December 2, 2025

Problem Set 11 Solutions

Exercise 1. (a) Let $D_3 = \langle r, s \mid r^3 = 1, s^2 = 1, srs = r^{-1}\rangle$ be the dihedral group of order 6. Describe the irreducible complex representations of D_3 and compute its character table. (Use Ex. 1, PS 6 and Ex. 1, PS 7).

(b) Decompose ρ_{lreg} the left regular representation $\mathbb{C}[D_3]$ into a direct sum or irreducible representations. Similarly, consider the right regular representation ρ_{rreg} of $\mathbb{C}[D_3]$ by multiplication on the right and decompose it into a direct sum of irreducible representations.

(c) As an associative algebra $\mathbb{C}[D_3]$ is isomorphic to a direct sum of matrix algebras. This decomposition provides a basis in $\mathbb{C}[D_3]$ given by the matrix elements $\{a^V_{ij}\}_{V\in \mathrm{Irr}}$ of $\mathrm{End}(V)$, which is consistent with the decomposition of ρ_{lreg} and ρ_{rreg}. Express this basis in terms of the basis $\{g\}_{g\in D_3}$.

(d) For each irreducible V decompose the representation ρ_{ad} of D_3 acting on $\mathrm{End}(V)$ by $\rho_{ad}(g)(f)(v) = \rho_V(g) \circ f(\rho_V(g^{-1})v)$ as a direct sum of irreducible representations. *Hint:* show that $V \simeq V^*$ for all irreducible V of D_3 and use characters.

(e) Consider the adjoint action of D_3 on $\mathbb{C}[D_3]$: $\rho_{ad}(g)(h) = ghg^{-1}$. Use (d) to decompose ρ_{ad} into a direct sum of irreducible representations.

(f) Find the center of the algebra $\mathbb{C}[D_3]$.

Solution 1. (a) According to the argument in PS 7, Ex.1 there are three irreducible representations of D_3: the trivial 1-dimensional V_0, the 1-dimensional sign representation V_s and the 2-dimensional irreducible representation V_2 given by the symmetries of an equilateral triangle on a plane. Set $\xi = e^{2\pi i/3}$. We have

$$\rho_0(r) = \rho_0(s) = 1, \quad \rho_s(r) = 1, \quad \rho_s(s) = -1, \quad \rho_2(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho_2(r) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$$

The conjugacy classes in D_3 are $\{(1), (r, r^2), (s, sr, sr^2)\}$. The character table is given by

	(1)	(r, r^2)	(s, sr, sr^2)
V_0	1	1	1
V_s	1	1	-1
V_2	2	-1	0

(b) By density theorem we have

$$\mathbb{C}[D_3] \simeq \mathrm{End}(V_0) \oplus \mathrm{End}(V_s) \oplus \mathrm{End}(V_2).$$

Each direct summand is a two-sided ideal in $\mathbb{C}[D_3]$. Each $\mathrm{End}(V_i) \simeq V_i^{\oplus d_i}$, where each column of $\mathrm{End}(V_i)$ is a subrepresentation isomorphic to V_i. (see for example Ex. 4, PS 4). Therefore, we have

$$\rho_{lreg} \simeq V_0 \oplus V_s \oplus V_2^{\oplus 2}.$$

Similarly, each $\mathrm{End}(V_i)$ acts on itself by right multiplication and since it is a simple algebra, it decomposes into a direct sum of irreducible representations, all isomorphic to V_i, since V_i is the only irreducible representation of $\mathrm{End}(V_i)$. For the right action, each row of the matrix of $\mathrm{End}(V_i)$ is a subrepresentation, isomorphic to V_i. So we have

$$\rho_{rreg} \simeq V_0 \oplus V_s \oplus V_2^{\oplus 2}.$$

(c) The algebra $\mathbb{C}[D_3]$ is isomorphic as an associative algebra to the following direct sum of matrix algebras,

$$\mathbb{C}[D_3] \simeq \mathrm{End(V_0)} \oplus \mathrm{End(V_s)} \oplus \mathrm{End(V_2)} \simeq \mathrm{Mat}_1(\mathbb{C}) \oplus \mathrm{Mat}_1(\mathbb{C}) \oplus \mathrm{Mat}_2(\mathbb{C}).$$

