Final exam

January 14, 2025

N°

Last name :

First name :

- No documents are allowed at the exam.
- Twice during the exam you are allowed to consult the book by Etingof and the typed list of theorems available at the front desk, for a total of 5 minutes.
- No electronic devices are allowed.
- Please provide clear, concise and easily readable arguments.
- You can answer in English or in French, but please do not mix the two languages.
- Scrap paper will not be graded.

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Question	1	2	3	4	5	6	7	8	Total
max	10	12	12	12	14	12	6	7	85
score									

Question 1. (10 pts)

(a) Let G be a finite group. Suppose that every complex irreducible representation of G is one-dimensional. Show that G is abelian.

(b) Let C_n be the cyclic group of order $n \geq 1$. Show that the algebra $\mathbb{C}[C_n]$ has a basis $\{e_i\}_{i=1}^n$ such that $e_i e_j = \delta_{ij} e_i$. Express $1 \in \mathbb{C}[C_n]$ in this basis.

Question 2. (12 pts) Let G be a finite group and $H \subset G$ be a proper subgroup.

(a) Let V_0 be the trivial representation of H. Show that the induced representation $\mathrm{Ind}_H^G V_0$ is not irreducible. *Hint:* Use Frobenius reciprocity.

(b) Describe all possible one-dimensional representations of the symmetric group S_k $k \geq 2$. Justify your answer. *Hint:* Recall that transpositions generate S_k and consider the possible action of a transposition in a one-dimensional representation.

(c) Let $S_k \subset S_{k+1}$ be a subgroup of permutations of a subset of k elements in the group of permutations of $k+1$ elements, where $2 \leq k < n$. Let U be a one-dimensional representation of S_k. Can $\mathrm{Ind}_{S_k}^{S_{k+1}} U$ be irreducible? Justify your answer.

Question 3. (12 pts)

(a) Let $D_6 = \langle r, s \mid r^6 = 1, s^2 = 1, srs = r^{-1} \rangle$ be the dihedral group of symmetries of the regular hexagon. Describe the conjugacy classes in D_6.

(b) Describe the irreducible representations of D_6 and construct its character table.

(c) Consider the group $D_3 = \langle r, s \mid r^3 = 1, s^2 = 1, srs = r^{-1} \rangle$. Its character table is given by

	(1)	(r, r^2)	(s, sr, sr^2)
V_0	1	1	1
V_s	1	1	-1
V_2	2	-1	0

Compute the characters of $\mathrm{Ind}_{D_3}^{D_6} V_0$ and $\mathrm{Ind}_{D_3}^{D_6} V_2$ and decompose them into a direct sum of irreducible representations of D_6.

Question 4. (12 pts) Let Q_8 denote the group of quaternions, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with the defining relations

$$i = jk = -kj, \quad j = ki = -ik, \quad k = ij = -ji, \quad -1 = i^2 = j^2 = k^2.$$

The character table of Q_8 is given by

	1	-1	$\pm i$	$\pm j$	$\pm k$
$\lvert C_g \rvert$	1	1	2	2	2
χ_{11}	1	1	1	1	1
χ_{12}	1	1	-1	1	-1
χ_{13}	1	1	1	-1	-1
χ_{14}	1	1	-1	-1	1
χ_2	2	-2	0	0	0

(a) Compute the tensor power $V_2^{\otimes 3}$ and $V_2^{\otimes 4}$ of the 2-dimensional representation V_2 and decompose the obtained representations into a direct sum of irreducibles.

(b) Recall that for any representation V we have $V \otimes V = S^2V \oplus \wedge^2 V$ and that the characters of the symmetric and exterior tensor square of V are given by $\chi_{S^2V}(g) = \frac{1}{2}\left((\chi_V(g))^2 + \chi_V(g^2)\right)$, $\chi_{\wedge^2 V}(g) = \frac{1}{2}\left((\chi_V(g))^2 - \chi_V(g^2)\right)$. Compute the decomposition into irreducibles of the representations S^2V_2 and $\wedge^2 V_2$ for the two-dimensional irreducible representation V_2.

(c) Recall that an irreducible representation is of complex type if $V \not\cong V^*$, of real type if the trivial representation is a direct summand of S^2V, and of quaternionic type if the trivial representation is a direct summand of $\wedge^2 V$. Determine the type of each of the irreducible representations of Q_8.

Question 5. (14 pts)

(a) Let G be a finite group and $e \in G$ the unit of the group, and suppose that $\chi_V(g) = \chi_V(e)$ for some nontrivial element $g \neq e$ and some nontrivial complex irreducible representation V. Show that G has a nontrivial normal subgroup.

(b) A representation V of G is called *faithful*, if the homomorphism $\rho : G \to \mathrm{GL}(V)$ has trivial kernel. Show that V of G is faithful if and only if $\chi_V(g) = \chi_V(e)$ implies $g = e$, where $e \in G$ is the unit of the group.

(c) Let S_n be the symmetric group, $n \geq 3$. Give an example of a nontrivial Specht module V_λ such that $\chi_{V_\lambda}(g) = \chi_{V_\lambda}(e)$ for some element $g \neq e$.

(d) Conclude from (a) and (c) that S_n, $n \geq 3$ contains a proper nontrivial normal subgroup. Describe this subgroup.

Question 6. (12 pts)

(a) Let G be a finite group and $\mathbf{1} \subset G$ the trivial subgroup of one element. Show that the left regular representation of G is isomorphic to the induced representation from the trivial representation V_0 of the trivial subgroup:

$$\mathbb{C}[G] \simeq \mathrm{Ind}_{\mathbf{1}}^{G} V_0.$$

(b) Decompose the left regular representation $\mathbb{C}[S_4]$ as a direct sum of Specht modules.

(c) Let $V_{(2,1)}$ be the Specht module of S_3 corresponding to the partition $(2,1)$. Decompose $\mathrm{Ind}_{S_3}^{S_5} V_{(2,1)}$ into a sum of Specht modules for S_5. (You can describe the Specht module V_λ by the Young diagram Y_λ, where λ is a partition of 5).

Question 7. (6 pts) Let Y_λ be a Young diagram and $h(i,j)$ the length of the hook to the right and down from the square (i,j) in Y_λ. Show that for any $n \geq 2$ we have

$$\sum_{\lambda} \frac{n!}{\prod_{i \leq \lambda_j} (h_{ij})^2} = 1,$$

where the sum is over all partitions λ of n.

Question 8. (7 pts). (Yes/No)

Which of the following numbers always divide the order of a finite group G:

(a) Dimension of a complex irreducible representation.

(b) Number of inequivalent irreducible representations over $\mathbb{C}$.

(c) Number of elements in a conjugacy class in G.

(d) Number of conjugacy classes in G.

(e) Number of elements in the centralizer of an element in G.

(f) Order of an element in G.

(g) Square of the dimension of a complex irreducible representation.

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