Fall 2025

Representation Theory I

1 Motivations

Symmetries of objects are described by groups. However the question of classification of all groups or objects they act on is a hard one. Representation theory studies linear action of groups on vector spaces, thus linearizing the question and bringing it to the domain of linear algebra. Instead of studying a group, we study group homomorphisms $G \to \mathrm{GL}(V)$ for a vector space V. One of the problems of representation theory is to classify the vector spaces where the group can act linearly and the ways it can act on a given vector space. For example a group of symmetries of a regular tetrahedron acts by matrices in the three dimensional real vector space.

In guise of a quick advertisement, here are a few examples of what representation theory can achieve in mathematics

- Many properties of the symmetric group S_n can be understood using its representation theory, which is extensively studied.
- The proof of Fermat's Last Theorem by Andrew Wiles uses methods of representation theory.
- The nature of the largest sporadic simple finite group, the Monster, was understood with the help of its representations.

1.1 Organization of the course

Here is the list of lectures by topic and main result. On the right you can see the goals of various parts of the course.

Lecture 1.	Definitions	$\Longrightarrow$ Basics of representation theory
Lecture 2.	Schur's lemma	
Lecture 3.	Maschke't theorem Weyl's unitary trick	$\Longrightarrow$ Semisimplicity of $\mathbb{C}[G]$
Lecture 4.	Density theorem	$\Longrightarrow$ Structure of $\mathbb{C}[G]$
Lecture 5.	Structure theorem for finite dimensional algebras	
Lecture 6.	Characters of representations	$\Longrightarrow$ Character determines a complex representation of G
Lecture 7.	Orthogonality relations for characters	
Lecture 8.	Tensor products and tensor powers of representations	
Lecture 9.	dim$V$ divides $\lvert G\rvert$ Burnside's theorem	$\Longrightarrow$ Applications to group theory
Lecture 10.	Induced representations Frobenius reciprocity	$\Longrightarrow$ Construction of new representations
Lecture 11.	Classification of the irreducible representations of S_n	$\Longrightarrow$ Representations of S_n over $\mathbb{C}$
Lecture 12.	Character of an induced representation of S_n	
Lecture 13.	Character of a Specht module and the hook length formula	
Lecture 14.	Schur-Weyl duality	$\Longrightarrow$ Relation between representations of S_n and representations of $\mathrm{GL}(V)$

1.2 First motivational example

Suppose you have a group of people around a round table with a bowl of peanuts in front of each person. Denote the weight of peanuts by a_i where $i = 1, \dots n$. They play the following game: at a given moment each person divides their peanuts into two equal parts and gives these to their neighbors on the left and on the right. Let us call this operation

M:

$$M(a_i) = \frac{1}{2}(a_{i-1} + a_{i+1}),$$

where the indices are counted modulo n. Question: what will happen after a large number N of iterations? Does the limit $\lim_{N\to\infty} M^N(a_i)$ exist? Does it depend on the index i or on the number of people around the table? Let us consider two examples: $n = 3$

$$\begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \longrightarrow \begin{array}{ccc} & 0 & \\ \frac{1}{2} & & \frac{1}{2} \end{array} \longrightarrow \begin{array}{ccc} & \frac{1}{2} & \\ \frac{1}{4} & & \frac{1}{4} \end{array} \longrightarrow \begin{array}{ccc} & \frac{1}{4} & \\ \frac{3}{8} & & \frac{3}{8} \end{array} \longrightarrow \begin{array}{ccc} & \frac{3}{8} & \\ \frac{5}{16} & & \frac{5}{16} \end{array} \longrightarrow \dots$$

Example $n = 4$:

$$\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \longrightarrow \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array} \longrightarrow \begin{array}{cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array} \longrightarrow \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array} \longrightarrow \dots$$

To solve this question, we can use methods of representation theory. Let V be the vector space of states of this system, where a vector $(a_1, a_2, \dots a_n)$ shows the amount of peanuts in each bowl at a given moment. Let $\{e_i\}_{i=1}^n$ be the standard basis of V. Let $C_n = \langle t : t^n = 1\rangle$ be the cyclic group of n elements and define group homomorphisms $L : C_n \to \mathrm{GL}(V)$ and $R : C_n \to \mathrm{GL}(V)$ by setting

$$L : e_i \to e_{i-1}, \qquad R : e_i \to e_{i+1},$$

where the indices are counted modulo n. Then one can notice that $M = \frac{1}{2}(L + R)$ and study the action of M in terms of the *representations* of C_n given by L and R on V. *Hint:* Find the invariant subspaces of operators L and R on V and consider the action of M of these subspaces. We will return to this question later.

1.3 Second motivational example

Suppose we need to find the dimension of the space of homogeneous polynomials of degree n in two variables, that satisfy the conditions

$$\begin{cases} p(u, w) = p(u + w, -u) \\ p(u, w) = -p(w, u) \end{cases}$$

We can approach this question from the viewpoint of representation theory of a group generated by the given transformations of the variables. Let U be the two dimensional vector space with the basis $\{u, w\}$. Then the given symmetries of the basis in U can be written as

$$a \cdot \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} w \\ u \end{pmatrix}; \qquad b \cdot \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u + w \\ -u \end{pmatrix}$$

Do a and b generate a nice group? We have

$$a^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1, \qquad b^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \implies b^6 = 1,$$

$$aba^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = b^{-1}.$$

These are the relations of the dihedral group $D_6 = \langle a, b \ : \ a^2 = 1, b^6 = 1, aba^{-1} = b^{-1}\rangle$. We can extend its action to the vector space V of homogeneous polynomials of degree n in variable u, w with the basis $\{u^n, u^{n-1}w, \dots uw^{n-1}, w^n\}$. We can apply then the theory of representations of finite groups to this question. More theory needs to be developed to obtain a solution.

In this course we will mostly consider representations of finite groups in finite dimensional complex vector spaces. However we will start with a more general setting.

2 Representations of associative algebras

Definition 2.1. An associative algebra A over a field k is a k-vector space with a bilinear map $\cdot : A \times A \to A$ such that

(1) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any $a, b, c \in A$,

(2) There exist $1 \in A$ such that $1 \cdot a = a \cdot 1 = a$ for any $a \in A$.

Below we will normally write ab for the product $a \cdot b$ of elements of an associative algebra.

Definition 2.2. A homomorphism of associative algebras $f : A \to B$ is a k-linear map such that

(1) $f(ab) = f(a)f(b)$ for any $a, b \in A$,

(2) $f(1_A) = 1_B$.

A homomorphism $f : A \to B$ is an isomorphism of algebras if there exists an algebra homomorphism $g : B \to A$ such that $f \circ g = \mathrm{Id}_B$, $g \circ f = \mathrm{Id}_A$. Equivalently, $f : A \to B$ is an algebra isomorphism if it is an algebra homomorphism and an isomorphism of vector spaces.

2.1 Examples of associative algebras

(1) $A = k$ is a 1-dimensional associative algebra.

(2) The algebra $\mathrm{End}_k(V)$ of linear transformations $V \to V = k^n$ is the matrix algebra $\mathrm{Mat}_n(k)$, with $\dim(\mathrm{Mat}_n(k)) = n^2$.

(3) Free algebra $A = k\langle x_1, x_2, \dots x_n\rangle$ with elements given by k-linear combinations of words in the alphabet $\{x_1, x_2, \dots x_n\}$ and the multiplication given by concatenation of words. This is a infinite dimensional algebra over k.

(4) Group algebra $k[G]$ of a finite group G. Elements are of the form $\sum_{g_i \in G} \lambda_i g_i$, where $\{g_i\}_{i=1}^n$ is the list of elements of G and $\lambda_i \in k$ for all $i = 1 \dots n$. The product is determined by the group law $g \cdot h = gh \in G$ and extended by linearity on $k[G]$. The dimension of $k[G]$ equals to the order of the group G.

Definition 2.3. An algebra A is commutative if $ab = ba$ for any $a, b \in A$.

Example (1) is commutative. Example (2) is commutative if and only if $\dim(V) = 1$. Example (3) is commutative if and only if $n = 1$ and $A = k\langle x_1\rangle = k[x_1]$ is a polynomial algebra. Example (4) is commutative if and only if G is abelian.

2.2 Representations of associative algebras

Definition 2.4. A representation of an associative algebra A over the field k is a pair (V, ρ), where V is a k-vector space and $\rho : A \to \mathrm{End}_k(V)$ is an algebra homomorphism. Explicitly, we have

$$\rho(ab) = \rho(a)\rho(b) \quad \forall a, b \in A, \qquad \rho(1_A) = \mathrm{Id}_V.$$

Definition 2.5. Let (V_1, ρ_1) and (V_2, ρ_2) be two representations of an algebra A over k. A homomorphism $\phi : V_1 \to V_2$ of representations is a k-linear map that commutes with the action of A:

$$\phi(\rho_1(a)v) = \rho_2(a)\phi(v) \quad \forall a \in A, v \in V.$$

Thus we have a commutative diagram

$$\begin{tikzcd} V_1 \arrow[r, "\phi"] \arrow[d, "\rho_1(a)"] & V_2 \arrow[d, "\rho_2(a)"] \\ V_1 \arrow[r, "\phi"] & V_2 \end{tikzcd}$$

The same property can be expressed by the equality

$$\phi \circ \rho_1 = \rho_2 \circ \phi.$$

Then ϕ is called an *intertwiner* of the representations ρ_1 and ρ_2. Homomorphisms of representations $(V_1, \rho_1) \to (V_2, \rho_2)$ form a vector space $\mathrm{Hom}_A(V_1, V_2)$. A homomorphism $\phi : V_1 \to V_2$ is an isomorphism of representations if there exists a homomorphism $\psi : V_2 \to V_1$ such that $\phi \circ \psi = \mathrm{id}_{V_2}$ and $\psi \circ \phi = \mathrm{Id}_{V_1}$.

Definition 2.6. A subrepresentation of (V, ρ) is a subspace $W \subset V$ that is closed under the action of A, meaning that

$$\rho(a)W \subset W \quad \forall a \in A.$$

Examples: $0 \subset V$ and $V \subset V$ are subrepresentations for any representation (V, ρ).

3 Representations of groups vs. representations of algebras

Definition 3.1. A representation of a group G over a field k is a couple (V, ρ), where V is a k-vector space and $\rho : G \to \mathrm{GL}(V)$ a group homomorphism, meaning that

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2) \quad \forall g_1, g_2 \in G,$$

$$\rho(1) = \mathrm{Id}_V.$$

If $\rho : G \to \mathrm{GL}(V)$ is injective, the representation (V, ρ) is called *faithful*.

Lemma 3.2. *[PS 1]. A representation of G in a vector space V over k uniquely determines the representation in V of the group algebra $k[G]$, and conversely.*

3.1 Examples of representations of groups and algebras

(1) Trivial representation $1 : G \to \mathrm{GL}(V)$ given by $g \to \mathrm{Id}_V$ for any $g \in G$.

(2) Let G be a finite group, $V = k[G]$ the vector space of the group algebra with the basis $\{g\}_{g \in G}$. Then $\rho : G \to \mathrm{GL}(V)$ given by $\rho(h)g = hg$ is a representation of G called *the left regular representation.*

(3) More generally, if A is an associative algebra, then $\rho : A \to \mathrm{End}_k(A)$ given by $\rho(a)b = ab$ for all $a, b \in A$ is the left regular representation of A.

(4) In particular, if $A = k[G]$, then the left regular representation of G on $k[G]$ defines according to Lemma 3.2 the left regular representation of the algebra A.

4 Irreducible and indecomposable representations

Below we will consider representations of associative algebras, but the same theory applies to representations of groups.

Definition 4.1. A representation (V, ρ) of an algebra is *irreducible* if $0 \subset V$ and $V \subset V$ are the only subrepresentations in V.

It follows that any one-dimensional representation is irreducible.

Definition 4.2. Let (V_1, ρ_1) and (V_2, ρ_2) be two representations of an algebra A. Then $(V_1 \oplus V_2, \rho_1 \oplus \rho_2)$ is also a representation of A, defined by

$$\rho(a) = \begin{pmatrix} \rho_1(a) & 0 \\ 0 & \rho_2(a) \end{pmatrix} \qquad \forall a \in A.$$

In this case $V_1 \subset V_1 \oplus V_2$ and $V_2 \subset V_1 \oplus V_2$ are subrepresentations.

Definition 4.3. A nonzero representation (V, ρ) of an algebra is *indecomposable* if it is not isomorphic to a direct sum of two nonzero subrepresentations.

In particular, an irreducible representation is indecomposable. The converse is false in general. If we have a representation of the form

$$\rho(a) = \begin{pmatrix} \rho_1(a) & \star \\ 0 & \rho_2(a) \end{pmatrix} \qquad \forall a \in A,$$

then the subspace $V_1 \subset V$ is invariant with respect to the action of A, and so it is a subrepresentation, but V_2 is not A-invariant.

The main questions of representation theory of an algebra A (or a group G) are:

1. Classify the irreducible representations of A (or G) over a field k.
2. Classify the indecomposable representations of A (or G) over a field k.

