\lambda = a_\lambda b_\lambda$ with

$$a_\lambda = \frac{1}{|P_\lambda|} \sum_{g \in P_\lambda} g; \quad b_\lambda = \frac{1}{|Q_\lambda|} \sum_{g \in Q_\lambda} (-1)^g g.$$

Here the subgroups $P_\lambda \in S_n$ and $Q_\lambda \in S_n$ are the stabilizers respectively of rows and columns of a Young tableau T_λ of shape λ.

(a) In class we showed that $c_\lambda^2 = x(\lambda) c_\lambda$, where $x(\lambda) \in \mathbb{Q}$ is a coefficient. Find $x(\lambda)$. *Hint:* Consider the action of c_λ in the regular representation of S_n.

(b) Let $C = \sum_{i<j} (ij) \in \mathbb{C}[S_n]$ be the sum of all transpositions. Show that C acts on the Specht module V_λ by multiplication by the scalar $z(\lambda) = \sum_{j=1}^{p} \sum_{i=1}^{\lambda_j} (i - j)$. (The integer $z(\lambda)$ is called the *content* of the Young diagram of shape λ.)

Exercise 3. Let $G = SL(2, \mathbb{F}_q)$ be the group of 2×2 matrices of determinant 1 with coefficients in the field of q elements $\mathbb{F}_q$ ($q \geq 3$ a prime). Consider the 2-dimensional $\mathbb{F}_q$-vector space V with the basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

(a) Find the order of G.

(b) Show that G acts transitively on $V \setminus \{(0, 0)\}$.

(c) Find the stabilizer $N \subset G$ of e_1.

(d) Show that the representation (F, ρ) of G in the complex vector space F of functions $f : V \setminus \{(0, 0)\} \to \mathbb{C}$ given by

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(x, y) = f(dx - by, -cx + ay)$$

is isomorphic to the induced representation of G from the trivial representation of N.

(e) Use Frobenius reciprocity to deduce that ρ is not irreducible.

Exercise 4. Let $K \subset G$ be a subgroup, and $\mathbb{C}_\chi$ a one-dimensional representation of K with character $\chi : K \to \mathbb{C}^\star$. Consider the central idempotent corresponding to χ:

$$e_\chi = \frac{1}{|K|} \sum_{g \in K} \chi(g)^{-1} g \quad \in \mathbb{C}[K].$$

Show that the induced representation $\mathrm{Ind}_K^G \mathbb{C}_\chi$ is naturally isomorphic to $\mathbb{C}[G]e_\chi$, with the action of G in $\mathbb{C}[G]e_\chi$ by left multiplication.