Representation Theory - I MATH-314 **fall 2024** **Anna Lachowska**

Final exam

January 14, 2025

N° 0

Last name :

First name :

- No documents are allowed at the exam.
- Twice during the exam you are allowed to consult the book by Etingof and the typed list of theorems available at the front desk, for a total of 5 minutes.
- No electronic devices are allowed.
- Please provide clear, concise and easily readable arguments.
- You can answer in English or in French, but please do not mix the two languages.
- Scrap paper will not be graded.

Leave this space blank

Question	1	2	3	4	5	6	7	8	Total
max	10	12	12	12	14	12	6	7	85
score									

Question 1. (10 pts)

(a) Let G be a finite group. Suppose that every complex irreducible representation of G is one-dimensional. Show that G is abelian.

(b) Let C_n be the cyclic group of order $n \geq 1$. Show that the algebra $\mathbb{C}[C_n]$ has a basis $\{e_i\}_{i=1}^n$ such that $e_i e_j = \delta_{ij} e_i$. Express $1 \in \mathbb{C}[C_n]$ in this basis.

(a) Consider $\mathbb{C}[G]$. By the structure theorem for finite dimensional semisimple algebras (by Maschke's theorem $\mathbb{C}[G]$ is semisimple), we have $\mathbb{C}[G] \simeq \bigoplus_{i=1}^n \operatorname{End} V_i$ where $\{V_i\}_{i=1}^n$ is a complete list of inequivalent irreducible representations.

Since $\dim V_i = 1 \ \forall i \Rightarrow \operatorname{End} V_i \simeq \operatorname{Mat}_1(\mathbb{C}) \ \forall i$

$\Rightarrow \underline{\mathbb{C}[G] \simeq \bigoplus_{i=1}^n \operatorname{Mat}_1(\mathbb{C})} \simeq \bigoplus_{i=1}^n \mathbb{C}$, where each copy of $\mathbb{C}$ is a two-sided ideal in $\mathbb{C}[G]$.

Let $\underset{v_i \neq 0}{v_i \in \mathbb{C}}$ for i-th copy of $\mathbb{C} \Rightarrow v_i v_j = 0,\ i \neq j$ and $v_i^2 = k_i v_i$

Then $g_i = \sum a_{ij} v_j$ for any group element $g_i \in G$, since $\{v_i\}$ is a basis in $\mathbb{C}[G]$.

Therefore we have: $g_i g_j = \left(\sum a_{ik} v_k\right)\left(\sum b_{je} v_e\right) = g_j g_i \quad \forall g_i, g_j \in G$.

$\Rightarrow G$ is abelian.

(b) C_n is abelian $\Rightarrow$ by Schur's lemma all irreducible representations of C_n over $\mathbb{C}$ are 1-dimensional. ($\rho(g)$ commutes with the action of C_n on an irreducible $V \Rightarrow \rho(g)$ acts by $\lambda \mathrm{Id}$, $\lambda \neq 0$ since $g^n = 1$ $\Rightarrow \rho(g)$ acts by a scalar $\forall g \in C_n$ $\Rightarrow V$ decomposes into a $\oplus$ of one-dimensional subrepresent.)

$\Rightarrow \underline{\mathbb{C}[C_n] \simeq \bigoplus_{i=1}^n \operatorname{Mat}_1(\mathbb{C})}$ as above

$\Rightarrow \exists$ a basis in $\mathbb{C}[C_n]$ with $\{e_i\}_{i=1}^n$ s.t. $e_i e_j = 0,\ i \neq j$

After rescaling $e_i^2 = e_i \quad \Rightarrow e_i e_j = \delta_{ij} e_i$

$1 \in \mathbb{C}[C_n]$ is such that $1 \cdot \sum a_{ij} e_i = \sum a_{ij} e_i \Leftrightarrow 1 e_i = e_i \ \forall i$

$\Rightarrow 1 = \sum x_{ij} e_j \Rightarrow \sum x_{ij} e_j e_i = x_{ii} e_i = e_i \Rightarrow x_{ii} = 1 \ \forall i$

$\Rightarrow 1 = \sum_{i=1}^n e_i$

Question 2. (12 pts) Let G be a finite group and $H \subset G$ be a proper subgroup.