4.1 Irreducible representations of the dihedral group D_4

Consider the dihedral group

$$D_4 = \langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^4 = 1 \rangle.$$

We are interested in classifying all irreducible representations of D_4 over $\mathbb{C}$. Note that it suffices to define the $\rho : D_r \to \mathrm{End}(V)$ on the generators s_1, s_2 only, and check that $\rho(s_1)$ and $\rho(s_2)$ satisfy the defining relations of the group. Then we can extend ρ on all group elements using the homomorphism property.

Let us start with representations of dimension 1.

$$\rho(s_1) = \lambda_1, \quad \rho(s_2) = \lambda_2 \quad \text{, such that } \lambda_1^2 = \lambda_2^2 = 1, \;\; (\lambda_1\lambda_2)^4 = 1.$$

It follows that $\lambda_1, \lambda_2 \in \{\pm 1\}$. Therefore, we obtain 4 1-dimensional representations

$$\begin{cases} \rho_0(s_1) = \rho_0(s_2) = 1 & \implies \rho(g) = 1 \;\; \forall g \in D_4, \;\; \text{trivial representation} \\ \rho_1(s_1) = \rho_1(s_2) = -1 & \implies \rho(g) = (-1)^{l(g)} \;\; \forall g \in D_4, \;\; \text{sign representation} \\ \rho_2(s_1) = 1, \; \rho_2(s_2) = -1 & \\ \rho_3(s_1) = -1, \; \rho_3(s_2) = 1 & \end{cases}$$

They are all pairwise non-isomorphic. Indeed, a homomorphism of representations $\phi : \mathbb{C} \to \mathbb{C}$ is given by a nonzero complex number $\mu \in \mathbb{C}^*$. If ϕ is an intertwiner between for example ρ_2 and ρ_3, then $\rho_2(\mu s_1) = \mu\rho_3(s_1)$, therefore $\mu = -\mu$, a contradiction. So we have 4 non-isomorphic 1-dimensional irreducible representations of D_4.

Now suppose $\dim(V) = 2$. We are looking for $\rho D_4 \to \mathrm{GL}(\mathbb{C}^2)$, such that $\rho(s_1)^2 = \rho(s_2)^2 = 1$. For example, we can consider

$$\rho(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(s_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

It is easy to check that the matrices satisfy the relations in D_4. We see also that the $\rho(s_1)$ has eigenvectors $(1,1)$ and $(1,-1)$ whie $\rho(s_2)$ has eigenvectors $(1,0)$ and $(0,1)$, so that the two operators do not have a common invariant subspace in V. This means that the representation $(V\rho)$ is irreducible.

Is there another non-isomorphic 2-dimensional irreducible representation of D_4? First we notice that the eigenvalues for both $\rho(s_1)$, $\rho(s_2)$ have to be 1 and -1, since otherwise if one of these operators has a repeated eigenvalue, then the operator is a scalar and the resulting representation is not irreducible. Therefore $\det\rho(s_1) = \det\rho(s_2) = -1$, and $\det\rho(s_1 s_2) = 1$.

Since we have $\rho(s_1 s_2)^4 = 1$, the eigenvalues of $\rho(s_1 s_2) \in \{\pm i, \pm 1\}$. The condition $\det\rho(s_1 s_2) = 1$ imposes the choice of eigenvalues i and $-i$, because otherwise we would have to have a repeated eigenvalue ± 1 with multiplitity 2 which leads again to a scalar operator and a representation that is not irreducible. Therefore,

$$\rho(s_1 s_2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

in some basis, which is a rotation by $\pi/2$ in a complex plane. Since $\rho(s_1)$ and $\rho(s_2)$ both have eigenvalues 1 and -1, they are both reflections such that their product is a rotation by $\pi/2$, therefore any irreducible 2-dimensional representation of D_4 is isomorphic to the one we constructed.

Later we will see that $\{V_0, V_1, V_2, V_3, V\}$ is a complete list of irreducible inequivalent representations of D_4.

5 Schur's lemma

Proposition 5.1. *Let V_1, V_2 be representations of an algebra A over any field. Let $\phi : V_1 \to V_2$ be a nonzero homomorphism of representations. Then*

(1) If V_1 is irreducible, then ϕ is injective.

(2) If V_2 is irreducible, then ϕ is surjective.

(3) If V_1 and V_2 are irreducible, then ϕ is an isomorphism of irreducible representations.

Proof. (1) Consider the subspace $\ker\phi \subset V_1$. It is a subrepresentation: let $v \in \ker\phi$. Then

$$\phi(\rho_1(a)v) = \rho_2(a)\phi(v) = \rho_2(a) \cdot 0 = 0 \qquad \implies \qquad \rho_1(a)v \in \ker\phi.$$

Suppose that V_1 is irreducible. Then either $\ker\phi = 0$, or $\ker\phi = V_1$. The second option is ruled out since $\phi \neq 0$. Therefore we have $\ker\phi = 0$ and $\phi : V_1 \to V_2$ is injective.

(2) Consider the subspace $\mathrm{Im}\phi \subset V_2$. It is a subrepresentation: let $u \in \mathrm{Im}\phi$. Then there exist $v \in V_1$ such that $\phi(v) = u$. Therefore we have

$$\rho_2(a)u = \rho_2(a)\phi(v) = \phi(\rho_1(a)v) \in \mathrm{Im}\phi.$$

Suppose that V_2 is irreducible. Then either $\mathrm{Im}\phi = 0$, or $\mathrm{Im}\phi = V_2$. The first option is ruled out since $\phi \neq 0$. Therefore we have $\mathrm{Im}\phi = V_2$ and $\phi : V_1 \to V_2$ is surjective.

(3) Suppose that both V_1 and V_2 are irreducible. Then ϕ is both injective and surjective, which means that $\phi : V_1 \to V_2$ is an isomorphism of representations.

☐

Corollary 5.2. *If V_1 and V_2 are irreducible representations of an algebra A over any field and $\mathrm{dim}V_1 \neq \mathrm{dim}V_2$, there is no nonzero intertwiner between them.*

Corollary 5.3. *(Schur's lemma over an algebraically closed field).*
Let V be an irreducible finite dimensional representation of A over an algebraically closed field k, and $\phi : V \to V$ an intertwiner. Then $\phi = \lambda \mathrm{Id}_V$ for some $\lambda \in k$.

Proof. Since k is algebraically closed, $\phi : V \to V$ has an eigenvalue $\lambda \in k$. Then $(\phi - \lambda\mathrm{Id}) : V \to V$ commutes with the action of A, therefore it is also an intertwiner. Since V is irreducible, by Schur's lemma we have either $\phi - \lambda\mathrm{Id} = 0$, or $\phi - \lambda\mathrm{Id} : V \to V$ is an isomorphism. The second options is impossible since $\det(\phi - \lambda\mathrm{Id}) = 0$. Therefore we have $\phi = \lambda\mathrm{Id}$. ☐

Schur's lemma over algebraically closed fields stays true for countably-dimensional representations: if V is an irreducible countably dimensional representation, and $\phi : V \to V$ is an intertwiner, then ϕ is a scalar operator.

However, Corollary 5.3 fails in general over non-algebraically closed fields. For example let $A = \mathbb{C}$ as an $\mathbb{R}$-algebra, and $V = \mathbb{C}$ a representation of A. Then V is irreducible: any $\mathbb{R}$-subspace of the form $\{az\}_{a\in\mathbb{R}}$ is not invariant with respect of the $\mathbb{C}$-action. But an intertwiner $\phi : \mathbb{C} \to \mathbb{C}$ does not have to be a real scalar operator: multiplication by any $x \in \mathbb{C}^*, x : \mathbb{C} \to \mathbb{C}$ is an intertwiner.

Corollary 5.4. *Let A be a commutative algebra (or G an abelian group) over an algebraically closed field k. Then every finite dimensional irreducible representation of A (or G) over k is one-dimensional.*

Proof. Let $v \in V$, then

$$\rho(a)(\rho(b)v) = \rho(ab)v = \rho(ba)v = \rho(b)(\rho(a)v) \qquad \forall a, b \in A.$$

Therefore $\rho(a) : V \to V$ is an intertwiner for any $a \in A$. By Schur's lemma $\rho(a) = \lambda\mathrm{Id}_V$ for any $a \in A$, therefore every subspace in V is A-invariant. Thus if V is irreducible, it is one-dimensional. The same argument works for the representations of an abelian group. ☐

Example 5.5. Let us classify the irreducible representations of the cyclic group $C_n = \langle t : t^n = 1\rangle$. The group is abelian, therefore all irreducible representations are one-dimensional. We need to find $\lambda \in \mathbb{C}^*$ such that $\rho(t) = \lambda \implies (\rho(t))^n = \lambda^n = 1$. Then $\lambda \in \{1, \xi, \xi^2, \ldots, \xi^{n-1}\}$, where $\xi = e^{\frac{2\pi i}{n}}$ is the n-th primitive root of unity. The representations V_{ξ^i} and V_{ξ^j} for are inequivalent for $i \neq j$ (mod n). Indeed, $\mu(\rho_i(t)1) = \rho_j(t)(\mu \cdot 1)$ implies $\mu\xi^i = \xi^j\mu$ and $\xi^i = \xi^j$.

We can also consider the left regular representation of C_n and decompose it into a direct sum of irreducible subrepresentations. The left regular representation is given by $\rho(t) \cdot \mathbb{C}[C_n] = t \cdot \mathbb{C}[C_n]$, in the basis $\{1, t, \ldots t^{n-1}\}$ the action is given by $\rho(t)t^k = t \cdot t^k = t^{k+1}$. Then we have

$$\rho(t) = \begin{pmatrix} 0 & 1 & & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & & & & \\ \ldots & & & 0 & 1 \\ 1 & 0 & & & 0 \end{pmatrix}$$

in this basis, which is equivalent to the diagonal matrix

$$\rho(t) = \begin{pmatrix} 1 & 0 & & \ldots & 0 \\ 0 & \xi & 0 & \ldots & \\ \ldots & 0 & \xi^2 & & \\ \ldots & & & \ldots & 0 \\ 0 & & & 0 & \xi^{n-1} \end{pmatrix}$$

in the basis of eigenvectors with eigenvalues $\xi^k, k = 0, 1, \dots n-1$. Therefore we have a direct sum decomposition of the left regular representation

$$\mathbb{C}[C_n] \simeq \bigoplus_{i=0}^{n-1} V_{\xi^i}, \qquad \rho_i(t) = \xi^i \text{ in } V_{\xi^i}.$$

Note that the left regular representation is a direct sum of all irreducible representations of C_n, each with multiplicity one.

5.1 Representations of C_n and the first motivational example

Let us consider two representations of the group $C^n = \langle t : t^n = 1\rangle$ in the vector space $V = \mathbb{C}^n$ given by cyclic permutations of the basis:

$$L : e_i \to e_{i-1}, \qquad R : e_i \to e_{i+1},$$

where the indices are counted modulo n. Let $a = (a_1, a_2, \dots a_n) \in V$. Then we have

$$L(t)(a) = \begin{pmatrix} 0 & 1 & & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & & & \\ \dots & & & 0 & 1 \\ 1 & 0 & & & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ \dots \\ a_n \end{pmatrix} = \begin{pmatrix} a_2 \\ a_3 \\ \dots \\ \dots \\ a_1 \end{pmatrix}, \qquad R(t)(a) = \begin{pmatrix} 0 & & & \dots & 1 \\ 1 & 0 & & \dots & 0 \\ 0 & 1 & 0 & \dots & \\ \dots & & & & \\ \dots & & & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ \dots \\ a_n \end{pmatrix} = \begin{pmatrix} a_n \\ a_1 \\ \dots \\ \dots \\ a_{n-1} \end{pmatrix}$$

Let $\xi = e^{\frac{2\pi i}{n}}$. The characteristic equation of $L(t)$ and $R(t)$ is $\lambda^n - 1 = 0$ which leads to the solutions $\{1, \xi, \dots \xi^{n-1}\}$ for the eigenvalues. The eigenvectors of the form $\{v_k = (1, \xi^k, \xi^{2k}, \dots \xi^{(n-1)k})\}_{k=0}^{n-1}$ correspond to the eigenvalues ξ^k for $L(t)$ and to ξ^{-k} for $R(t)$. Note that each eigenvector spans a one-dimensional irreducible subrepresentation for both $L(t)$ and $R(t)$. Recall that we are interested in the iterated action of the operator $M = \frac{1}{2}(L(t) + R(t))$, for which v_k spans an invariant subspace for each $k = 0, \dots n-1$. More precisely, we have

$$Mv_k = \frac{1}{2}(L(t) + R(t))v_k = \left(\frac{\xi^k + \xi^{-k}}{2}\right) v_k = \cos\left(\frac{2\pi k}{n}\right) v_k.$$

Let us consider the limit

$$\lim_{N\to\infty} M^N v_k = \lim_{N\to\infty} \cos^N \left(\frac{2\pi k}{n}\right) v_k = 0, \quad \text{if } \ k \neq 0, k \neq \frac{n}{2}.$$

Otherwise,

$$M^N v_0 = v_0, \qquad M^N v_{\frac{n}{2}} = (-1)^N v_{\frac{n}{2}} \quad \forall N \in \mathbb{N}.$$

To answer our original question we need to find the action of M^N in the original basis. We have

$$\begin{pmatrix} 1 & 1 & & \dots & 1 \\ 1 & \xi & \xi^2 & \dots & \xi^{n-1} \\ 1 & \xi^2 & \xi^4 & \dots & \xi^{2(n-1)} \\ \dots & & & & \\ 1 & \xi^{n-1} & & & \xi^{(n-1)^2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \dots \\ \dots \\ e_n \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ \dots \\ \dots \\ v_{n-1} \end{pmatrix}.$$

Inverting the matrix, we obtain e_i as a linear combination of the eigenvectors $\{v_k\}_{k=0}^{n-1}$.