Since we know how each matrix algebra decomposes with respect to the left and right regular action of D_3, it is easy to compute the matrix elements. With respect to the left and right regular actions, the component corresponding to End(V$_0$) is the trivial representation, given by the sum of all group elements, and for the sign representations, the sum of group elements with their signs, where we use the isomorphism $D_3 \simeq S_3$:

$$\mathrm{End}(V_0) = \mathrm{Span}\left(\sum_{g\in D_3} g\right), \qquad \mathrm{End}(V_s) = \mathrm{Span}\left(\sum_{g\in S_3} \varepsilon(g)g\right) = \mathrm{Span}(1+r+r^2-s-sr-sr^2).$$

To find the matrix elements of $\mathrm{End}(V_2)$ we recall from (b) that the columns and the rows of $\mathrm{End}(V_2)$ are isomorphic to V_2 with respect to the left (resp. right) action of the generators. We can first find the eigenspace of $\rho_{lreg}(r)$ with the eigenvalue ξ:

$$r\cdot v = r\cdot(a+br+cr^2+ds+esr+fsr^2) = \xi(a+br+cr^2+ds+esr+fsr^2)$$

therefore $v = a(1+\xi^2 r+\xi r^2)+d(s+\xi sr+\xi^2 sr^2)$. In a similar way we can find the eigenvector of $\rho_{rreg}(r)$ with eigenvalue ξ:

$$w\cdot r = (a'+b'r+c'r^2+d's+e'sr+f'sr^2)\cdot r = \xi(a'+b'r+c'r^2+d's+e'sr+f'sr^2).$$

Therefore, $w = a'(1+\xi^2 r+\xi^2 r^2)+d'(s+\xi^2 sr+\xi sr^2)$. To have a common eigenvector for $\rho_{lreg}(r)$ and $\rho_{rreg}(r)$ with the same eigenvalue ξ, we must have $d=d'=0$ and $a=a'$. We can take $a_{11} = (1+\xi^2 r+\xi r^2)$, which is first element of the matrix of $\mathrm{End}(V_2)$. Note that the action of s swaps two basis elements of V_2. Then acting by s on the right and on the left of a_{11}, we obtain the next element along the first column $s\cdot a_{11} = a_{21} = (s+\xi^2 sr+\xi sr^2)$ and along the first row $a_{11}\cdot s = (s+\xi sr+\xi^2 sr^2) = a_{12}$. Finally, the common eigenvector of $\rho_{lreg}(r)$ and $\rho_{rreg}(r)$ with eigenvalue ξ^2 is $a_{22} = (1+\xi r+\xi^2 r^2)$. You can check that $s\cdot a_{12} = a_{22}$ and $a_{21}\cdot s = a_{22}$. Finally we have the following matrix:

$$\mathrm{End}(V_2) = \begin{pmatrix} 1+\xi^2 r+\xi r^2 & s+\xi sr+\xi^2 sr^2 \\ s+\xi^2 sr+\xi sr^2 & 1+\xi r+\xi^2 r^2 \end{pmatrix}.$$

Notice that $\{\sum_{g\in D_3} g,\ \sum_{g\in D_3}\varepsilon(g)g,\ a_{11},\ a_{12},\ a_{21},\ a_{22}\}$ form a basis in $\mathbb{C}[D_3]$. Moreover, we have that each endomorphism algebra is a two-sided ideal in $\mathbb{C}[D_3]$.