(a) Let V_0 be the trivial representation of H. Show that the induced representation $\mathrm{Ind}_H^G V_0$ is not irreducible. *Hint:* Use Frobenius reciprocity.

(b) Describe all possible one-dimensional representations of the symmetric group S_k $k \geq 2$. Justify your answer. *Hint:* Recall that transpositions generate S_k and consider the possible action of a transposition in a one-dimensional representation.

(c) Let $S_k \subset S_{k+1}$ be a subgroup of permutations of a subset of k elements in the group of permutations of $k+1$ elements, where $2 \leq k < n$. Let U be a one-dimensional representation of S_k. Can $\mathrm{Ind}_{S_k}^{S_{k+1}} U$ be irreducible? Justify your answer.

(a) Frobenius reciprocity: Let W_0 be the trivial representation of G.

Then $\mathrm{Hom}_G(W_0, \mathrm{Ind}_H^G V_0) = \mathrm{Hom}_H(\underset{\underset{V_0}{\shortparallel}}{\mathrm{Res}_H^G W_0}, V_0) = \mathrm{Hom}_H(V_0, V_0) = \mathbb{C}$

$\Rightarrow$ $\mathrm{Ind}_H^G V_0 \simeq W_0 \oplus W$ contains a trivial subrepresentation.

$\dim \mathrm{Ind}_H^G V_0 = [G:H] \cdot \dim V_0 > \dim V_0$ since $H \subset G$ is proper

$\Rightarrow$ $W \neq 0$. $\Rightarrow$ $\mathrm{Ind}_H^G V_0$ is not irreducible.

(b) Let V be a 1-dimensional representation of S_k. Since the transpositions generate S_k it is enough to define the action of the transpositions in V.

Note that $gtg^{-1} = t'$ where t, t' are transpositions and $g \in S_k$

$\Rightarrow$ $\rho(g)\rho(t)\rho(g)^{-1} = \rho(g)^{-1}\rho(g)\rho(t) = \rho(t')$ in V since $\dim V = 1$.

$\Rightarrow$ the action of all transpositions in V is by the same $\overset{\rho(t)}{\text{number}} \in \mathbb{C}$.

We have $\rho(t)^2 = \rho(t^2) = \rho(1) = 1$ $\Rightarrow$ $\rho(t) = \pm 1$

If $\rho(t) = 1$ $\Rightarrow$ $V = V_0$ trivial

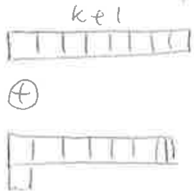
If $\rho(t) = -1$ $\Rightarrow$ $V = V_{sign}$ the sign representation.

$\Rightarrow$ only two possibilities: V_0 and V_{sign}.

(c) $S_k < S_{k+1}$ $\Rightarrow$ $\mathrm{Ind}_{S_k}^{S_{k+1}} U$ is the direct sum of Specht modules V_λ where λ is obtained from the Young diagram for U by adding one square. (theorem from the course)

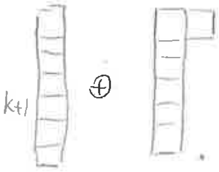
If $\mathcal{U} = \mathcal{U}_0$ trivial =>

$\mathcal{U} \to \mathrm{Ind}_{S_k}^{S_{k+1}}\mathcal{U} =$



If $\mathcal{U} = \mathcal{U}_{sign}$ sign =>

$\to \mathrm{Ind}_{S_k}^{S_{k+1}}\mathcal{U} =$



In both cases $\underline{\mathrm{Ind}_{S_k}^{S_{k+1}}\mathcal{U} \text{ is not irreducible}}$

Question 3. (12 pts)

(a) Let $D_6 = \langle r, s \mid r^6 = 1, s^2 = 1, srs = r^{-1} \rangle$ be the dihedral group of symmetries of the regular hexagon. Describe the conjugacy classes in D_6.