Let n be odd. Then we have

$$e_i = \frac{1}{n} v_0 + \sum_{k=1}^{n-1} b_k v_k,$$

where $b_k \in \mathbb{C}$ are some constants that do not play the role in our computations, because $M^N v_k \to 0$ when $N \to \infty$ for $k > 0$. For every $i = 1, \dots n$ we have $\lim_{N\to\infty} M^N e_i = \frac{1}{n} v_0 = \frac{1}{n} \sum_i^n e_i$. Then we get

$$\lim_{N\to\infty} M^N (\sum_{i=1}^n a_i e_i) = \frac{1}{n} (\sum_{i=1}^n a_i) v_0 = \frac{1}{n} (\sum_{i=1}^n a_i)(\sum_i^n e_i).$$

Therefore, if n is odd, in the limit when the number of iterations goes to infinity, each bowl will have the same amount of peanuts equal to $1/n$ of the total amount.

Now let n be even. Then we have

$$e_i = \frac{1}{n}v_0 + (-1)^i\frac{1}{n}v_{\frac{n}{2}} + \sum_{k=1}^{n-1} b_k v_k,$$

where $b_k \in \mathbb{C}$ and $M^N v_k \to 0$ when $N \to \infty$ for $k \neq 0, k \neq \frac{n}{2}$. We have $M^N(e_i) = \frac{1}{n}v_0 + (-1)^{i+N}\frac{1}{n}v_{\frac{n}{2}} + \varepsilon$, where ε is negligeable when N is large. Using the inverse transformation of bases, we get

$$M^N(\sum_{i=1}^{n} a_i e_i) = \frac{1}{n}(\sum_{i=1}^{n} a_i)v_0 + (-1)^N\frac{1}{n}(\sum_{i=1}^{n}(-1)^i a_i)v_{\frac{n}{2}} + \varepsilon.$$

Since $v_{\frac{n}{2}} = \frac{1}{n}\sum_{i=1}^{n}(-1)^i e_i$, we have finally

$$M^N(\sum_{i=1}^{n} a_i e_i) = \frac{1}{n}(\sum_{i=1}^{n} a_i)(\sum_j e_j) + (-1)^N\frac{1}{n}(\sum_{i=1}^{n}(-1)^i a_i)(\sum_j(-1)^j e_j) + \varepsilon.$$

Therefore, if n is even, at the limit when $N \to \infty$ we will have values $\frac{2}{n}\sum_{i \text{ even}} a_i$ and $\frac{2}{n}\sum_{i \text{ odd}} a_i$ at all even and odd vertices respectively, alternating at each step.

6 Complete reducibility of complex representations of finite groups

6.1 Maschke's theorem

Definition 6.1. A representation V of A is *completely reducible* if it is isomorphic to a direct sum of irreducible representations:

$$V \simeq W_1 \oplus \ldots \oplus W_n, \qquad W_i \text{ irreducible}.$$

Definition 6.2. An algebra A is semisimple over k if every finite dimensional representation of A over k is completely reducible.

Theorem 6.3. *(Maschke's theorem)*
Let G be a finite group and k a field such that the characteristic of k does not divide $|G|$. Then the group algebra $k[G]$ is semisimple: if V is a finite dimensional representation of G, and $W \subset V$ is a subrepresentation, then there exists a subrepresentation $W' \subset V$ such that $V \simeq W \oplus W'$ as representations of G.

Proof. Choose a complement vector space $\hat{W}$ such that $W \oplus \hat{W} = V$, and let $P : V \to W$ be the projector along $\hat{W}$, in particular $P|_{\hat{W}} = 0$ and $P|_W = \mathrm{Id}_W$. Let

$$\overline{P} = \frac{1}{|G|}\sum_{g\in G}\rho(g)P\rho(g^{-1}),$$

where $\rho : k[G] \to \mathrm{End}(V)$ and let $W' = \ker\overline{P}$. If $x \in W$, taking into account that $P|_W = \mathrm{Id}_W$, and that W is a subrepresentation, we have

$$\overline{P}(x) = \frac{1}{|G|}\sum_{g\in G}\rho(g)P\rho(g^{-1})(x) = \frac{1}{|G|}\sum_{g\in G}\rho(g)\rho(g^{-1})(x) = \frac{1}{|G|}\sum_{g\in G}x = \frac{|G|}{|G|}x = x.$$

Therefore, $\overline{P}|_W = \mathrm{Id}_W$.

If $y \in V$, then

$$\overline{P}(y) = \frac{1}{|G|}\sum_{g\in G}\rho(g)P\rho(g^{-1})(y) \in W,$$

and therefore $\overline{P}^2 = \overline{P}$ is a projector onto W.

Now suppose $z \in W' = \ker\overline{P}$. Then

$$\overline{P}\rho(h)z = \frac{1}{|G|}\sum_{g\in G}\rho(g)P\rho(g^{-1}h)z = \frac{1}{|G|}\sum_{k\in G}\rho(hk)P\rho(k^{-1})z = \rho(h)\overline{P}(z) = 0$$

where we denoted $k^{-1} = g^{-1}h$, $g = hk$. The last equality holds because $z \in W' = \ker\overline{P}$. We have shown that for any $z \in W'$, we have $\rho(h)z \in W'$, which means that $W' \subset V$ is a subrepresentation, and $V \simeq W \oplus W'$ is a direct sum of subrepresentations. □

Definition 6.4. Let V be a representation of A and $W \subset V$ a subrepresentation. Let V/W be the quotient space of W-cosets of the form $\{v+W\}_{v\in V}$. Then V/W is a representation of A by setting $\rho(a)(v+W) = \rho(a)v + W$. This is well defined because If $u \in v+W$, we have $\rho(a)(u+W) = \rho(a)u + W = \rho(a)v + \rho(a)(u-v) + W = \rho(a)v + W$ for any $a \in A$.

By Maschke's theorem any finite dimensional representation of G over a filed whose characteristic does not divide $|G|$ is completely reducible. Therefore any quotient representation V/W of G over k is also a subrepresentation $W' \simeq V/W$, and $V \simeq W \oplus W'$. In particular, the left regular representation $k[G]$ is isomorphic to a direct sum of irreducible representations of G.

Exercise 6.5. *If G is a finite group and k a field such that the characteristic of k does not divide $|G|$, then any irreducible representation of G occurs as a direct summand in the left regular representation $k[G]$.*

6.2 Representations of the polynomial algebra $\mathbb{C}[x]$.

All irreducible representations of $A = \mathbb{C}[x]$ are one-dimensional by Schur's lemma. Setting $\rho(x) = \lambda \in \mathbb{C}$ defines the pairwise non-isomorphic irreducible representations $\{V_\lambda\}_{\lambda\in\mathbb{C}}$.

To find the indecomposable representations of A, suppose $\dim(V) = n$. Then to define a representation of A it suffices to define a matrix $\rho(x) : V \to V$, and any matrix gives rise to a representation. An isomorphism of representation is given by a conjugation with an element from $\mathrm{GL}(V)$. Therefore the indecomposable finite dimensional representations of A are classified by matrices up to conjugation, or by the Jordan normal blocks

$$\rho(x) = \begin{pmatrix} \lambda & 1 & & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & & & & \\ \dots & & & 0 & 1 \\ 0 & 0 & & & \lambda \end{pmatrix} = J_{n,\lambda},$$

where $J_{n,\lambda}$ is the Jordan block of eigenvalue $\lambda \in \mathbb{C}$ of size $n \times n$.

The algebra $\mathbb{C}[x]$ is not semisimple. Indeed, we can consider the left regular representation of $\mathbb{C}[x]$, and the subspace of the ideal $\langle x^3 \rangle = x^3\mathbb{C}[x] \subset \mathbb{C}[x]$ which is obviously a subrepresentation. Let $V = \mathbb{C}[x]/x^3\mathbb{C}[x]$ the quotient representation of dimension 3 with a basis $\{1, x, x^2\}$. The action of x in V is given by

$$\rho(x) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = J_{0,3}$$

which is indecomposable. In particular the linear span of x^2, denoted by $\mathcal{L}\{x^2\} \subset V$, is a subrepresentation where x acts by zero. So V is indecomposable, but not irreducible. Thus the algebra $\mathbb{C}[x]$ is not semisimple. In fact we have the chain of subrepresentations $\mathcal{L}\{x^2\} \subset \mathcal{L}\{x, x^2\} \subset V$.

6.3 Converse to Maschke's theorem

Theorem 6.6. *Let k be a field and G a finite group. If any finite dimensional representation of G is completely reducible, then the characteristic of k does not divide $|G|$.*

Proof. Consider the left regular representation $V = k[G]$. Let $\phi : V \to k$ be a linear map, $\phi(g) = 1$ for all $g \in G$. Then $\phi : k[G] \to k$ a homomorphism from $k[G]$ to the trivial representation. Just like in the proof of Schur's lemma, we can show that the subspace $\ker\phi \subset k[G]$ is G-invariant. Suppose $V = k[G]$ is completely reducible, then $V = \ker\phi \oplus U$. We can define a basis in $\ker\phi$ given by $\{g-1\}_{g\in G,\, g\neq 1}$, with the dimension $\dim(\ker\phi) = |G| - 1$. Therefore $\dim(U) = 1$, and $U = \mathcal{L}(u)$ is spanned by a single vector $u \in k[G]$. We have

$$u = \sum_{g\in G} \mu_g g, \qquad \mu_g \in k \quad \forall g \in G.$$

Let $x \in G$, then

$$\rho(x)u - u = \sum_{g\in G} \mu_g xg - \sum_{g\in G} \mu_g g = \sum_{g\in G} \mu_g (xg - g) \in \ker\phi \cap U = 0.$$

The inclusion $\sum_{g\in G} \mu_g (xg-g) \in \ker\phi$ holds because the elements $(xg - g) \in \ker\phi$ for all $x, g \in G$. Therefore we have $\rho(x)u = u$ for all $x \in G$.

Let us compute the coefficient of $1 \in G$ in the element $u = \rho(x)u$: it is equal to $\mu_1 = \mu_{x^{-1}}$ for any $x \in G$, therefore $u = \mu_1 \sum_{g\in G} g$. Then $\phi(u) = \mu_1\phi(\sum_{g\in G} g) = \mu_1|G| \neq 0$ since $u \notin \ker\phi$. This shows that $|G| \neq 0$ in k.

□

Example 6.7. Consider the regular representation of the cyclic group $C_3 = \langle t : t^3 = 1\rangle$ over the field $\mathbb{F}_3$. In the basis $\{1, t, t^2\}$ we have

$$\rho(t) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation is $-\lambda^3 + 1 = 0$ with the only solution for the eigenvalue $\lambda = 1$ in $\mathbb{F}_3$. The only eigenvector is $(1, 1, 1)$, which spans a subrepresentation. We can find the generalized eigenvectors v', v'' such that $(A - \lambda \mathrm{Id})^2 v' = 0$ and $(A - \lambda \mathrm{Id})^3 v'' = 0$. Then we get

$$\mathcal{L}(v) \subset \mathcal{L}(v, v') \subset \mathcal{L}(v, v', v'') = V = \mathbb{F}_3[C_3],$$

so that V is an indecomposable non-irreducible representation. We have

$$\rho(t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $\{v, v', v''\}$. The left regular representation $\mathbb{F}_3[C_3]$ is not semisimple.

Exercise 6.8. *Check that the left regular representation of C_3 over the field $\mathbb{F}_5$ is semisimple and find its decomposition into the direct sum of irreducibles.*

6.4 Another viewpoint on complete reducibility: Weyl's unitary trick

Definition 6.9. A representation ρ of G in a $\mathbb{C}$-vector space V is *unitary* if V has a hermitian inner product invariant under the action of G:

$$\langle v, w\rangle = \langle \rho(g)v, \rho(g)w\rangle \qquad \forall v, w \in V, \ \ \forall g \in G.$$

Proposition 6.10. *Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of a finite group G. Then there exists an inner product $\langle\ ,\ \rangle$ on V that is G-invariant.*

Proof. Let $\langle\ ,\ \rangle_0$ be any hermitian inner product on V, for example $\langle u, v\rangle_0 = u^T \overline{v}$ for any $u, v \in V$. Then

$$\langle u, v\rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)u, \rho(g)v\rangle_0$$

is G-invariant. Indeed,

$$\langle \rho(h)u, \rho(h)v\rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(gh)u, \rho(gh)v\rangle_0 = \langle u, v\rangle.$$

□

Therefore, any $\mathbb{C}$-representation of a finite group is unitary.

Theorem 6.11. *(Weyl's unitary trick)*
Finite dimensional unitary representation of any group is completely reducible.

