

List of theorems

Below and at the exam all representations are supposed to be complex and finite dimensional.

Theorem 1. (PS 1) A representation of a finite group G is equivalent to a representation of the group algebra $\mathbb{C}[G]$.

Theorem 2. (Lecture 2) (Schur's lemma).

Let V_1, V_2 be representations of an algebra A . Let $\varphi : V_1 \rightarrow V_2$ be a nonzero homomorphism of representations. Then

- (a) If V_1 is irreducible, then φ is injective.
- (b) If V_2 is irreducible, then φ is surjective.
- (c) If V_1 and V_2 are irreducible representations over \mathbb{C} , then $\varphi = \lambda \text{Id}$, where $\lambda \in \mathbb{C}^*$.

Theorem 3. (Lecture 2). Every irreducible finite dimensional representation of a commutative algebra is one-dimensional.

Theorem 4. (Lecture 3) (Maschke's theorem).

Let G be a finite group. Then every finite dimensional complex representation of G is completely reducible, meaning that it is isomorphic to a direct sum of irreducible representations. The algebra $\mathbb{C}[G]$ is semisimple.

Theorem 5. (Lecture 3) (Weyl's unitary trick).

- (a) Every finite dimensional complex representation of a finite group is unitary.
- (b) Every finite dimensional unitary representation of any group is completely reducible.

Theorem 6. (Lecture 4). Let A be an associative algebra.

- (a) If $I \subset A$ is a left ideal, then $I \subset A$ is a subrepresentation of the left regular representation, and A/I is a quotient representation.
- (b) V is cyclic if and only if $V \simeq A/I$ for some left ideal $I \subset A$.
- (c) V is irreducible if and only if every vector $v \in V$ is cyclic.
- (d) V is irreducible if and only if $V \simeq A/I_{\max}$ for some maximal left ideal I_{\max} .

Theorem 7. (Lecture 4) (Density lemma)

Let V_i , $1 \leq i \leq m$ be irreducible finite dimensional complex representations of an algebra A . Let $V \simeq \bigoplus_{i=1}^m V_i^{\oplus n_i}$. Suppose that $W \subset V$ is a subrepresentation. Then $W \simeq \bigoplus_{i=1}^m V_i^{\oplus r_i}$, where $r_i \leq n_i$ for all i , and the inclusion $\varphi_i : V_i^{\oplus r_i} \rightarrow V_i^{\oplus n_i}$ is given by a constant $r_i \times n_i$ matrix X_i with complex entries.

Theorem 8. (Lecture 4) (Density theorem)

Let (V, ρ) be an irreducible finite dimensional complex representation of A .

- (a) Let $\{v_1, \dots, v_n\} \subset V$ be linearly independent vectors, and let $\{w_1, \dots, w_n\} \subset V$ be any set of vectors. Then there exist $a \in A$ such that $\rho(a)v_i = w_i$ for all $1 \leq i \leq n$.
- (b) The map $\rho : A \rightarrow \text{End}(V)$ is surjective.

Theorem 9. (PS 4) (Lecture 5) An irreducible representation of a matrix algebra $\text{Mat}_n(\mathbb{C})$ is isomorphic to \mathbb{C}^n . Irreducible representations of a finite direct sum of matrix algebras $A = \bigoplus_{i \in I} \text{Mat}_{n_i}(\mathbb{C})$ are $\{\mathbb{C}^{n_i}\}_{i \in I}$.

Theorem 10. (Lecture 5) (Structure theorem for finite dimensional algebras).

A finite dimensional complex algebra A has only finitely many inequivalent irreducible representations. Each irreducible representation is finite dimensional and

$$A/\text{Rad}(A) \simeq \bigoplus_{i=1}^n \text{End}(V_i),$$

where $\{V_i\}_{i=1}^n$ is a complete list of inequivalent irreducible representations of A .

Theorem 11. (Lecture 5) (Structure theorem for semisimple finite dimensional algebras).

Let A be a finite dimensional complex algebra. Then

$$A \text{ is semisimple} \iff A \simeq \bigoplus_{i=1}^n \text{Mat}_{n_i}(\mathbb{C}).$$

Theorem 12. (Lecture 5) (Linear independence of characters).

Let A be a complex associative algebra.

- (a) Characters of inequivalent irreducible finite dimensional representations of A are linearly independent.
- (b) If A is a finite dimensional semisimple algebra, then the irreducible characters form a basis in $(A/[A, A])^*$.

Theorem 13. (Lecture 6) (Linear independence of characters of a finite group)

Let G be a finite group. The characters of complex irreducible representations of G form a basis in the space of complex-valued class functions on G .

Theorem 14. (Lecture 6). Let G be a finite group. The number of isomorphism classes of irreducible representations of G equals to the number of conjugacy classes of G . Any finite dimensional complex representation of G is determined by its character: $\chi_V = \chi_W \iff V \simeq W$.