(b) Describe the irreducible representations of D_6 and construct its character table.

(c) Consider the group $D_3 = \langle r, s \mid r^3 = 1, s^2 = 1, srs = r^{-1} \rangle$. Its character table is given by

	(1)	(r, r^2)	(s, sr, sr^2)
V_0	1	1	1
V_s	1	1	-1
V_2	2	-1	0

Compute the characters of $\mathrm{Ind}_{D_3}^{D_6} V_0$ and $\mathrm{Ind}_{D_3}^{D_6} V_2$ and decompose them into a direct sum of irreducible representations of D_6.

(a) Conjugacy classes of D_6 : $\{ (1) ; (r^3) \; (r, r^5) \; (r^2, r^4), (s, sr^2, sr^4)$

central

$(sr, sr^3, sr^5) \}$

$srs = r^{-1}, \quad rsr^{-1} = sr^{-2}$ $\quad \underbrace{6 \text{ conj. classes}}$

(b) $K = \{1, r^2, r^4\}$ is a normal subgroup in $D_6 \Rightarrow D_6/K \simeq (r, s : r^2 = 1, s^2 = 1, rs = sr)$

$= C_2 \times C_2$ abelian

$\Rightarrow C_2 \times C_2$ has 4 1-dim representations where s, r act by ± 1

Sum of squares formula $\Rightarrow |D_6| = 12 = 1 + 1 + 1 + 1 + d_1^2 + d_2^2 \Rightarrow d_1 = d_2 = 2$

Action of D_6 by symmetries of a regular hexagon gives 2 more representation of dim = 2. $\rho(r) = \mathrm{diag}(e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}})$ $\rho(r) = \mathrm{diag}(e^{\frac{\pi i}{3}}, e^{\frac{-\pi i}{3}})$ (see PS2 for classification of representations of D_n)

	1	r^3	r, r^5	r^2, r^6	s, sr^2, sr^4	sr, sr^3, sr^5
V_{++}	1	1	1	1	1	1
V_{+-}	1	1	1	1	-1	-1
V_{-+}	1	-1	-1	1	1	-1
V_{--}	1	-1	-1	1	-1	1
$V_{\pi/3}$	2	-2	1	-1	0	0
$V_{2\pi/3}$	2	2	-1	-1	0	0
$\mathrm{Ind}\, V_2$	4	0	0	-2	0	0

One can check the orthogonality of characters in the table.

(c) $D_3 \subset D_6$, generated by $\{r^2, s\}$: $D_3 = \langle 1, r^2, r^4, sr^2, sr^4, s \rangle$

Which irreducible representations of D_6 reduce to V_0 over D_3 ?

$\rho(r^2) = 1, \rho(s) = 1 \Rightarrow V_{++}, V_{-+}$: Frobenius reciprocity:

$\mathrm{Hom}_{D_6}(V_{++}, \mathrm{Ind}_{D_3}^{D_6} V_0) = \mathrm{Hom}_{D_3}(\mathrm{Res}_{D_3}^{D_6} V_{++}, V_0) = \mathbb{C}$

$\mathrm{Hom}_{D_6}(V_{-+}, \mathrm{Ind}_{D_3}^{D_6} V_0) = \mathrm{Hom}_{D_3}(\mathrm{Res}_{D_3}^{6\, D_6} V_{-+}, V_0) = \mathbb{C}$

$\Rightarrow \underbrace{\mathrm{Ind}_{D_3}^{D_6} V_0 = V_{++} \oplus V_{-+}}_{\dim = 2 = [D_6 : D_3]}$