(d) We have $\chi_{V^*}(g) = \overline{\chi_V(g)}$. Since the characters of all irreducible representations of D_3 are real, each of them is self-dual. Alternatively, we can notice from the character table that $\chi_0^2 = \chi_0$, $\chi_s^2 = \chi_0$ and $\chi_2^2 = \chi_2+\chi_0+\chi_s$. Therefore, each $V\otimes V$ contains a trivial representation, and therefore $V\otimes V \simeq V^*\otimes V$, and $V\simeq V^*$. Recall (Lecture 6) that for an irreducible representation V, $\mathrm{End}(V)$ with the action of ρ_{ad} is isomorphic to the representation $V\otimes V^*$. Since in our case $V^*\simeq V$, we have $\mathrm{End}(V)\simeq V\otimes V$ and

$$\mathrm{End}(V_0)\simeq V_0\otimes V_0\simeq V_0, \quad \mathrm{End}(V_s)\simeq V_s\otimes V_s\simeq V_0, \quad \mathrm{End}(V_2)\simeq V_2\otimes V_2\simeq V_0\oplus V_s\oplus V_2.$$

(e) Note that the action of the adjoint representation on $\mathbb{C}[D_3]\simeq \mathrm{End}(V_0)\oplus\mathrm{End}(V_s)\oplus\mathrm{End}(V_2)$ is given by the adjoint action on each of the direct summands, computed in (d). Then we have

$$\rho_{ad}\simeq V_0\oplus V_0\oplus V_0\oplus V_s\oplus V_2\simeq V_0^{\oplus 3}\oplus V_s\oplus V_2.$$

(f) Since the action of ρ_{ad} on $\mathbb{C}[D_3]$ is given by $\rho_{ad}(g)h = ghg^{-1}$, we conclude that the center is spanned by the trivial isotypical component of this representation. From (e) we have

$$Z(\mathbb{C}[D_3])\simeq(\rho_{ad})^G\simeq V_0^{\oplus 3}.$$

Therefore, the center is 3-dimensional. We also know that $\sum_{h\in C} h$ for any conjugacy class $C\subset D_3$ is a central element. Therefore, the center is spanned by

$$\{1,\ r+r^2,\ s+sr+sr^2\} = \{1,\ (123)+(132),\ (12)+(23)+(13)\},$$

where the second presentation uses the group isomorphism $D_3\simeq S_3$. We can also notice that the traces of the matrices in the matrix presentation of $\mathbb{C}[D_3]$ computed in (c) provide another basis in the center, namely

$$1+r+r^2+s+sr+sr^2, \quad 1+r+r^2-s-sr-sr^2, \quad 2-r-r^2.$$

This basis has the property that the product of any two distinct elements is zero. After a renormalization we can have

$$e_1 = \frac{1}{6}(1+r+r^2+s+sr+sr^2), \quad e_2 = \frac{1}{6}(1+r+r^2-s-sr-sr^2), \quad e_3 = \frac{1}{3}(2-r-r^2).$$

Then $e_ie_j = \delta_{ij}e_i$. Central elements with this property are called the *central idempotents*.

Exercise 2. The purpose of this exercise is to illustrate the statements used in the proof of Burnside's theorem. Let $G = A_4$, the alternating group of 4 elements.

(a) We have proved in class that if V is an irreducible representation of G and C a conjugacy class in G such that $\gcd(|C|, \dim(V)) = 1$, then for any $g \in C$ we have either $\chi_V(g) = 0$, or $\rho_V(g) = \lambda \,\mathrm{Id}_V$. For each nontrivial conjugacy class in A_4 and irreducible representation satisfying the condition $\gcd(|C|, \dim(V)) = 1$, find whether $g \in C$ acts as a scalar in V or has zero character.

(b) We also proved that if G has a conjugacy class C of a prime power order, then G has a proper nontrivial normal subgroup H defined by $H = \langle ab^{-1},\ a, b \in C\rangle \lhd G$. Find all conjugacy classes of prime power order in A_4 and construct the corresponding normal subgroups.

Solution 2. (a) We start with the character table of A_4 given for example in the course. The second line shows the number of elements in each conjugacy class.

	1	(12)(34)	(123)	(132)
$\lvert C\rvert$	1	3	4	4
V_0	1	1	1	1
V_ξ	1	1	ξ	ξ^2
V_{ξ^2}	1	1	ξ^2	ξ
V_3	3	-1	0	0

First note that any 1-dimensional representation of G has the property $\gcd(|C|, \dim(V)) = 1$ and that any group element acts by a nonzero scalar given by a root of unity on a 1-dimensional representation, because $(\rho_V(g))^n = \mathrm{Id}$ for $n = \mathrm{order}(g)$. Therefore, for any C, an element $g \in C$ act by a scalar on any 1-dimensional representation.