Proof. Let V be a finite dimensional unitary representation of G and W a nontrivial irreducible subrepresentation, $W \subset V$. Let $w \in W$ and $v \in W^\perp$, where $W^\perp$ is the orthogonal complement of W in V with respect to the G-invariant inner product $\langle\ ,\ \rangle$. Then

$$\langle \rho(g)v, w\rangle = \langle \rho(g)v, \rho(g)\rho(g^{-1})w\rangle = \langle v, \rho(g^{-1})w\rangle = 0.$$

Therefore $\rho(g)v \in W^\perp$ for any $v \in W^\perp$, any $g \in G$. Therefore $W^\perp \subset V$ is a subrepresentation and $V = W \oplus W^\perp$ as representations of G. Continuing with the decomposition of $W^\perp$ if necessary, we end up with a decomposition into the irreducible components, because V is finite dimensional. Moreover, the decomposition into the irreducible components is orthogonal with respect to the G-invariant hermitian inner product. □

Since any complex representation of a finite group is unitary, this provides an alternative proof of complete reducibility of a finite dimensional complex representation of a finite group.

6.5 Digression: Weyl's unitary trick for a continuous compact group

Let G be a compact Lie group (a compact topological space that is a group such that the group multiplication and taking the inverse are continuous maps). Let (V, ρ) be a complex finite dimensional representation of G. Similarly to the case of a finite group, starting with any Hermitian inner product $\langle\ ,\ \rangle_0$ on V, we can average it over the action of G to obtain a G-invariant inner product:

$$\langle v, w\rangle = \int_G \langle \rho(g)v, \rho(g)w\rangle_0 \, dg,$$

where dg is a G-invariant measure on G. This means that for any $x \in G$, we have $d(gx) = dg$.

Theorem 6.12. *A compact Lie group has a G-invariant measure called the Haar measure.*

This is a difficult result in general and we will accept it here without a proof. Assuming the existence of a G-invariant measure, we have

$$\langle \rho(x)v, \rho(x)w\rangle = \int_G \langle \rho(gx)v, \rho(gx)w\rangle \, dg = \int_G \langle \rho(g')v, \rho(g')w\rangle \, dg' = \langle v, w\rangle.$$

Theorem 6.13. *Any finite dimensional representation of a compact Lie group over $\mathbb{C}$ is completely reducible.*

Proof. Same as in the case of a finite group, see Theorem 6.11. □

6.6 Digression: representations of the group $U(1)$

Representations of the group $U(1)$. Let $U(1) \simeq S^1$ denote the group of rotations in a plane (a circle group):

$$U(1) = \{e^{i\theta},\ \theta \in [0, 2\pi[\} \subset \mathbb{C}.$$

Theorem 6.14. *If $\rho : U(1) \to \mathrm{GL}(n, \mathbb{C})$ is a representation of $U(1)$ in $V = \mathbb{C}^n$, then*

$$V \simeq V_1 \oplus V_2 \oplus \ldots \oplus V_n,$$

where $\rho_k(e^{i\theta}) = e^{im_k\theta}$, with $m_k \in \mathbb{Z}$.

Proof. **Claim 1**. Irreducible unitary representations of $U(1)$ are one-dimensional and parametrized by $\mathbb{Z}$. More precisely they are given by $\{\rho(e^{i\theta}) = e^{in\theta}\}_{n\in\mathbb{Z}}$.
Indeed, by Schur's lemma, all irreducible representations over $\mathbb{C}$ of an abelian group $U(1)$ are one-dimensional. Therefore we have $\rho(e^{i\theta}) = \lambda \in \mathbb{C}^*$. Since we are looking for the unitary representations, we have

$$\langle x, y\rangle = \langle \rho(e^{i\theta})x, \rho(e^{i\theta})y\rangle = \langle \lambda x, \lambda y\rangle = \lambda\overline{\lambda}\langle x, y\rangle = |\lambda|^2\langle x, y\rangle,$$

therefore $|\lambda|^2 = 1$. Let $\rho(e^{i\theta}) = e^{it(\theta)}$, then we have by the homomorphism property of the representation $t(0) = 0$, $t(\theta_1 + \theta_2) = t(\theta_1) + t(\theta_2)$, therefore $t(\theta) = \alpha\theta$ is a linear function. We also have

$$\rho(e^{i(\theta+2\pi)}) = \rho(e^{i\theta}) \qquad \Longrightarrow \qquad e^{i\alpha\theta} = e^{i\alpha(\theta+2\pi)} = e^{i\alpha\theta}e^{i2\pi\alpha},$$

therefore $\alpha \in \mathbb{Z}$. We obtain that $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ is the complete list of inequivalent irreducible representations of $U(1)$.

Claim 2. The representation $V = \mathbb{C}^n$ admits a $U(1)$-invariant inner product.
Indeed, let $\langle\ ,\ \rangle_0$ be any Hermitian inner product on V and set

$$\langle u, v\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle \rho(e^{i\theta})u, \rho(e^{i\theta})v\rangle_0 \, d\theta,$$

then we have

$$\langle \rho(e^{i\xi})u, \rho(e^{i\xi})v\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle \rho(e^{i(\theta+\xi)})u, \rho(e^{i(\theta+\xi)})v\rangle_0 \, d(\theta + \xi) = \langle u, v\rangle.$$

Here we used the $U(1)$-invariance of the measure: if ξ is fixed, then $d(\theta + \xi) = d\theta$.

Since any complex finite dimensional representation V of $U(1)$ is unitary, by Weyl's unitary trick it is completely reducible. Then $V \simeq \oplus_{k=1}^n V_k$, where each (ρ_k, V_k) is one dimensional and $\rho_k(e^{i\theta}) = e^{im_k\theta}$, $m_k \in \mathbb{Z}$.

□

7 Algebras, ideals, modules

Definition 7.1. Let A be an associative algebra, and M a vector space over k. A left A-module structure on M is given by a map $A \times M \to M$ given by $(a, m) \to am \in M$, satisfying the conditions

(1) $(ab)m = a(bm)$

(2) $1m = m$

(3) $(a + b)m = am + bm$

(4) $a(m + n) = am + an$

This is equivalent to defining a representation $\rho : A \to \mathrm{End}(M)$ of A on M: (1) is the homomorphism property, (2) is the property $\rho(1) = \mathrm{Id}_M$, (3) is the linearity $\rho(a + b) = \rho(a) + \rho(b)$, and (4) states that $\rho(a) \in \mathrm{End}(M)$ is a linear transformation. An A-module M where the only submodules are 0 and M is called *simple*. This is equivalent to the representation (M, ρ) being irreducible.

Definition 7.2. A subspace $I \subset A$ is a left ideal if $aI \subset I$ for all $a \in A$.
A subspace $J \subset A$ is a right ideal if $Ja \subset I$ for all $a \in A$.
A subspace $I \subset A$ is a two-sided ideal if it is a left ideal and a right ideal.
A subspace $J \subset A$ is a maximal ideal if there is no ideal $I \subset A$ such that $J \subset I \subset A$, and $J \neq I, I \neq A$.

Definition 7.3. A representation V of A is *cyclic* if there exist a vector $v \in V$ that generates V as an A-representation: $\rho(A)v = V$.

Proposition 7.4. *(1) A subspace $I \subset A$ is a left ideal if and only if $I \subset A$ is a subrepresentation of the left regular representation of A.*

(2) A representation V is cyclic if and only if V is isomorphic to a quotient representation A/I, where $I \subset A$ is a left ideal in A.

(3) A representation V is irreducible if and only if every nonzero vector $v \in V$ is cyclic.

(4) A representation V is irreducible if and only if $V \simeq A/I_{max}$ for a maximal left ideal I_{max}.

Proof. (1) follows by definition. For (2), see [PS4]. Let us prove (3). If $v \in V$, then $Av \subset V$ is a nonzero submodule. If V is irreducible, then the module V over A is simple, and $Av = V$, which means that v is a cyclic vector. Conversely, if $W \subset V$ is a nontrivial submodule, then for any vector $w \in W$ we have $Aw \subset W \neq V$, and w is not cyclic.
To show (4) it is enough to notice that if $I \subset J \subset A$ are two unequal nontrivial ideals in A, then $J/I \subset A/I$ is a submodule. □

We have the following diagram

$$\begin{tikzcd}
\{A/I\} \arrow[r, leftrightarrow] & \{\text{cyclic } A-\text{modules}\} \arrow[r, leftrightarrow] & \{\text{cyclic } A-\text{representations}\} \\
\{A/I_{max}\} \arrow[u, phantom, "\cup"] \arrow[r, leftrightarrow] & \{\text{simple } A-\text{modules}\} \arrow[u, phantom, "\cup"] \arrow[r, leftrightarrow] & \{\text{irreducible } A-\text{representations}\} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

8 Density theorem

Lemma 8.1. *Let V_i, $1 \leq i \leq m$ be irreducible finite dimensional pairwise nonisomorphic representations of an algebra A over an algebraically closed field. Let $V = \oplus_{i=1}^{n} V_i^{\oplus n_i}$. Suppose that $W \subset V$ is a subrepresentation of A. Then $W \simeq \oplus_{i=1}^{n} V_i^{\oplus r_i}$ with $r_i \leq n_i$ $\forall i$. For each i the inclusion map $\phi_i : V_i^{\oplus r_i} \to V_i^{\oplus n_i}$ is given by a constant $r_i \times n_i$ matrix X_i:*

$$\phi_i(v_1, \dots, v_{r_i}) = (v_1, \dots, v_{r_i}) \left(\begin{array}{ccc} & & \\ & X_i & \\ & & \end{array} \right) = (u_1, \dots, u_{n_i}) \in V_i^{\oplus n_i}.$$

The subrepresentation $V_i^{\oplus n_i} \subset V$ is called the *isotypical component* of V of type V_i.

Proof. The proof is by induction on $n = \sum_{i=1}^{m} n_i$.
By Schur's lemma, the homomorphisms $V_j \to V_i$ are zero for $i \neq j$, so we only need to consider the inclusion of isotypical components.
Base: If $n = 1$, then $V = V_i$. Then if $W \subset V_i$ is a nonzero subrepresentation, then $W \simeq V_i$ is irreducible, and the isomorphism is given by a single scalar by Schur's lemma.
Induction step: Let $P \subset W$ be irreducible. Then $P \simeq V_i$ and we have $\phi(P) \to V_i^{\oplus n_i}$ given by $v \to (vq_1, \ldots, vq_{n_i})$, where q_j are scalars.
This construction is best explained by an example. Suppose we have an irreducible 2-dimensional representation $V_i = \mathcal{L}(v_0, v_1)$ with the action of an algebra element $a \in A$ given by $av_0 = v_1$ and we are looking for the representation homomorphisms $\phi : V_i \to V_i^{(1)} \oplus V_i^{(2)}$, where $V_i^{(1)} = \mathcal{L}(u_0, u_1)$ and $V_I^{(2)} = \mathcal{L}(w_0, w_1)$ are identical copies of the same representation with the action $au_0 = u_1$ and $aw_0 = w_1$. Then $\phi(v_0) = \alpha u_0 + \beta w_0$ determines the homomorphism because we have $\phi(av_0) = \alpha a u_0 + \beta a w_0 = \alpha u_1 + \beta w_1 = \phi(v_1)$. In this example the matrix $X_i = (\alpha \;\; \beta)$.
Choose a matrix $F \in \mathrm{GL}_{n_i}(k)$ such that $(q_1, \ldots, q_{n_i})F = (1, 0, 0, \ldots, 0)$. Then F acts on $V_i^{\oplus n_i}$ by sending $(v_1, \ldots v_{n_i}) \to (v_1 \ldots v_{n_i})F$ and we have

$$P \xrightarrow{\phi} V_i^{\oplus n_i} \xrightarrow{F} V_i^{\oplus n_i} F \simeq V_i \oplus W'.$$

Then we can use the induction hypothesis for W' which has $n-1$ total irreducible components. By construction, we get an inclusion $V_i^{\oplus r_i} \to V_i^{\oplus n_i}$ given by an $r_i \times n_i$- matrix X_i. ☐

Theorem 8.2. *(Density theorem - I)*
Let V be an irreducible finite dimensional representation of A over an algebraically closed field. Let $\{v_1, \ldots v_n\}$ be linearly independent vectors in V. Then for any set $\{w_1, \ldots w_n\}$ of vectors in V, there exist an element $a \in A$ such that $\rho(a)v_i = w_i \; \forall i$.

Proof. Assume the contrary. Then $T : A \to V^{\oplus n}$ given by $a \to (\rho(a)v_1, \rho(a)v_2, \ldots, \rho(a)v_n)$ is such that $\mathrm{Im}T \subset V^{\oplus n}$ is a proper subrepresentation. By Lemma, the inclusion $\mathrm{Im}T \to V^{\oplus n}$ is given by a $r \times n$ matrix X, where $r < n$.
Let $a = 1$. Then $\mathrm{Im}T$ contains the vector $(v_1, v_2, \ldots, v_n)$, therefore we can find a vector $(u_1, \ldots, u_r) \in V^{\oplus r}$ such that

$$(u_1, \ldots, u_r)X = (v_1, \ldots, v_n) \in \mathrm{Im}T \subset V^{\oplus n}.$$

Let $(t_1, \ldots, t_n)$ be a nonzero vector such that

$$\left(\quad X \quad \right) \begin{pmatrix} t_1 \\ \cdot \\ \cdot \\ t_n \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ 0 \end{pmatrix}.$$

Such a vector exists because $r < n$. Then we have

$$\sum_{i=1}^{n} t_i v_i = (v_1, \ldots, v_n) \begin{pmatrix} t_1 \\ \cdot \\ \cdot \\ t_n \end{pmatrix} = (u_1, \ldots, \ldots u_r) \left(\quad X \quad \right) \begin{pmatrix} t_1 \\ \cdot \\ \cdot \\ t_n \end{pmatrix} = 0.$$

This is a contradiction since $\{v_i\}_{i=1}^{n}$ are linearly independent vectors. ☐

Theorem 8.3. *(Density theorem - II)*
Let V be an irreducible finite dimensional representation of an algebra A over an algebraically closed field. Then the map

$$\rho : A \to \mathrm{End}(V)$$

is surjective.