$\text{Ind}_{D_3}^{D_6} V_2 = W$. Frobenius formula:

$$\chi(g) = \sum_{\substack{xgx^{-1}\in D_3 \\ x\in\{1,r\}\ \text{cosets}}} \chi_{V_2}(xgx^{-1}) \Rightarrow \chi(1) = \chi_{V_2}(1) + \chi_{V_2}(1) = 4$$

$$\chi(r) = 0$$

$$\chi(r^2) = \chi_{V_2}(r^2) + \chi_{V_2}(r^2) = -2$$

$$\chi(s) = \cancel{\chi}_{V_2}(s) + \chi_{V_2}(s) = 0$$

$$\chi(r^3) = 0$$

$$\chi(sr) = 0$$

From the character table we have:

$$\underbrace{\text{Ind}_{D_3}^{D_6} V_2 \simeq V_{\pi/3} \oplus V_{2\pi/3}}$$

Question 4. (12 pts) Let Q_8 denote the group of quaternions, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with the defining relations

$$i = jk = -kj, \quad j = ki = -ik, \quad k = ij = -ji, \quad -1 = i^2 = j^2 = k^2.$$

The character table of Q_8 is given by

	1	-1	$\pm i$	$\pm j$	$\pm k$
$\lvert C_g \rvert$	1	1	2	2	2
χ_{11}	1	1	1	1	1
χ_{12}	1	1	-1	1	-1
χ_{13}	1	1	1	-1	-1
χ_{14}	1	1	-1	-1	1
χ_2	2	-2	0	0	0

(a) Compute the tensor power $V_2^{\otimes 3}$ and $V_2^{\otimes 4}$ of the 2-dimensional representation V_2 and decompose the obtained representations into a direct sum of irreducibles.

(b) Recall that for any representation V we have $V \otimes V = S^2V \oplus \wedge^2 V$ and that the characters of the symmetric and exterior tensor square of V are given by $\chi_{S^2V}(g) = \frac{1}{2}\left((\chi_V(g))^2 + \chi_V(g^2)\right)$, $\chi_{\wedge^2 V}(g) = \frac{1}{2}\left((\chi_V(g))^2 - \chi_V(g^2)\right)$. Compute the decomposition into irreducibles of the representations S^2V_2 and $\wedge^2 V_2$ for the two-dimensional irreducible representation V_2.

(c) Recall that an irreducible representation is of complex type if $V \not\simeq V^*$, of real type if the trivial representation is a direct summand of S^2V, and of quaternionic type if the trivial representation is a direct summand of $\wedge^2 V$. Determine the type of each of the irreducible representations of Q_8.

(a) We compute the characters of $V_2^{\otimes 3}$ and $V_2^{\otimes 4}$: $\chi_{V_1 \otimes V_2}(g) = \chi_{V_1}(g)\chi_{V_2}(g)$

	1	-1	$\pm i$	$\pm j$	$\pm k$
$\chi_{V_2^{\otimes 3}}$	8	-8	0	0	0
$\chi_{V_2^{\otimes 4}}$	16	16	0	0	0

$\Rightarrow \chi_{V^{\otimes 3}}(g) = \chi_V^3(g)$

$\chi_{V^{\otimes 4}}(g) = \chi_V^4(g)$

$\Rightarrow \underline{V_2^{\otimes 3} \simeq 4\, V_2}$

$\underline{V_2^{\otimes 4} \simeq 4(V_{11} \oplus V_{12} + V_{13} + V_{14})}$

Also can compute $\langle \chi_{V^{\otimes 3}}, \chi_W \rangle$ etc.

(b)

χ_{S^2V}	3	3	-1	-1	-1
$\chi_{\wedge^2 V}$	1	1	1	1	1

$\chi_{S^2V}(1) = \frac{1}{2}(4+2) = 3$ $\quad \chi_{S^2V}(\pm i) = \frac{1}{2}(0-2) = -1$

$\chi_{S^2V}(-1) = \frac{1}{2}(4+2) = 3$ $\quad \pm j$, $\pm k$

$\Rightarrow \underline{S^2 V \simeq V_{12} \oplus V_{13} \oplus V_{14}},$

$\chi_{\wedge^2 V}(1) = \frac{1}{2}(4-2) = 1$ $\quad \chi_{\wedge^2 V}(\pm i) = \frac{1}{2}(0+2) = 1$

