November 25, 2025

Problem Set 10

Exercise 1. Consider the group algebra $\mathbb{C}[G]$ of a finite group G. We know from the course that the the regular representation $\mathbb{C}[G]_{reg}$ decomposes as a direct sum

$$\mathbb{C}[G] = \bigoplus_{i=1}^{r} V_i^{\oplus \dim V_i}$$

where $\{V_i\}_{i=1}^r$ are the inequivalent irreducible representations of G. In particular, $\mathbb{C}[G]_{reg}$ always contains the trivial subrepresentation of the group with multiplicity 1. The following exercise gives a description of the complement to the trivial representation in $\mathbb{C}[G]_{reg}$ in the basis of group elements.

(a) Define a subspace N in $\mathbb{C}[G]$ by

$$N = \left\{ \sum_{g\in G} a_g g \ : \ a_g \in \mathbb{C}, \ \sum_{g\in G} a_g = 0 \right\}.$$

Show that $N \subset \mathbb{C}[G]$ is a $\mathbb{C}[G]$-submodule in the left regular module $\mathbb{C}[G]$ over itself and find its dimension.

(b) Consider the quotient module, $M = \mathbb{C}[G]/N$. Find its dimension, introduce a basis and describe its structure.

Exercise 2. A group is nilpotent if its ascending central series terminates in the whole group. The ascending central series is the sequence of normal subgroups

$$1 = Z_0 \subset Z_1 \subset \ldots \subset Z_i \subset \ldots,$$

where $Z_{i+1} = \{x \in G : xyx^{-1}y^{-1} \in Z_i \ \forall y \in G\}$. In particular, Z_1 is the center of G.

(a) Show that a nilpotent group is solvable.

(b) Give an example of a solvable group that is not nilpotent.

(c) Show that a group of order p^k, where p is a prime, is nilpotent.
(This is another way to show that a group of order p^k is solvable – an easy special case of Burnside's theorem).

Exercise 3. Show that if V is an irreducible complex representation of a finite group G, and $\dim V > 1$, then there is an element $g \in G$ such that $\chi_V(g) = 0$. *Hint:*

(a) Use orthonormality of characters to show that the arithmetic mean of the numbers $|\chi_V(g)|^2$ for $g \neq e$ is strictly less than 1. Deduce that

$$\beta = \prod_{g\neq e} |\chi_V(g)|^2 < 1.$$

(b) Consider the map $G \to G$ defined by $g \to g^j$, where j is any positive integer such that $\gcd(j, |G|) = 1$. Show that it is a bijection, and deduce that it leaves the number β fixed.

(c) Show that $\beta \in \mathbb{Z}[\xi]$, where $\xi = e^{2\pi i/|G|}$ is the $|G|$-th root of unity. Use (b) to show that $\beta = 0$.