Proof. Let $\phi \in \mathrm{End}(V)$ and $\{v_i\}_{i=1}^{n}$ a basis in V. Let $\{\phi(v_1), \phi(v_2), \ldots, \phi(v_n)\} = \{w_i\}_{i=1}^{n}$. Then by the above theorem there exist $a \in A$ such that $\rho(a)v_i = w_i \; \forall i$. Therefore, there exist $a \in A$ such that $\rho(a) = \phi$. ☐

Theorem 8.4. *(Density theorem - III)*
Let $V_1 \oplus V_2 \oplus \ldots \oplus V_r$, where V_i are irreducible finite dimensional pairwise non-isomorphic representation of an algebra A over an algebraically closed field. Then the map

$$\bigoplus_{i=1}^{r} \rho_i : A \to \bigoplus_{i=1}^{n} \mathrm{End}(V_i)$$

is surjective.

Proof. Let V is an irreducible representation of an algebra A. Then $\operatorname{End} V$ also has a structure of an A-representation by left multiplication: if $\varphi \in \operatorname{End} V$, then $\varphi(v) \in V$ for $v \in V$, so we define $(a \cdot \varphi)(v) = a\varphi(v)$. Let $\dim V = n$. Let $\{v_1, \ldots, v_n\}$ be a basis in V. Then the map $F : \operatorname{End} V \to V^{\oplus n}$ defined by

$$F(\varphi) = (\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_n))$$

is an isomorphism of A-representations, so $\operatorname{End} V \cong V^{\oplus n}$.

So for each $i = 1, \ldots r$ we know that $\oplus_{i=1}^{r} \operatorname{End}(V_i)$ is semisimple as a representation of A, in particular it is isomorphic to $\oplus_{i=1}^{r} V_i^{\oplus d_i}$, where $d_i = \dim V_i$. Then by Lemma 8.1 we have that the image of A is a direct sum of images in each $\operatorname{End}(V_i)$. We have by Theorem 8.3 that the image of A in $\operatorname{End}(V_i)$ equals to $\operatorname{End}(V_i)$. Therefore, A surjects onto $\bigoplus_{i=1}^{n} \operatorname{End}(V_i)$. ☐

9 Structure of Finite Dimensional Algebras

9.1 Direct sum of algebras

Definition 9.1. Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ be a direct sum of vector spaces which are also algebras. Let $1_j \in A_j$ be the identity element for each j. Then the identity in A is $1 = 1_1 + 1_2 + \cdots + 1_n$. The multiplication in A is defined by setting $a_i a_j = 0$ for all $i \neq j$, where $a_i \in A_i$ and $a_j \in A_j$. In particular, $1_i 1_j = \delta_{ij} 1_i$. The A_i's are two-sided ideals in A.

Proposition 9.2. *Let $A = A_1 \oplus \cdots \oplus A_n$ be a direct sum of algebras. Any representation V of A decomposes as a direct sum $V = \bigoplus_{i=1}^{n} V_i$ of subrepresentations, where V_i is a representation of A_i, and A_j acts by zero on V_i for $i \neq j$. If V is an irreducible representation of A, then $1_i V$ is an irreducible representation of A_i for exactly one $i \in \{1, \ldots, n\}$, and $1_j V = 0$ for all $j \neq i$.*

Proof. Let V be a representation of A. Let $V_i = 1_i V$. Then V_i is a representation of A_i. For any $a_i \in A_i$,

$$\rho(a_i)V_i = \rho(a_i)\rho(1_i)V = \rho(a_i 1_i)V = \rho(a_i)V.$$

Conversely, if V_i is a representation of A_i for each i, then $V_1 \oplus V_2 \oplus \cdots \oplus V_n$ is a representation of A with the block-diagonal action:

$$\rho(a_1, a_2, \ldots, a_n)(v_1, \ldots, v_n) = (\rho_1(a_1)v_1, \rho_2(a_2)v_2, \ldots, \rho_n(a_n)v_n).$$

For any $v \in V$, we have $1v = v = \sum_{i=1}^{n} 1_i v$. Let $v_i = 1_i v \in V_i$. Then $V = \sum_{i=1}^{n} V_i$. To see that the sum is direct, note that $1_j V = 1_j \sum_{i=1}^{n} 1_i V = \sum_{i=1}^{n} 1_j 1_i V = 1_j^2 V = 1_j V = V_j$. The decomposition $V = \sum_{i=1}^{n} V_i$ is uniquely determined. Thus, $V = \bigoplus_{i=1}^{n} 1_i V$.

If V is irreducible, then there must be exactly one i such that $1_i V \neq 0$. If there were more than one, say $1_i V \neq 0$ and $1_j V \neq 0$ for $i \neq j$, then V would have nontrivial subrepresentations V_i and V_j, contradicting irreducibility. So, for some unique i, $1_i V = V$ and $1_j V = 0$ for $j \neq i$. Then V as a representation of A is irreducible if and only if it is irreducible as a representation of A_i. ☐

Let us recall what we understand about the algebra of square $n \times n$ matrices over a field k.

(a) $V \cong k^n$ is irreducible for $\operatorname{Mat}_n(k)$ (it is cyclic).

(b) $\operatorname{Mat}_n(k) \cong V^{\oplus n}$ as a left regular representation.

(c) Left ideals in $\operatorname{Mat}_n(k) \longleftrightarrow$ subrepresentations in $V^{\oplus n}$. This correspondence preserves the order defined by inclusions.

(d) By Density theorem the subrepresentations of $V^{\oplus n}$ are isomorphic to $V^{\oplus r}$, for $r \leq n$. A maximal nontrivial subrepresentation is isomorphic to $V^{\oplus(n-1)}$.

(e) This implies that any irreducible representation of $\operatorname{Mat}_n(k)$ is isomorphic to $\operatorname{Mat}_n(k)/V^{\oplus(n-1)} \cong V^{\oplus n}/V^{\oplus(n-1)} \cong V \cong k^n$ (cf. [PS4]).

Corollary 9.3. *The irreducible representations of $\bigoplus_{i=1}^{r} \operatorname{Mat}_{n_i}(k)$ are $V_1 \approx k^{n_1}, V_2 \approx k^{n_2}, \ldots, V_r \approx k^{n_r}$.*

9.2 Finite dimensional algebras

Definition 9.4. The *radical* of a finite dimensional algebra A, denoted $\operatorname{Rad}(A)$, is the set of all elements of A that act by 0 in all irreducible representations of A. $\operatorname{Rad}(A)$ is a two-sided nilpotent ideal (see [PS5]).

Theorem 9.5. *A finite dimensional algebra A over an algebraically closed field k has only finitely many inequivalent irreducible representations. Each irreducible representation is finite dimensional, and*

$$A/\operatorname{Rad}(A) \cong \bigoplus_{i=1}^{n} \operatorname{End}(V_i)$$

where $\{V_i\}_{i=1}^{n}$ is a complete list of inequivalent irreducible representations of A.

Proof. If V is a representation of A and $v \in V$ is a nonzero vector, then $A \cdot v$ is a subrepresentation. Since $\dim A < \infty$, every irreducible representation of A is finite dimensional.

Let $\{V_1, V_2, \dots\}$ be the set of inequivalent irreducible representations. Consider the map $\bigoplus_i \rho_i : A \to \bigoplus_i \operatorname{End} V_i$. By Density-3 theorem 8.4, this map is surjective. The dimension of the image is $\sum_i \dim \operatorname{End} V_i$. Since the map is a homomorphism from A, we must have $\dim A \geq \sum_i \dim \operatorname{End} V_i$. This implies there can be only a finite number, say n, of inequivalent irreducible representations.

The map $\bigoplus_{i=1}^{n} \rho_i : A \to \bigoplus_{i=1}^{n} \operatorname{End} V_i$ is surjective. By the first isomorphism theorem,

$$A/\ker\left(\bigoplus_{i=1}^{n} \rho_i\right) \cong \operatorname{Im}\left(\bigoplus_{i=1}^{n} \rho_i\right) = \bigoplus_{i=1}^{n} \operatorname{End} V_i.$$

The kernel is

$$\ker\left(\bigoplus_{i=1}^{n} \rho_i\right) = \{x \in A : \rho_i(x) = 0 \text{ for all } i = 1, \dots, n\},$$

which is precisely the definition of the radical of A. Therefore, $A/\operatorname{Rad}(A) \cong \bigoplus_{i=1}^{n} \operatorname{End} V_i$.

□

Corollary 9.6. *For a finite dimensional algebra A over an algebraically closed field k,*

$$\dim A = \sum_{i=1}^{n} (\dim V_i)^2 + \dim \operatorname{Rad}(A).$$

Example 9.7. Let $A = \mathbb{C}[x]/\langle x^n \rangle$ for a fixed $n \in \mathbb{N}$. A basis for A is $\{1, x, \dots, x^{n-1}\}$. For an irreducible representation V, by Schur's lemma we must have $\dim V = 1$. Let $\rho : A \to \operatorname{End}(V) \cong \mathbb{C}$. Then $\rho(x) = \lambda$ for some $\lambda \in \mathbb{C}$. The relation $x^n = 0$ in A implies $\rho(x^n) = \rho(x)^n = \lambda^n = 0$, so $\lambda = 0$. Thus, there is a unique irreducible representation V_0, where $\rho(x) = 0$. Using the dimension formula:

$$\underbrace{\dim A}_{n} = \underbrace{(\dim V_0)^2}_{1^2} + \dim \operatorname{Rad}(A).$$

This implies $\dim \operatorname{Rad}(A) = n - 1$. Indeed, $\operatorname{Rad}(A) = \langle x \rangle$.

9.3 Semisimple Algebras

Definition 9.8. An algebra A is *semisimple* if every finite dimensional representation of A is completely reducible (i.e., decomposes as a direct sum of irreducible representations).

Theorem 9.9 (Structure of Semisimple Finite Dimensional Algebras)**.** *Let A be a finite dimensional algebra over an algebraically closed field k. Then*

$$A \text{ is semisimple} \iff A \cong \bigoplus_{i=1}^{r} \operatorname{Mat}_{n_i}(k).$$

Proof. ($\Rightarrow$) Assume A is semisimple. Consider the left regular representation of A on itself. Since A is semisimple, this representation decomposes into a direct sum of irreducibles: $A \cong \bigoplus_{i=1}^{r} V_i^{\oplus n_i}$ for some irreducible representations V_i. Now consider the algebra of A-module endomorphisms of A, $\operatorname{End}_A(A)$.

$$\operatorname{End}_A(A) = \operatorname{End}_A\left(\bigoplus_{i=1}^{r} V_i^{\oplus n_i}\right).$$

By Schur's lemma, since the V_i are distinct irreducible representations over an algebraically closed field,

$$\mathrm{Hom}_A(V_i, V_j) = \begin{cases} 0 & i \neq j \\ k & i = j \end{cases}.$$

This implies that $\mathrm{End}_A(\bigoplus_{i=1}^r V_i^{\oplus n_i}) \cong \bigoplus_{i=1}^r \mathrm{End}_A(V_i^{\oplus n_i})$. Furthermore, $\mathrm{Hom}_A(V_i^{\oplus n_i}, V_i^{\oplus n_i}) \cong \mathrm{Mat}_{n_i}(\mathrm{Hom}_A(V_i, V_i)) \cong \mathrm{Mat}_{n_i}(k)$. For example, $\mathrm{End}_A(V \oplus V) = \begin{pmatrix} \mathrm{Hom}_A(V,V) & \mathrm{Hom}_A(V,V) \\ \mathrm{Hom}_A(V,V) & \mathrm{Hom}_A(V,V) \end{pmatrix} \cong \mathrm{Mat}_2(k)$. So, we have

$$\mathrm{End}_A(A) \cong \bigoplus_{i=1}^r \mathrm{Mat}_{n_i}(k).$$

We know that $\mathrm{End}_A(A) \cong A^{\mathrm{op}}$, the opposite algebra of A, where multiplication is reversed: $a^{\mathrm{op}}b = ba$. The isomorphism is given by mapping an endomorphism φ to $\varphi(1)$ (see [PS5]).

$$\Rightarrow A^{\mathrm{op}} \cong \bigoplus_{i=1}^r \mathrm{Mat}_{n_i}(k).$$

The opposite algebra of a direct sum is the direct sum of the opposite algebras. Also, $(\mathrm{Mat}_n(k))^{\mathrm{op}} \cong \mathrm{Mat}_n(k)$ via the transpose map $M \mapsto M^T$, which satisfies $(AB)^T = B^T A^T$.

$$\Rightarrow A \cong \bigoplus_{i=1}^r \mathrm{Mat}_{n_i}(k).$$

($\Leftarrow$) Let $A = \bigoplus_{i=1}^r \mathrm{Mat}_{n_i}(k)$. First, we show that $\mathrm{Mat}_n(k)$ is a semisimple algebra. Any finite dimensional representation of $\mathrm{Mat}_n(k)$ decomposes into a direct sum of irreducible representations, all of which are isomorphic to the standard representation $V \cong k^n$ (see [PS5]). A hint for this is to use the basis of matrix units $\{E_{ij}\}_{1\le i,j\le n}$ and the identity decomposition $1 = \sum_{i=1}^n E_{ii}$. Since a direct sum of semisimple algebras is semisimple, $A = \bigoplus_{i=1}^r \mathrm{Mat}_{n_i}(k)$ is semisimple. ☐

Corollary 9.10. *A finite dimensional algebra A over an algebraically closed field is semisimple if and only if* $\mathrm{Rad}(A) = \{0\}$.