$-1 \quad \frac{1}{2}(4-2) = 1$ $\quad \pm j$, $\pm k$

$\Rightarrow \underline{\wedge^2 V \simeq V_{11}}.$

(c) If $\dim V = 1 \Rightarrow V \otimes V \simeq S^2 V$; Since the characters of $V_{11}, V_{12}, V_{13}, V_{14}$ are real

$\Rightarrow \overline{\chi_{V_{xx}}} = \chi_{V_{xx}} \Rightarrow V_{xx}^{*} \simeq V_{xx}$ in each case;

also $\chi_{V_{xx}}^2(g) = 1 \ \forall g \in Q_8 \Rightarrow V_0 \simeq V_{xx} \otimes V_{xx}$

for $xx \in \{11, 12, 13, 14\}$

=> representations $V_{11}, V_{12}, V_{13}, V_{14}$ are of real type. $V_0 \subset S^2 V$

In (b) we computed: $S^2 V_2 \simeq V_{12} \oplus V_{13} \oplus V_{14}$

$\Lambda^2 V_2 \simeq V_{11} \simeq V_0$ the trivial representation

=> $V_0 < \Lambda^2 V_2$ => V_2 is of quaternionic type.

Question 5. (14 pts)

(a) Let G be a finite group and $e \in G$ the unit of the group, and suppose that $\chi_V(g) = \chi_V(e)$ for some nontrivial element $g \neq e$ and some nontrivial complex irreducible representation V. Show that G has a nontrivial normal subgroup.

(b) A representation V of G is called *faithful*, if the homomorphism $\rho : G \to \mathrm{GL}(V)$ has trivial kernel. Show that V of G is faithful if and only if $\chi_V(g) = \chi_V(e)$ implies $g = e$, where $e \in G$ is the unit of the group.

(c) Let S_n be the symmetric group, $n \geq 3$. Give an example of a nontrivial Specht module V_λ such that $\chi_{V_\lambda}(g) = \chi_{V_\lambda}(e)$ for some element $g \neq e$.

(d) Conclude from (a) and (c) that S_n, $n \geq 3$ contains a proper nontrivial normal subgroup. Describe this subgroup.

(a) Let $\chi_V(e) = \chi_V(g) = \dim V$, $\chi_V(g) = \sum_{i=1}^{\dim V} \xi_i$ where ξ_i are roots of 1

$\left|\sum_{i=1}^{\dim V} \xi_i\right| \leq \dim V$, with equality possible $\Leftrightarrow \xi_i = \xi_j$,

then $\sum \xi_i = \dim V(\xi_i) \Rightarrow \xi_i = 1$

$\Rightarrow \rho(g) = Id \Rightarrow g \in \ker\rho$, $\rho : G \to GL(V)$ group homomorphism.

$\ker\rho \lhd G$ is normal in G; $\exists g \neq e : g \in \ker\rho \Rightarrow \ker\rho$ is nontrivial

$\Rightarrow \underline{\ker\rho \lhd G}$ and $\ker\rho \neq \{e\}$.

(b) If $\exists g \in G : \chi_V(g) = \chi_V(e) \Rightarrow \ker\rho : G \to GL(V)$ is nontrivial, $g \in \ker\rho$ by (a)

Therefore, If V is faithful $\Rightarrow$ $\chi_V(g) = \chi_V(e)$ implies $g = e$.

If V is not faithful $\Rightarrow \exists g \neq e : \rho_V(g) = Id \Rightarrow \chi_V(g) = \dim V = \chi_V(e)$.

(c) Let $\underline{V_\lambda = V_{sign}}$, then by definition $\rho_{sign}(g) = (-1)^g = 1$ $\forall$ even g,

for example for any 3-cycle or any product of 2 ba

(d) $\exists$ a 3-cycle $(123) \subset S_n$, $n \geq 3 \Rightarrow g = (123)$ acts trivially on the V_{sign}

and $\chi_{V_{sign}}((123)) = 1 = \chi_{V_s}(1) \Rightarrow S_n$, $n \geq 3$ contains a normal subgroup.