Theorem 15. (Lecture 6). Let G be a finite group. Then $\mathbb{C}[G]$ is semisimple and

$$\mathbb{C}[G] \simeq \bigoplus_{i=1}^r \text{End}(V_i),$$

where $\{V_i\}_{i=1}^r$ is the complete list of inequivalent irreducible representations of G , and r is the number of its conjugacy classes.

Theorem 16. (Lecture 6). Let V, W be finite dimensional complex representations of a finite group G . The character of the dual representation V^* is $\chi_{V^*}(g) = \overline{\chi_V(g)}$. The character of the tensor product $V \otimes W$ is $\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$.

Theorem 17. (Lecture 6) Let V, W be finite dimensional complex representations of a finite group G . Then

$$W \otimes V^* \simeq \text{Hom}(V, W)$$

as a representation of G .

Theorem 18. (Lecture 7) (1st orthogonality relation).

For any finite dimensional complex representations V, W of G we have

$$(\chi_V, \chi_W) := \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)} = \dim \text{Hom}_G(V, W).$$

If V, W are irreducible, then

$$(\chi_V, \chi_W) = \begin{cases} 1, & V \simeq W \\ 0, & V \not\simeq W \end{cases}$$

Theorem 19. (Lecture 7) (2nd orthogonality relation).

Let $g, h \in G$ and $Z_g = \{t \in G : tgt^{-1} = g\}$ the centralizer subgroup of g . Then

$$\sum_{V \in \text{Irr}(G)} \chi_V(g) \overline{\chi_V(h)} = \begin{cases} |Z_g| & \text{if } g \text{ is conjugate to } h \\ 0 & \text{otherwise} \end{cases}$$

Theorem 20. (Lecture 7) (3rd orthogonality relation).

- (a) Matrix elements of inequivalent irreducible representations of G are orthogonal under the form

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

- (b)

$$(t_{ij}^V, t_{kl}^V) = \delta_{i,k} \delta_{j,l} \frac{1}{\dim(V)}.$$

Theorem 21. (Lecture 8) (Symmetric and exterior product of vector spaces).

- (a) Let $\{e_i\}_{i=1}^k$ be a basis in V . Then $S^n V$ has a basis $\{e_{i_1} e_{i_2} \dots e_{i_n}\}_{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq k}$ and $\dim(S^n V) = \binom{n+k-1}{n}$.
- (b) Let $\{e_i\}_{i=1}^k$ be a basis in V . Then $\wedge^n V$ has a basis $\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n}\}_{1 \leq i_1 < i_2 < \dots < i_n \leq k}$, where $n \leq k$, and $\dim(\wedge^n V) = \binom{k}{n}$.

Theorem 22. (Lecture 8) (Action of G on $V^{\otimes n}$).

Let V be a finite dimensional complex representation of G . The action of G on $V^{\otimes n}$ commutes with the action of S_n by permutation of factors. Every S_n -isotypical component in $V^{\otimes n}$ is a subrepresentation of G in $V^{\otimes n}$.

Theorem 23. (Lecture 8) (Frobenius-Schur indicator).

Let V be an irreducible complex representation of a finite group G .

- If $V \not\simeq V^*$, then V is of complex type;
- If $V \simeq V^*$ and $V_0 \subset S^2 V$, then V is of real type;
- If $V \simeq V^*$ and $V_0 \subset \wedge^2 V$, then V is of quaternionic type.

Then the number of involutions $\{g \in G : g^2 = 1\}$ equals to $\sum_{V \text{ real}} \dim(V) - \sum_{V \text{ quaternionic}} \dim(V)$.

Theorem 24. (Lecture 9)

Let G be a finite group and V a complex irreducible representation of G . Then $\dim(V)$ divides $|G|$.

Theorem 25. (Lecture 9)

Let V be an irreducible complex representation of a finite group G , and C a conjugacy class in G such that $\gcd(|C|, \dim(V)) = 1$. Then either $\chi_V(g) = 0 \forall g \in C$, or any $g \in C$ acts as a scalar in V .

Theorem 26. (Lecture 9) (Burnside's theorem).

A group of order $p^a q^b$, where p, q are primes, is solvable.

Theorem 27. (Lecture 10) (Frobenius formula for the character of an induced representation).

Let $H \subset G$ be a subgroup and $\{x_\sigma\}_{\sigma \in H \backslash G}$ representatives of right cosets of H in G . Then the character χ of $\text{Ind}_H^G V$ is given by

$$\chi(g) = \sum_{\sigma \in H \backslash G : x_\sigma g x_\sigma^{-1} \in H} \chi_V(x_\sigma g x_\sigma^{-1}) = \frac{1}{|H|} \sum_{x \in G : x g x^{-1} \in H} \chi_V(x g x^{-1}).$$

Theorem 28. (Lecture 10) (Frobenius reciprocity).