Now consider V_3 of dimension 3. We have $\gcd(|C|, 3) = 1$ for $C = C_{(123)}$ and $C = C_{(132)}$. Looking at the character table, in both cases we have $\chi_{V_3}((123)) = \chi_{V_3}((132)) = 0$.

(b) Consider $C_{(12)(34)}$ of order 3. It gives rise to the nontrivial proper normal subgroup

$$H_1 = \langle ab^{-1}, a, b \in \{(12)(34), (13)(24), (14)(23)\}\rangle = \{1, (12)(34), (13)(24), (14)(23)\} = K \lhd A_4.$$

Thus we obtain the Klein subgroup which is normal in A_4. Consider $C_{(123)}$ of order 2^2. It gives rise to the nontrivial proper normal subgroup

$$H_2 = \langle ab^{-1}, a, b \in \{(123), (214), (341), (432)\}\rangle = \{1, (12)(34), (13)(24), (14)(23)\} = K \lhd A_4.$$

Since the products are of the form $(123)(214)^{-1} = (13)(24)$, we obtain the Klein subgroup in A_4. Similarly, starting from the conjugacy class $C_{(132)}$ of order 2^2, we obtain the same Klein subgroup $K \lhd A_4$.
Remark In fact, because K is the only nontrivial proper normal subgroup in A_4, we can deduce without a computation that every conjugacy class of prime power order gives rise to the same group K.

Exercise 3. The purpose of this exercise is to compute concrete examples of induced representations and illustrate the Frobenius reciprocity. Consider the group $D_3 = \langle r, s \mid r^3 = 1, s^2 = 1, srs = r^{-1}\rangle$ and the subgroups $C_3 = \{1, r, r^2\} \subset D_3$ and $C_2 = \{1, s\} \subset D_3$.

(a) Use the character formula for the induced representation to decompose into the irreducible components the representation $\mathrm{Ind}_{C_3}^{D_3} V$ for each irreducible representation V of C_3.

(b) Use the Frobenius reciprocity to decompose into the irreducible components the representation $\mathrm{Ind}_{C_2}^{D_3} V$ for each irreducible representation V of C_2.

Solution 3. (a) Let us first recall the classification of the irreducible representations of $C_3 = \langle r \mid r^3 = 1\rangle$. There are exactly 3 inequivalent irreducible representations of the cyclic group: V_0, V_ξ, V_{ξ^2} where $\rho(r) = 1, \xi, \xi^2$ respectively. Consider the right cosets with respect to C_3: $\{\sigma_1 = C_3 1,\ \sigma_s = C_3 s\}$. For $g \in D_3$, we have $\sigma_i g = \sigma_i$ if and only if g is in the conjugacy classes $\{(1), (r, r^2)\}$. According to the Frobenius character formula for an induced representation $\mathrm{Ind}_{C_3}^{D_3}(V)$ we have

$$\chi(g) = \sum_{\sigma_i\colon\, \sigma_i g = \sigma_i} \chi_V(x_\sigma g x_\sigma^{-1}).$$

So we have for $\mathrm{Ind}_{C_3}^{D_3} V$ where $V = V_0, V_\xi, V_{\xi^2}$:

$$\chi(1) = \chi_V(1) + \chi_V(1) = 2, \quad \chi(s) = 0,$$

the second equality follows because the action of s permutes the right C_3-cosets. If $V = V_0$, we have

$$\chi(r) = \chi_V(r) + \chi_V(srs) = \chi_V(1) + \chi_V(r^{-1}) = 2.$$

If $V = V_\xi$ or $V = V_{\xi^2}$, we have

$$\chi(r) = \chi_V(r) + \chi_V(srs) = \chi_V(1) + \chi_V(r^{-1}) = \xi + \xi^2 = -1.$$

Finally we have the following characters, that we have added as extra lines in the character table of D_3 (see Ex. 1 above):