Proof. From Theorem 9.5, we have $A/\mathrm{Rad}(A) \cong \bigoplus_{i=1}^r \mathrm{End}(V_i)$. A is semisimple if and only if $A \cong \bigoplus_{i=1}^r \mathrm{End}(V_i)$. ☐

Definition 9.11. A finite dimensional algebra A is *simple* if it is semisimple and has exactly one irreducible representation up to isomorphism. This is equivalent to $A \cong \mathrm{Mat}_n(k)$ for some n, and also equivalent to A having no two-sided ideals other than $\{0\}$ and A.

10 Characters of Representations

Definition 10.1. Let V be a finite dimensional representation of an algebra A. The *character* $\chi_V : A \to k$ is a linear function given by

$$\chi_V(a) = \mathrm{Tr}_V(\rho(a)).$$

Remark 10.2. The commutator subspace $[A, A] = \mathrm{span}\{xy - yx \mid x, y \in A\}$ is contained in the kernel of any character χ_V. This is due to the cyclic property of the trace:

$$\mathrm{Tr}_V(\rho(x)\rho(y)) = \mathrm{Tr}_V(\rho(y)\rho(x)) \implies \chi_V(xy - yx) = 0.$$

Thus, a character can be viewed as a linear functional $\chi_V : A/[A, A] \to k$.

Theorem 10.3. *Let A be an algebra over an algebraically closed field k.*

(a) Characters of distinct irreducible finite dimensional representations of A are linearly independent.

(b) If A is a finite dimensional semisimple algebra, then the characters of its irreducible representations, $\{\chi_{V_i}\}_{i=1}^r$, form a basis for the dual space $(A/[A, A])^*$.

Proof. (a) Let $V_1, \dots, V_r$ be inequivalent irreducible representations of A. By the Density-3 Theorem 8.4, the map

$$\rho = \rho_{V_1} \oplus \cdots \oplus \rho_{V_r} : A \to \operatorname{End}(V_1) \oplus \cdots \oplus \operatorname{End}(V_r)$$

is surjective. Suppose we have a linear combination of characters equal to zero: $\sum_{i=1}^r \lambda_i \chi_{V_i}(a) = 0$ for all $a \in A$. This means $\sum_{i=1}^r \lambda_i \operatorname{Tr}(\rho_{V_i}(a)) = 0$ for all $a \in A$. Since ρ is surjective, this is equivalent to $\sum_{i=1}^r \lambda_i \operatorname{Tr}(M_i) = 0$ for any tuple of matrices $(M_1, \dots, M_r)$ where $M_i \in \operatorname{End}(V_i)$. In particular, for a fixed $j \in \{1, \dots, r\}$, we can choose an element $a \in A$ such that $\rho_{V_j}(a) = M_j$ for any chosen matrix M_j and $\rho_{V_i}(a) = 0$ for $i \neq j$. For this choice of a, the sum becomes

$$\lambda_j \operatorname{Tr}(M_j) = 0.$$

Therefore all coefficients λ_j are zero, and the characters $\{\chi_{V_i}\}_{i=1}^r$ are linearly independent.

(b) We first show that $[\operatorname{Mat}_d(k), \operatorname{Mat}_d(k)] = \mathfrak{sl}_d(k) := \{M \in \operatorname{Mat}_d(k) \mid \operatorname{Tr}(M) = 0\}$. Clearly $[\operatorname{Mat}_d(k), \operatorname{Mat}_d(k)] \subset \mathfrak{sl}_d(k)$ since $\operatorname{Tr}(XY - YX) = 0$. For the other inclusion, consider the elementary matrices E_{ij}. We have

$$[E_{ij}, E_{jm}] = E_{im} \quad (i \neq m)$$
$$[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1}$$

These elements form a basis in $\mathfrak{sl}_d(k)$. Therefore, $[\operatorname{Mat}_d(k), \operatorname{Mat}_d(k)] = \mathfrak{sl}_d(k)$.

Since A is semisimple, it follows by the structure theorem 9.9 that $A \cong \operatorname{Mat}_{d_1}(k) \oplus \cdots \oplus \operatorname{Mat}_{d_r}(k)$. Then $[A, A] = [\operatorname{Mat}_{d_1}(k), \operatorname{Mat}_{d_1}(k)] \oplus \cdots \oplus [\operatorname{Mat}_{d_r}(k), \operatorname{Mat}_{d_r}(k)] = \mathfrak{sl}_{d_1}(k) \oplus \cdots \oplus \mathfrak{sl}_{d_r}(k)$. The dimensions of these components are $d_1^2 - 1, \dots, d_r^2 - 1$ respectively. This implies $\dim(A/[A, A]) = \dim A - \dim[A, A] = \sum_{i=1}^r d_i^2 - \sum_{i=1}^r (d_i^2 - 1) = r$.

There exists a unique irreducible representation of A for each component $\operatorname{Mat}_{d_i}(k)$, up to isomorphism. This implies there are exactly r irreducible representations of A. By part (1), we have r linearly independent characters. Since $\dim((A/[A, A])^*) = r$, these characters form a basis in $(A/[A, A])^*$. □

Example 10.4. Decomposition of a group algebra into a direct sum of matrix algebras. Let $A = \mathbb{C}[C_2]$ where $C_2 = \{1, s\}$ with $s^2 = 1$. The irreducible representations are the trivial representation V_0 ($\rho_0(1) = \rho_0(s) = 1$) and the sign representation V_s ($\rho_s(1) = 1, \rho_s(s) = -1$). The algebra decomposes as

$$A \cong \operatorname{Mat}_1(\mathbb{C}) \oplus \operatorname{Mat}_1(\mathbb{C}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, \qquad \text{where } e_1 = \frac{1+s}{2}, e_2 = \frac{1-s}{2}.$$

These are orthogonal idempotents:

$$\begin{aligned} e_1^2 &= \frac{1}{4}(1 + 2s + s^2) = \frac{1}{4}(1 + 2s + 1) = \frac{1+s}{2} = e_1, \\ e_2^2 &= \frac{1}{4}(1 - 2s + s^2) = \frac{1}{4}(1 - 2s + 1) = \frac{1-s}{2} = e_2, \\ e_1 e_2 &= \frac{1}{4}(1 - s^2) = \frac{1}{4}(1 - 1) = 0. \end{aligned}$$

Also, $e_1 + e_2 = \frac{1}{2}(1+s) + \frac{1}{2}(1-s) = 1$. The left regular representation of A in the basis $\{e_1, e_2\}$ is given by:

$$\begin{aligned} s \cdot e_1 &= s \cdot \frac{1}{2}(1+s) = \frac{1}{2}(s + s^2) = \frac{1}{2}(s+1) = e_1 \\ s \cdot e_2 &= s \cdot \frac{1}{2}(1-s) = \frac{1}{2}(s - s^2) = \frac{1}{2}(s-1) = -e_2 \end{aligned}$$

Thus, $\mathbb{C}e_1 \cong V_0$ and $\mathbb{C}e_2 \cong V_s$. This shows the decomposition of the left regular representation: $A \cong V_0 \oplus V_s$.

11 Characters of finite groups

Definition 11.1. If G is a finite group and V is a finite dimensional representation over a field k, its character $\chi_V : G \to k$ is defined as $\chi_V(g) = \operatorname{Tr}_V(\rho(g))$.

Note that V is also a representation of the group algebra $k[G]$, and the character $\chi_V(g)$ on G is the restriction of the character on $k[G]$.

Lemma 11.2. *The character $\chi_V : G \to k$ is a class function, i.e., it only depends on the conjugacy class in G.*

Proof. For any $g, h \in G$,

$$\chi_V(hgh^{-1}) = \operatorname{Tr}_V \rho(hgh^{-1}) = \operatorname{Tr}_V(\rho(h)\rho(g)\rho(h)^{-1}) = \operatorname{Tr}_V \rho(g) = \chi_V(g)$$

by the cyclicity of the trace. □

Definition 11.3. Let $F(G,k)$ be the space of k-valued functions on G. Let $F_c(G,k) \subset F(G,k)$ be the subspace of class functions.

Theorem 11.4. *The characters of irreducible representations of a finite group G over $\mathbb{C}$ form a basis in the space of class functions $F_c(G,\mathbb{C})$.*

Proof. By Maschke's theorem, the group algebra $A = \mathbb{C}[G]$ is semisimple. From Theorem 10.3, the characters of irreducible representations of A form a basis in $(A/[A,A])^*$. We identify $(A/[A,A])^*$ with $F_c(G,\mathbb{C})$. An element $\varphi \in (A/[A,A])^*$ is a linear functional on A that vanishes on commutators.

$$\begin{aligned}(A/[A,A])^* &= \{\varphi \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G],\mathbb{C}) : \varphi(xy-yx) = 0 \quad \forall x,y \in \mathbb{C}[G]\} \\ &\cong \{f \in F(G,\mathbb{C}) : f(gh) = f(hg) \quad \forall g,h \in G\}\end{aligned}$$

The condition $f(gh) = f(hg)$ for all $g,h \in G$ is equivalent to being a class function:

$$f(hgh^{-1}) = f(h(gh^{-1})) = f((gh^{-1})h) = f(g).$$

Thus, $(A/[A,A])^* \cong F_c(G,\mathbb{C})$. Since the irreducible characters form a basis for the former, they also form a basis for the latter. □

Corollary 11.5. *The number of isomorphism classes of irreducible representations of G over $\mathbb{C}$ is equal to the number of conjugacy classes in G.*

Corollary 11.6. *Any finite dimensional representation of G over $\mathbb{C}$ is completely determined up to isomorphism by its character.*

Proof. Let $V_1,\dots,V_r$ be the distinct irreducible representations of G. Any finite dimensional representation V can be written as $V \cong \bigoplus_{i=1}^r V_i^{\oplus n_i}$ for some non-negative integers n_i. The character of V is then $\chi_V = \sum_{i=1}^r n_i\chi_{V_i}$. If $W \cong \bigoplus_{i=1}^r V_i^{\oplus m_i}$ is another representation, then $\chi_W = \sum_{i=1}^r m_i\chi_{V_i}$. Since the irreducible characters $\{\chi_{V_i}\}_{i=1}^r$ form a basis of $F_c(G,\mathbb{C})$, the equality $\chi_V = \chi_W$ implies $\sum n_i\chi_{V_i} = \sum m_i\chi_{V_i}$, which in turn implies $n_i = m_i$ for all i. Therefore, $V \cong W$. □

Conclusions

1. $\mathbb{C}[G]$ is semisimple and $\mathbb{C}[G] \cong \bigoplus_{i=1}^r \mathrm{Mat}_{n_i}(\mathbb{C})$, where $\{V_i\}_{i=1}^r$ are the irreducible representations of G and $n_i = \dim V_i$.
2. The sum of the squares of the dimensions of the irreducible representations equals the order of the group: $|G| = \sum_{i=1}^r n_i^2$.

Example 11.7. The dihedral group D_4 has order 8, with presentation $\langle r,s \mid r^4 = s^2 = 1, srs = r^{-1}\rangle$. Its elements are $\{1,r,r^2,r^3,s,sr,sr^2,sr^3\}$. The conjugacy classes are $\{1\},\{r^2\},\{r,r^3\},\{s,sr^2\},\{sr,sr^3\}$. There are 5 conjugacy classes, so there must be 5 irreducible representations. The known irreducible representations are:

- V_0 (trivial): $\rho(s) = 1, \rho(r) = 1$ (1-dim)
- V_1: $\rho(s) = -1, \rho(r) = 1$ (1-dim)
- V_2: $\rho(s) = 1, \rho(r) = -1$ (1-dim)
- V_3: $\rho(s) = -1, \rho(r) = -1$ (1-dim)
- V_4 (standard): $\rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \rho(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (2-dim)

The sum of squares of dimensions is $1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 1+1+1+1+4 = 8 = |D_4|$. This confirms that these are all the inequivalent irreducible representations of D_4.

The character table for D_4 (with representatives from conjugacy classes) is:

Conj. Class Representative	$\{1\}$ 1	$\{r,r^3\}$ r	$\{r^2\}$ r^2	$\{s,sr^2\}$ s	$\{sr,sr^3\}$ sr
χ_0	1	1	1	1	1
χ_1	1	1	1	−1	−1
χ_2	1	−1	1	1	−1
χ_3	1	−1	1	−1	1
χ_4	2	0	−2	0	0

11.1 Dual representations

Definition 11.8. Let V be a finite dimensional representation of a finite group G. The dual space $V^* = \mathrm{Hom}(V, \mathbb{C})$ is also a representation of G, called the dual representation. The action is given by

$$(\rho^*(g)(L))(v) = L(\rho(g^{-1})v) \quad \text{for } L \in V^*, v \in V, g \in G.$$

One can check that $\rho^*(g_1 g_2) = \rho^*(g_1)\rho^*(g_2)$.