Let $H \subset S_n$ be the subgroup that contains all even permutations

Clearly $g_{ev} \cdot h_{ev} = k_{ev}$ since $S_n \to \mathbb{C}$
g_{ev}^{-1} is even $\qquad g \to (-1)^g$ is a group homomorphism

$\Rightarrow \underline{A_n \lhd S_n}$ is a subgroup, the alternating group.

Question 6. (12 pts)

(a) Let G be a finite group and $\mathbf{1} \subset G$ the trivial subgroup of one element. Show that the left regular representation of G is isomorphic to the induced representation from the trivial representation V_0 of the trivial subgroup:

$$\mathbb{C}[G] \simeq \mathrm{Ind}_{\mathbf{1}}^{G} V_0.$$

(b) Decompose the left regular representation $\mathbb{C}[S_4]$ as a direct sum of Specht modules.

(c) Let $V_{(2,1)}$ be the Specht module of S_3 corresponding to the partition $(2,1)$. Decompose $\mathrm{Ind}_{S_3}^{S_5} V_{(2,1)}$ into a sum of Specht modules for S_5. (You can describe the Specht module V_λ by the Young diagram Y_λ, where λ is a partition of 5).

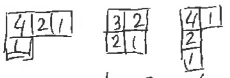
(a) $\mathbb{1} < G$ trivial subgroup.

$$\mathrm{Ind}_{\mathbb{1}}^{G} V_0 = \mathbb{C}[G] \underset{\mathbb{C}[\mathbb{1}]}{\otimes} V_0 \simeq \mathbb{C}[G] \otimes \mathbb{C} = \mathbb{C}[G]$$

(b) Method (1) : $\mathbb{C}[S_4] \simeq \oplus V_\lambda^{\oplus \dim V_\lambda} \simeq V_{(4)} \oplus V_{(3,1)}^{\oplus 3} \oplus V_{(2,2)}^{\oplus 2} \oplus V_{(2,1,1)}^{\oplus 3} + V_{(1,}$

4: (4), (3,1), (2,2), (2,1,1) (1,1,1,1)

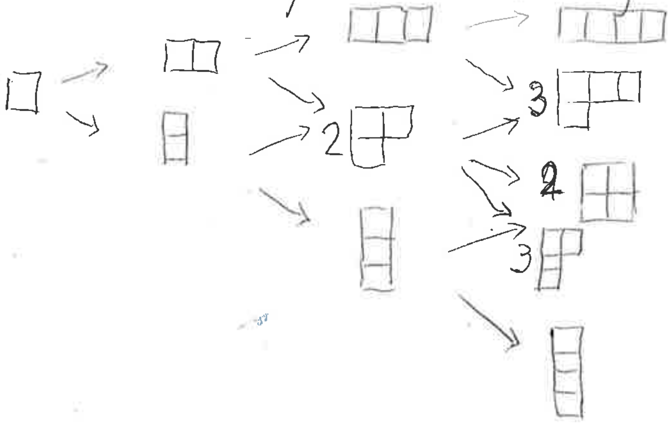
dim=1 dim=1



dim=3 dim=2 dim=3

Method (2) : Using (a) and transitivity of the induction:

$\mathbb{C}[S_4] = \mathrm{Ind}_{\mathbb{1}}^{S_4} V_0 \simeq \mathrm{Ind}_{S_3}^{S_4} \mathrm{Ind}_{S_2}^{S_3} \mathrm{Ind}_{\mathbb{1}}^{S_2} V_0$; at each step add a square to the Young diagram:

