December 2, 2025

Problem Set 11

Exercise 1. (a) Let $D_3 = \langle r, s \mid r^3 = 1, s^2 = 1, srs = r^{-1} \rangle$ be the dihedral group of order 6. Describe the irreducible complex representations of D_3 and compute its character table. (Use Ex. 1, PS 6 and Ex. 1, PS 7).

(b) Decompose ρ_{lreg} the left regular representation $\mathbb{C}[D_3]$ into a direct sum or irreducible representations. Similarly, consider the right regular representation ρ_{rreg} of $\mathbb{C}[D_3]$ by multiplication on the right and decompose it into a direct sum of irreducible representations.

(c) As an associative algebra $\mathbb{C}[D_3]$ is isomorphic to a direct sum of matrix algebras. This decomposition provides a basis in $\mathbb{C}[D_3]$ given by the matrix elements $\{a_{ij}^V\}_{V \in \mathrm{Irr}}$ of $\mathrm{End}(V)$, which is consistent with the decomposition of ρ_{lreg} and ρ_{rreg}. Express this basis in terms of the basis $\{g\}_{g \in D_3}$.

(d) For each irreducible V decompose the representation ρ_{ad} of D_3 acting on $\mathrm{End}(V)$ by $\rho_{ad}(g)(f)(v) = \rho_V(g) \circ f(\rho_V(g^{-1})v)$ as a direct sum of irreducible representations. *Hint:* show that $V \simeq V^*$ for all irreducible V of D_3 and use characters.

(e) Consider the adjoint action of D_3 on $\mathbb{C}[D_3]$: $\rho_{ad}(g)(h) = ghg^{-1}$. Use (d) to decompose ρ_{ad} into a direct sum of irreducible representations.

(f) Find the center of the algebra $\mathbb{C}[D_3]$.

Exercise 2. The purpose of this exercise is to illustrate the statements used in the proof of Burnside's theorem. Let $G = A_4$, the alternating group of 4 elements.

(a) We have proved in class that if V is an irreducible representation of G and C a conjugacy class in G such that $\gcd(|C|, \dim(V)) = 1$, then for any $g \in C$ we have either $\chi_V(g) = 0$, or $\rho_V(g) = \lambda \,\mathrm{Id}_V$. For each nontrivial conjugacy class in A_4 and irreducible representation satisfying the condition $\gcd(|C|, \dim(V)) = 1$, find whether $g \in C$ acts as a scalar in V or has zero character.

(b) We also proved that if G has a conjugacy class C of a prime power order, then G has a proper nontrivial normal subgroup H defined by $H = \langle ab^{-1},\ a, b \in C \rangle \lhd G$. Find all conjugacy classes of prime power order in A_4 and construct the corresponding normal subgroups.

Exercise 3. The purpose of this exercise is to compute concrete examples of induced representations and illustrate the Frobenius reciprocity. Consider the group $D_3 = \langle r, s \mid r^3 = 1, s^2 = 1, srs = r^{-1} \rangle$ and the subgroups $C_3 = \{1, r, r^2\} \subset D_3$ and $C_2 = \{1, s\} \subset D_3$.

(a) Use the character formula for the induced representation to decompose into the irreducible components the representation $\mathrm{Ind}_{C_3}^{D_3} V$ for each irreducible representation V of C_3.

(b) Use the Frobenius reciprocity to decompose into the irreducible components the representation $\mathrm{Ind}_{C_2}^{D_3} V$ for each irreducible representation V of C_2.