Remark 11.9. The natural pairing $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{C}$ is G-invariant:

$$\langle \rho^*(g)L, \rho(g)v \rangle = (\rho^*(g)L)(\rho(g)v) = L(\rho(g^{-1})\rho(g)v) = L(v) = \langle L, v \rangle.$$

Remark 11.10. We have $V^{**} \simeq V$ as a representation of G. Indeed, we have $V \simeq V^{**}$ as a vector space. Then for any $v \in V \simeq V^{**}$ and any $L \in V^*$ we have

$$L(\rho^{**}(g)v) = \rho^*(g^{-1})(L)(v) = L(\rho(g)v).$$

Remark 11.11. For a finite dimensional representation V, its dual V^* is irreducible if and only if V is irreducible. If $W \subset V$ is a subrepresentation, then its annihilator $W^\perp = \{\varphi \in V^* : \varphi(w) = 0 \text{ for all } w \in W\}$ is a subrepresentation of V^*. Indeed, for $\varphi \in W^\perp$, $g \in G$, and $w \in W$:

$$(\rho^*(g)(\varphi))(w) = \varphi(\rho(g^{-1})w) = 0,$$

since $\rho(g^{-1})w \in W$. Thus $\rho^*(g)(\varphi) \in W^\perp$.

Proposition 11.12. *The character of the dual representation is the complex conjugate of the original character:* $\chi_{V^*}(g) = \overline{\chi_V(g)}$.

Proof. In a chosen basis, the matrix of $\rho^*(g)$ is $(\rho(g^{-1}))^T$. Thus, $\chi_{V^*}(g) = \mathrm{Tr}(\rho(g^{-1})) = \chi_V(g^{-1})$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\rho(g)$. Since G is finite, g has finite order, say m. Then $\rho(g)^m = \rho(g^m) = \rho(1) = I$. This implies that the eigenvalues λ_i are m-th roots of unity. In particular, $|\lambda_i| = 1$ and $\lambda_i^{-1} = \overline{\lambda_i}$. The eigenvalues of $\rho(g^{-1})$ are $\lambda_1^{-1}, \dots, \lambda_n^{-1}$. Therefore,

$$\chi_{V^*}(g) = \chi_V(g^{-1}) = \sum_{i=1}^n \lambda_i^{-1} = \sum_{i=1}^n \overline{\lambda_i} = \overline{\sum_{i=1}^n \lambda_i} = \overline{\chi_V(g)}. \qquad \square$$

11.2 Tensor products of representations

Definition 11.13. Let V and W be two vector spaces over a field k. The tensor product $V \otimes W$ is the vector space spanned by elements of the form $v \otimes w$ for $v \in V, w \in W$, subject to the relations:

$$\begin{aligned}(v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\ (c \cdot v) \otimes w = v \otimes (c \cdot w) &= c(v \otimes w) \quad \text{for } c \in k.\end{aligned}$$

Remark 11.14. If V has basis $\{e_1, \dots, e_n\}$ and W has basis $\{f_1, \dots, f_m\}$, then $V \otimes W$ has basis $\{e_i \otimes f_j\}_{1 \le i \le n, 1 \le j \le m}$. The dimension is $\dim(V \otimes W) = (\dim V) \cdot (\dim W)$.

Definition 11.15. If V and W are representations of a group G, then their tensor product $V \otimes W$ is also a representation of G with the action defined by linear extension of

$$\rho_{V \otimes W}(g)(v \otimes w) = \rho_V(g)v \otimes \rho_W(g)w.$$

It is straightforward to check that $\rho_{V \otimes W}(g_1 g_2) = \rho_{V \otimes W}(g_1)\rho_{V \otimes W}(g_2)$.

Lemma 11.16. *The character of a tensor product is the product of the characters:*

$$\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g).$$

Proof. Let $A = (a_{ij})$ be the matrix of $\rho_V(g)$ in a basis $\{e_i\}$ of V, and $B = (b_{kl})$ be the matrix of $\rho_W(g)$ in a basis $\{f_k\}$ of W. Then the matrix of $\rho_{V \otimes W}(g)$ in the basis $\{e_i \otimes f_k\}$ is the Kronecker product $A \otimes B$. This is a block matrix:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{pmatrix}.$$

The trace of this matrix is

$$\operatorname{Tr}(\rho_{V\otimes W}(g)) = \sum_{i=1}^{n} a_{ii}\operatorname{Tr}(B) = \left(\sum_{i=1}^{n} a_{ii}\right)\operatorname{Tr}(B) = \operatorname{Tr}(A)\operatorname{Tr}(B) = \chi_V(g)\cdot\chi_W(g). \qquad \square$$

Definition 11.17. If V, W are representations of G, then $\operatorname{Hom}(V, W)$ is a G-representation with the action

$$\rho(g)(\varphi) = \rho_W(g)\circ\varphi\circ\rho_V(g^{-1}) \quad \text{for } \varphi\in\operatorname{Hom}(V,W).$$

Lemma 11.18. *If V, W are finite dimensional representations of G, then $W\otimes V^*\cong\operatorname{Hom}(V,W)$ as G-representations.*

Proof. Define the map $F : W\otimes V^*\to\operatorname{Hom}(V,W)$ by

$$F(w\otimes L)(v) = L(v)\cdot w \quad \text{for } w\in W, L\in V^*, v\in V.$$

This map is a well-known vector space isomorphism. In particular, if $\{e_i\}$ is a basis of V and $\{e^i\}$ a dual basis of L, so that $e^i(e_j)=\delta_{ij}$, and $w\in W$, then $\phi : V\to W$ that sends $\phi(e_i)=w$, $\phi(e_j)=0, j\neq i$ is given by $F(w\otimes e^i)=\phi$. We check that it is a G-module homomorphism (an intertwining operator). The action of $g\in G$ on an element $w\otimes L\in W\otimes V^*$ is

$$g\cdot(w\otimes L) = \rho_W(g)w\otimes\rho_{V^*}(g)L = \rho_W(g)w\otimes(L\circ\rho_V(g^{-1})).$$

Applying F to this transformed element gives a map from V to W:

$$F(g\cdot(w\otimes L))(v) = (L\circ\rho_V(g^{-1}))(v)\cdot\rho_W(g)w = L(\rho_V(g^{-1})v)\cdot\rho_W(g)w.$$

On the other hand, the action of g on the element $F(w\otimes L)\in\operatorname{Hom}(V,W)$ is

$$\begin{aligned}(g\cdot F(w\otimes L))(v) &= (\rho_W(g)\circ F(w\otimes L)\circ\rho_V(g^{-1}))(v)\\ &= \rho_W(g)(F(w\otimes L)(\rho_V(g^{-1})v))\\ &= \rho_W(g)(L(\rho_V(g^{-1})v)\cdot w)\\ &= L(\rho_V(g^{-1})v)\cdot\rho_W(g)w.\end{aligned}$$

Since $F(g\cdot(w\otimes L)) = g\cdot F(w\otimes L)$, F is a G-intertwining map. Since it is an isomorphism of vector spaces, it is an isomorphism of G-representations. $\square$

12 Orthogonality of Characters

12.1 First Orthogonality Relation

Theorem 12.1 (First Orthogonality Relation)**.** *For any two finite dimensional representations V, W of a finite group G over $\mathbb{C}$, we define the inner product of their characters as:*

$$(\chi_V,\chi_W) := \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\overline{\chi_W(g)}.$$

This inner product is equal to the dimension of the space of G-homomorphisms between them:

$$(\chi_V,\chi_W) = \dim\operatorname{Hom}_G(V,W).$$

In particular, if V and W are irreducible representations, then

$$(\chi_V,\chi_W) = \begin{cases} 1 & \text{if } V\cong W\\ 0 & \text{if } V\not\cong W\end{cases}$$

Proof. Let $P = \frac{1}{|G|}\sum_{g\in G} g$. We have

$$\begin{aligned}(\chi_V,\chi_W) &= \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\chi_{W^*}(g)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_{V\otimes W^*}(g) = \chi_{V\otimes W^*}\left(\frac{1}{|G|}\sum_{g\in G} g\right) = \chi_{V\otimes W^*}(P)\end{aligned}$$

We examine the properties of P:

(a) $P \in Z(\mathbb{C}[G])$ (the center of the group algebra):

$$hP = \frac{1}{|G|} \sum_{g \in G} hg = \frac{1}{|G|} \sum_{g' \in G} g' = P$$

and

$$Ph = \frac{1}{|G|} \sum_{g \in G} gh = \frac{1}{|G|} \sum_{g'' \in G} g'' = P$$

- $P(U) \subset U$ is a subrepresentation: $h(Pu) = (hP)u = P(u) \in P(U)$ for all $h \in G, u \in U$.
- $P : U \to P(U)$ is an intertwiner (a G-module homomorphism): $hP(u) = Ph(u)$ for all $h \in G, u \in U$.

(b) $P(U)$ is a trivial subrepresentation of U: for any $u \in U$, $h(Pu) = P(u)$ for all $h \in G$.

(c) $P^2 = P$:

$$\begin{aligned} P^2 &= \left(\frac{1}{|G|} \sum_{g_1 \in G} g_1 \right) \left(\frac{1}{|G|} \sum_{g_2 \in G} g_2 \right) = \frac{1}{|G|^2} \sum_{g_1, g_2 \in G} g_1 g_2 \\ &= \frac{1}{|G|^2} \sum_{g_1 \in G} \sum_{g \in G} g = \frac{1}{|G|^2} |G| \sum_{g \in G} g = \frac{1}{|G|} \sum_{g \in G} g = P \end{aligned}$$

Therefore P is a projector.

(d) If $u \in U_0 \subset U$ where U_0 is a trivial subrepresentation, then $gu = u$ for all $g \in G$.

$$P(u) = \frac{1}{|G|} \sum_{g \in G} gu = \frac{1}{|G|} \sum_{g \in G} u = \frac{1}{|G|} |G| u = u.$$

Thus P is a projector onto the (possibly multi-dimensional) trivial isotypical component of the representation U of G.

Now let us apply P to the representation $V \otimes W^*$. The trace of a projector is the dimension of its image. The image of P acting on $V \otimes W^*$ is the trivial isotypical component, or the subspace of G-invariants, $(V \otimes W^*)^G$.

$$(\chi_V, \chi_W) = \mathrm{Tr}_{V \otimes W^*}(P) = \dim (V \otimes W^*)^G.$$

We recall the canonical isomorphism of G-representations: $V \otimes W^* \cong \mathrm{Hom}(W, V)$. Under this isomorphism, the subspace of G-invariants corresponds to the space of G-module homomorphisms:

$$(V \otimes W^*)^G \cong \mathrm{Hom}_G(W, V).$$

Indeed, if $\phi \in (\mathrm{Hom}(W, V))^G$, then ϕ is a linear map $\phi : W \to V$ such that $\rho_V(g) \circ \phi \circ \rho_W(g^{-1}) = \phi$ for any $g \in G$. This condition is equivalent to $\rho_V(g) \circ \phi = \phi \circ \rho_W(g)$ for any $g \in G$. Therefore,

$$(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(W, V).$$

In particular, if V and W are irreducible, by Schur's Lemma:

$$\dim \mathrm{Hom}_G(W, V) = \begin{cases} 1, & \text{if } V \cong W \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof. □

Example 12.2. Let $G = D_4$ The character table for D_4 is given below. The numbers in the top row indicate the size of the conjugacy classes.