	(1)	(r, r^2)	(s, sr, sr^2)
V_0	1	1	1
V_s	1	1	-1
V_2	2	-1	0
$\mathrm{Ind}_{C_3}^{D_3} V_0$	2	2	0
$\mathrm{Ind}_{C_3}^{D_3} V_\xi$	2	-1	0
$\mathrm{Ind}_{C_3}^{D_3} V_{\xi^2}$	2	-1	0

Comparing the characters, we conclude

$$\mathrm{Ind}_{C_3}^{D_3} V_0 \simeq V_0 \oplus V_s, \qquad \mathrm{Ind}_{C_3}^{D_3} V_\xi \simeq \mathrm{Ind}_{C_3}^{D_3} V_{\xi^2} \simeq V_2.$$

(b) We will use Frobenius reciprocity. The group $C_2 = \{1, s\}$ has two inequivalent 1-dimensional representations V_0 and V_s, where s acts as ± 1 respectively. We have $\chi_{V_0}(1) = \chi_{V_s}(1) = 1$ and $\chi_{V_s}(1) = 1$, $\chi_{V_s}(s) = -1$. Looking at the character table for $S_3 \simeq D_3$, we conclude that

$$\chi_{\mathrm{Res}_{C_2}^{D_3} V_2}(1) = 2, \qquad \chi_{\mathrm{Res}_{C_2}^{D_3} V_2}(s) = 0.$$

$$\chi_{\mathrm{Res}_{C_2}^{D_3} V_0}(1) = 1, \qquad \chi_{\mathrm{Res}_{C_2}^{D_3} V_0}(s) = 1.$$

$$\chi_{\mathrm{Res}_{C_2}^{D_3} V_s}(1) = 1, \qquad \chi_{\mathrm{Res}_{C_2}^{D_3} V_s}(s) = -1.$$

Therefore we conclude that

$$\mathrm{Res}_{C_2}^{D_3} V_2 \simeq V_0 \oplus V_s, \quad \mathrm{Res}_{C_2}^{D_3} V_0 \simeq V_0, \quad \mathrm{Res}_{C_2}^{D_3} V_s \simeq V_s.$$

Using the Frobenius reciprocity we compute

$$\dim \mathrm{Hom}_{D_3}(V_2, \mathrm{Ind}_{C_2}^{D_3} V_0) = \dim \mathrm{Hom}_{C_2}(\mathrm{Res}_{C_2}^{D_3} V_2, V_0) = 1;$$

$$\dim \mathrm{Hom}_{D_3}(V_0, \mathrm{Ind}_{C_2}^{D_3} V_0) = \dim \mathrm{Hom}_{C_2}(\mathrm{Res}_{C_2}^{D_3} V_0, V_0) = 1;$$

$$\dim \mathrm{Hom}_{D_3}(V_s, \mathrm{Ind}_{C_2}^{D_3} V_0) = \dim \mathrm{Hom}_{C_2}(\mathrm{Res}_{C_2}^{D_3} V_s, V_0) = 0.$$

Therefore,

$$\mathrm{Ind}_{C_2}^{D_3} V_0 \simeq V_2 \oplus V_0.$$

Similarly,

$$\dim \mathrm{Hom}_{D_3}(V_2, \mathrm{Ind}_{C_2}^{D_3} V_s) = \dim \mathrm{Hom}_{C_2}(\mathrm{Res}_{C_2}^{D_3} V_2, V_s) = 1;$$

$$\dim \mathrm{Hom}_{D_3}(V_0, \mathrm{Ind}_{C_2}^{D_3} V_s) = \dim \mathrm{Hom}_{C_2}(\mathrm{Res}_{C_2}^{D_3} V_0, V_s) = 0;$$

$$\dim \mathrm{Hom}_{D_3}(V_s, \mathrm{Ind}_{C_2}^{D_3} V_s) = \dim \mathrm{Hom}_{C_2}(\mathrm{Res}_{C_2}^{D_3} V_s, V_s) = 1.$$

Therefore,

$$\mathrm{Ind}_{C_2}^{D_3} V_s \simeq V_2 \oplus V_s.$$