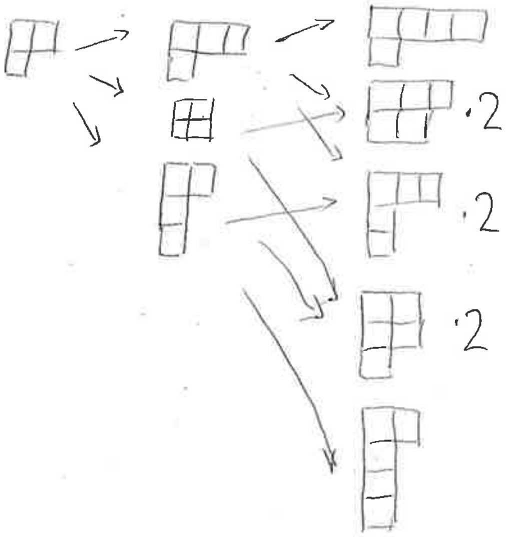


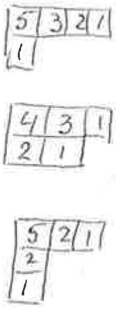
$\Rightarrow \mathbb{C}[S_4] \simeq V_4 \oplus V_{(3,1)}^{\oplus 3} \oplus V_{(2,2)}^{\oplus 2} \oplus$

$\oplus V_{(2,1,1)}^{\oplus 3} + V_{(1,1,1,1)}$

(c) Transitivity of the induction:

$$\operatorname{Ind}_{S_3}^{S_5} V_{(2,1)} = \operatorname{Ind}_{S_4}^{S_5} \operatorname{Ind}_{S_3}^{S_4} V_{(2,1)} \simeq \underbrace{V_{(4,1)} \oplus V_{(3,2)}^{\oplus 2} \oplus V_{(3,1,1)}^{\oplus 2} \oplus V_{(2,2,1)}^{\oplus 2} + V_{(2,1,1,1)}}$$





dim = 4

dim = 5

dim = 6

Check the dimensions: $\dim \operatorname{Ind}_{S_3}^{S_5} V_{(2,1)} = [S_5 : S_3] \cdot \dim V_{(2,1)} = 20 \cdot 2 = 40$

$\dim(\oplus) = 2 \cdot 4 + 4 \cdot 5 + 2 \cdot 6 = 8 + 20 + 12 = 40$:)

Question 7. (6 pts) Let Y_λ be a Young diagram and $h(i,j)$ the length of the hook to the right and down from the square (i,j) in Y_λ. Show that for any $n \geq 2$ we have

$$\sum_\lambda \frac{n!}{\prod_{i\leq\lambda_j}(h_{ij})^2} = 1,$$

where the sum is over all partitions λ of n.

We have: $\mathbb{C}[S_n] \simeq \bigoplus_\lambda V_\lambda^{\oplus \dim V_\lambda}$

$$\Rightarrow \dim \mathbb{C}[S_n] = n! = \sum_\lambda (\dim V_\lambda)^2$$

Hook length formula: $\dim V_\lambda = \frac{n!}{\prod_{i\leq\lambda_j} h(i,j)}$

$$\Rightarrow n! = \sum_\lambda \left(\frac{n!}{\prod_{i\in\lambda_j}(h(i,j))}\right)^2 \Rightarrow \underline{1 = \sum_\lambda \frac{n!}{\prod_{i\leq\lambda_j}(h_{ij})^2}} \; . \quad ▨$$

Question 8. (7 pts). (Yes/No)

Which of the following numbers always divide the order of a finite group G:

(a) Dimension of a complex irreducible representation. Y

(b) Number of inequivalent irreducible representations over $\mathbb{C}$. N: $G=Q_8$, 5 irred. repres.

(c) Number of elements in a conjugacy class in G. Y $|C_g| = [G: Z_g]$

(d) Number of conjugacy classes in G. N : $G=Q_8$ 5 conj classes

(e) Number of elements in the centralizer of an element in G. Y $Z_g \subset G$ is a subgroup

(f) Order of an element in G. Y

(g) Square of the dimension of a complex irreducible representation. N: $G=S_3$, $(\dim V_2)^2=4$