Let $H \subset G$ be a subgroup, V a representation of G , W a representation of H . Then

$$\text{Hom}_G(V, \text{Ind}_H^G W) \simeq \text{Hom}_H(\text{Res}_H^G V, W).$$

Theorem 29. (Lecture 11) (Induction and restriction).

Let $H \subset G$ be a subgroup, V a representation of G , W a representation of H . Then

$$\text{Ind}_H^G W \simeq \text{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], W) \simeq \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$$

$$\text{Res}_H^G V \simeq \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], V) \simeq \mathbb{C}[G] \otimes_{\mathbb{C}[G]} V.$$

Theorem 30. (PS 13). (Transitivity of the induction) Let $K \subset H \subset G$ be subgroups of a finite group G and V a complex representation of K . Then

$$\text{Ind}_H^G \text{Ind}_K^H V \simeq \text{Ind}_K^G V.$$

Theorem 31. (Lecture 11) (Specht module for S_n).

Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_p)$ be a partition of n and Y_λ a Young diagram with rows λ_i filled with numbers from 1 to n . Let P_λ be the subgroup of permutations along the columns and Q_λ the subgroup of permutations along the rows of Y_λ . Let $a_\lambda = \sum_{g \in P_\lambda} g$, $b_\lambda = \sum_{g \in Q_\lambda} (-1)^g g$, and $c_\lambda = a_\lambda b_\lambda$. Then

$$V_\lambda = \mathbb{C}[S_n] c_\lambda$$

is the Specht module corresponding to the partition λ . The Specht modules $\{V_\lambda\}_{\lambda \text{ partition of } n}$ form the complete set of inequivalent irreducible representations of S_n .

Theorem 32. (Lecture 12).

Let A be an associative algebra and M a left A -module. Let $e \in A$ be an idempotent: $e^2 = e$. Then

$$\text{Hom}_A(Ae, M) \simeq eM.$$

Theorem 33. (Lecture 12). (Induced representations of S_n)

Let λ be a partition of n and define

$$U_\lambda = \text{Ind}_{P_\lambda}^{S_n} \mathbb{C}_{\text{triv}} \simeq \mathbb{C}[S_n]a_\lambda.$$

Then $U_\lambda = \bigoplus_{\mu \geq \lambda} K_{\lambda, \mu} V_\mu$, where V_μ are the Specht modules and $K_{\lambda, \mu}$ the Kostka numbers, $K_{\lambda, \lambda} = 1$.

Theorem 34. (Lecture 12) (Character of U_λ).

Let C_i be the conjugacy class in S_n of cycle type $(i_1, i_2, \dots, i_l, \dots)$ where i_l is the number of cycles of length l . Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p)$ be a partition of n . Let $N \geq p$ and $\{x_1, \dots, x_N\}$ be variables. Then the character $\chi_{U_\lambda}(C_i)$ is equal to the coefficient of $x^\lambda = \prod_j x_j^{\lambda_j}$ in the polynomial

$$\prod_{m \geq 1} H_m(x)^{i_m}$$

where $H_m(x) = (x_1^m + x_2^m + \dots + x_N^m)$.

Theorem 35. (Lecture 12) (Character of the Specht module V_λ).

Let C_i be the conjugacy class in S_n of cycle type $(i_1, i_2, \dots, i_l, \dots)$ where i_l is the number of cycles of length l . Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p)$ be a partition of n . Let $N \geq p$ and $\{x_1, \dots, x_N\}$ be variables. Then the character $\chi_{V_\lambda}(C_i)$ is equal to the coefficient of $x^{\lambda+\rho} = \prod_j x_j^{\lambda_j + N - j}$ in the polynomial

$$\Delta(x) \prod_{m \geq 1} H_m(x)^{i_m},$$

where

$$\Delta(x) = \prod_{1 \leq i < j \leq N} (x_i - x_j), \quad H_m(x) = (x_1^m + x_2^m + \dots + x_N^m).$$

Theorem 36. (Lecture 13) (Hook length formula).

Let V_λ be the Specht module corresponding to the partition λ of n . Then

$$\dim V_\lambda = \frac{n!}{\prod_{i \in \lambda_j} h(i, j)},$$

where $h(i, j)$ is the number of squares in the hook to the right and down from the square (i, j) in Y_λ .

Theorem 37. (PS 13) (Induction from S_{n-1} to S_n)

Let V_λ denote the Specht module for S_n , where λ is a partition of n . Then

- (a) $\text{Res}_{S_{n-1}}^{S_n} V_\lambda \simeq \bigoplus_{\mu \in R(\lambda)} V_\mu$, where $R(\lambda)$ is the set of Young diagrams obtained by removing one square from Y_λ .
- (b) $\text{Ind}_{S_{n-1}}^{S_n} V_\mu \simeq \bigoplus_{\lambda \in A(\mu)} V_\lambda$, where $A(\mu)$ is the set of Young diagrams obtained by adding one square from Y_μ .