	1	2	1	2	2
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	-1	1	-1	1
χ_3	1	-1	1	1	-1
χ_4	2	0	-2	0	0

Let's verify the orthogonality for some pairs of characters.

$$\begin{aligned}
(\chi_2, \chi_3) &= \frac{1}{8}(1 \cdot 1 \cdot 1 + 2 \cdot (-1) \cdot (-1) + 1 \cdot 1 \cdot 1 + 2 \cdot (-1) \cdot 1 + 2 \cdot 1 \cdot (-1)) \\
&= \frac{1}{8}(1 + 2 + 1 - 2 - 2) = 0 \\
(\chi_4, \chi_3) &= \frac{1}{8}(1 \cdot 2 \cdot 1 + 2 \cdot 0 \cdot (-1) + 1 \cdot (-2) \cdot 1 + 2 \cdot 0 \cdot 1 + 2 \cdot 0 \cdot (-1)) \\
&= \frac{1}{8}(2 + 0 - 2 + 0 + 0) = 0 \\
(\chi_4, \chi_4) &= \frac{1}{8}(1 \cdot 2 \cdot 2 + 2 \cdot 0 \cdot 0 + 1 \cdot (-2) \cdot (-2) + 2 \cdot 0 \cdot 0 + 2 \cdot 0 \cdot 0) \\
&= \frac{1}{8}(4 + 0 + 4 + 0 + 0) = 1
\end{aligned}$$

Corollary 12.3. *A representation V of G is irreducible if and only if $(\chi_V, \chi_V) = 1$.*

Proof. Any representation V can be decomposed into a direct sum of irreducible representations V_i: $V = \bigoplus_i V_i^{\oplus n_i}$, where $\{V_i\}$ are the inequivalent irreducible representations of G. The character of V is then $\chi_V = \sum_i n_i \chi_{V_i}$. We compute the inner product of χ_V with itself:

$$(\chi_V, \chi_V) = \left(\sum_i n_i \chi_{V_i}, \sum_j n_j \chi_{V_j} \right) = \sum_{i,j} n_i n_j (\chi_{V_i}, \chi_{V_j})$$

By the first orthogonality relation, $(\chi_{V_i}, \chi_{V_j}) = \delta_{ij}$. Thus,

$$(\chi_V, \chi_V) = \sum_i n_i^2 (\chi_{V_i}, \chi_{V_i}) = \sum_i n_i^2.$$

If $(\chi_V, \chi_V) = 1$, then $\sum_i n_i^2 = 1$. Since n_i are non-negative integers, this implies that there is a unique index k for which $n_k = 1$ and all other $n_i = 0$. Thus $V \cong V_k$, which is irreducible. Conversely, if V is irreducible, it is isomorphic to some V_k, so $n_k = 1$ and all other $n_i = 0$, which gives $(\chi_V, \chi_V) = 1^2 = 1$. □

12.2 Second Orthogonality Relation

Theorem 12.4 (Second orthogonality relation)**.** *Let $g, h \in G$ and let $Z_g = \{t \in G : tgt^{-1} = g\}$ be the centralizer of g in G. Then*

$$\sum_{V \in \mathrm{Irr}(G)} \chi_V(g) \overline{\chi_V(h)} = \begin{cases} |Z_g| & \text{if } g \text{ is conjugate to } h \\ 0 & \text{otherwise} \end{cases}$$

where the sum is over all inequivalent irreducible representations of G.

Proof. We have $\overline{\chi_V(h)} = \chi_{V^*}(h)$. Thus, the sum is

$$\sum_{V \in \mathrm{Irr}(G)} \chi_V(g)\chi_{V^*}(h) = \sum_{V \in \mathrm{Irr}(G)} \mathrm{Tr}(\rho_V(g)) \,\mathrm{Tr}(\rho_{V^*}(h)) = \sum_{V \in \mathrm{Irr}(G)} \mathrm{Tr}(\rho_V(g) \otimes \rho_{V^*}(h)).$$

Recall that $\bigoplus_{V \in \mathrm{Irr}(G)} V \otimes V^* \cong \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{End}(V) \cong \mathbb{C}[G]$ as G-modules, where the action of G on $\mathrm{End}(V)$ is defined as follows: $A \in \mathrm{End}(V)$ is transformed to $\rho_V(g) A \rho_V(h)^{-1}$. So we have the trace of the linear map $F : \mathbb{C}[G] \to \mathbb{C}[G]$ defined by $x \mapsto gxh^{-1}$ for $x \in \mathbb{C}[G]$. We compute this trace in the basis of group elements $\{k\}_{k \in G}$. The matrix of F in this basis is a permutation matrix. The trace is the number of basis elements k fixed by the map, i.e., the number of $k \in G$ such that $gkh^{-1} = k$.

$$\mathrm{Tr}_{\mathbb{C}[G]}(F) = |\{k \in G \mid gkh^{-1} = k\}|.$$

The condition $gkh^{-1} = k$ is equivalent to $g = khk^{-1}$. If g is not conjugate to h, there is no such $k \in G$, so the set is empty and the trace is 0. If g is conjugate to h, say $g = k_0 h k_0^{-1}$ for some $k_0 \in G$, then the set of all k satisfying the condition is $\{k \in G \mid khk^{-1} = g = k_0 h k_0^{-1}\}$. This is equivalent to $k_0^{-1} k h (k_0^{-1} k)^{-1} = h$, which means $k_0^{-1} k$ is in the centralizer of h, Z_h. The set of solutions is the coset $k_0 Z_h$. The size of this set is $|Z_h|$. Since g and h are conjugate, their centralizers have the same size, $|Z_g| = |Z_h|$. Thus, the trace is $|Z_g|$ if g and h are conjugate, and 0 otherwise. □

Example 12.5. $G = D_4$ The character table with conjugacy class representatives and centralizer sizes.

	1	2	1	2	2
Class Rep.	1	r	r^2	s	sr
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	-1	1	-1	1
χ_3	1	-1	1	1	-1
χ_4	2	0	-2	0	0
$\lvert Z_g\rvert$	8	4	8	4	4

The second orthogonality relation states that the columns of the character table are orthogonal. For example, for columns r and s:

$$\sum_i \chi_i(r)\overline{\chi_i(s)} = 1\cdot 1 + 1\cdot(-1) + (-1)\cdot(-1) + (-1)\cdot 1 + 0\cdot 0 = 1 - 1 + 1 - 1 + 0 = 0.$$

For the column r with itself:

$$\sum_i \chi_i(r)\overline{\chi_i(r)} = 1^2 + 1^2 + (-1)^2 + (-1)^2 + 0^2 = 1 + 1 + 1 + 1 + 0 = 4 = |Z_r|.$$

If we normalize the columns by dividing by $\sqrt{|Z_g|}$, we get an orthonormal set of vectors.

	C_1	C_r	C_{r^2}	C_s	C_{sr}
$\frac{\chi_0}{\sqrt{\lvert Z_g\rvert}}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\chi_1}{\sqrt{\lvert Z_g\rvert}}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{8}}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{\chi_2}{\sqrt{\lvert Z_g\rvert}}$	$\frac{1}{\sqrt{8}}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{8}}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{\chi_3}{\sqrt{\lvert Z_g\rvert}}$	$\frac{1}{\sqrt{8}}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{2}$	$-\frac{1}{2}$
$\frac{\chi_4}{\sqrt{\lvert Z_g\rvert}}$	$\frac{2}{\sqrt{8}}$	0	$-\frac{2}{\sqrt{8}}$	0	0

The rows and columns of this matrix are orthonormal.

12.3 Third Orthogonality Relation

Definition 12.6. Let V be an irreducible representation of G, and let $\{v_1, \ldots, v_n\}$ be an orthonormal basis in V with respect to a G-invariant Hermitian inner product. The *matrix elements* of the representation ρ_V are the functions $t^V_{ij} : G \to \mathbb{C}$ defined by

$$t^V_{ij}(g) = \langle \rho_V(g)v_i, v_j\rangle.$$

Theorem 12.7 (Third Orthogonality Relation)**.** *The matrix elements of irreducible representations satisfy the following orthogonality relations in* $F(G, \mathbb{C})$*:*

(a) If V *and* W *are non-isomorphic irreducible representations, their matrix elements are orthogonal:*

$$(t^V_{ij}, t^W_{kl}) = 0 \quad \textit{for all } i, j, k, l.$$

(b) For a single irreducible representation V*:*

$$(t^V_{ij}, t^V_{kl}) = \frac{1}{\dim V}\delta_{ik}\delta_{jl}. \tag{1}$$

Proof. Let $\{v_i\}$ and $\{w_k\}$ be orthonormal bases in irreducible representations V and W respectively. Let $\{w^*_k\}$ be the dual basis in W^*, defined by $w_k^*(u) = \langle u, w_k \rangle$, where $\langle \ , \ \rangle$ is a G-invariant Hermitian inner product.

$$\begin{aligned}
(t_{ij}^V, t_{kl}^W) &= \frac{1}{|G|} \sum_{g \in G} t_{ij}^V(g) \overline{t_{kl}^W(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \langle g v_i, v_j \rangle \overline{\langle g w_k, w_l \rangle} \\
&= \frac{1}{|G|} \sum_{g \in G} \langle g v_i, v_j \rangle \langle w_l, g w_k \rangle \\
&= \frac{1}{|G|} \sum_{g \in G} \langle g v_i, v_j \rangle \langle g^{-1} w_l, w_k \rangle \\
&= \langle P(v_i \otimes w_k^*), v_j \otimes w_l^* \rangle_{V \otimes W^*}
\end{aligned}$$

where $P = \frac{1}{|G|} \sum_{g \in G} g$ is the projector onto the trivial subrepresentation.

(a) If $V \ncong W$, then $V \otimes W^* \cong \mathrm{Hom}(W, V)$ contains no trivial subrepresentation, since $\dim \mathrm{Hom}_G(\mathbb{C}, \mathrm{Hom}(W, V)) = \dim \mathrm{Hom}_G(W, V) = 0$. Thus P acts as the zero operator on $V \otimes W^*$, which implies $(t_{ij}^V, t_{kl}^W) = 0$.

(b) If $V \cong W$, we consider the action of P on $V \otimes V^*$. We have $\dim(V \otimes V^*)^G = \dim \mathrm{Hom}_G(V, V) = 1$. The one-dimensional space of invariants is spanned by the element corresponding to the identity map Id_V. In terms of the basis, this element is $\sum_m v_m \otimes v_m^*$. The projector P maps $V \otimes V^*$ onto this one-dimensional subspace. Let's compute $P(v_i \otimes v_k^*)$. This is an element in $(V \otimes V^*)^G$, so it must be a multiple of the identity element:

$$P(v_i \otimes v_k^*) = c_{ik} \sum_m v_m \otimes v_m^*.$$

The inner product is then

$$(t_{ij}^V, t_{kl}^V) = \langle P(v_i \otimes v_k^*), v_j \otimes v_l^* \rangle = \left\langle c_{ik} \sum_m v_m \otimes v_m^*, v_j \otimes v_l^* \right\rangle = c_{ik} \delta_{jl}.$$

To find c_{ik}, we use the isomorphism $V \otimes V^* \simeq \mathrm{Hom}(V, V)$ again and take the trace of the operator $v_i \otimes v_k^* : v \mapsto v_i \langle v, v_k \rangle$. Its trace is $\langle v_i, v_k \rangle = \delta_{ik}$. The trace is invariant under the action of P, because P projects $V \otimes V^*$ onto the linear span of $(\sum_m v_m \otimes v_m^*)$ parallel to $\{\sum_{ik} a_{ik} v_i \otimes v_k^* \ : \sum_m a_{mm} = 0\}$. We have $\mathrm{Tr}(P(v_i \otimes v_k^*)) = \mathrm{Tr}(v_i \otimes v_k^*) = \delta_{ik}$. Also, $\mathrm{Tr}(c_{ik} \sum_m v_m \otimes v_m^*) = c_{ik} \sum_m \mathrm{Tr}(v_m \otimes v_m^*) = c_{ik} \sum_m \delta_{mm} = c_{ik} \dim V$. Thus, $c_{ik} \dim V = \delta_{ik}$, which gives $c_{ik} = \frac{1}{\dim V} \delta_{ik}$. Substituting this back, we get

$$(t_{ij}^V, t_{kl}^V) = \frac{1}{\dim V} \delta_{ik} \delta_{jl}.$$

This completes the proof. □

Corollary 12.8. *The set of all matrix elements* $\{\sqrt{\dim V}\ t_{ij}^V\}_{V \in \mathrm{Irr}(G), 1 \le i,j \le \dim V}$ *forms an orthonormal basis for the space of functions* $F(G, \mathbb{C})$.

Proof. The previous theorem shows that these functions are orthogonal. The number of such functions is

$$\sum_{V \in \mathrm{Irr}(G)} (\dim V)^2.$$

We know from the structure of the group algebra $\mathbb{C}[G]$ that $\mathbb{C}[G] \cong \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{End}(V)$. Taking dimensions, we get $|G| = \sum_{V \in \mathrm{Irr}(G)} (\dim \mathrm{End}(V)) = \sum_{V \in \mathrm{Irr}(G)} (\dim V)^2$. The space of functions $F(G, \mathbb{C})$ has dimension $|G|$. Since we have found $|G|$ orthogonal functions, they form a basis. □

Example 12.9. Example: Let $G = D_4$ Let $V = V_4$ be the 2-dimensional irreducible representation.

$$\rho_4(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_4(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Other matrices are: $\rho_4(r^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\rho_4(r^3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\rho_4(sr) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\rho_4(sr^2) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $\rho_4(sr^3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The identity element is $\rho_4(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let's check $(t^4_{12}, t^4_{12}) = \frac{1}{\dim V_4}\delta_{11}\delta_{22} = \frac{1}{2}$. The values of $t^4_{12}(g)$ for $g \in \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ are $\{0, 1, 0, -1, 1, 0, -1, 0\}$.

$$\begin{aligned}
(t^4_{12}, t^4_{12}) &= \frac{1}{8}\sum_{g\in D_4} |t^4_{12}(g)|^2 \\
&= \frac{1}{8}(0^2 + 1^2 + 0^2 + (-1)^2 + 1^2 + 0^2 + (-1)^2 + 0^2) \\
&= \frac{1}{8}(0 + 1 + 0 + 1 + 1 + 0 + 1 + 0) = \frac{4}{8} = \frac{1}{2}.
\end{aligned}$$

This matches the formula.

Let's check $(t^3_{11}, t^4_{12}) = 0$. The representation ρ_3 is 1-dimensional, with $\rho_3(s) = 1, \rho_3(r) = -1$. So $t^3_{11}(g) = \chi_3(g)$. The values of $t^3_{11}(g)$ for $g \in \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ are $\{1, -1, 1, -1, 1, -1, 1, -1\}$. The values of $t^4_{12}(g)$ are $\{0, 1, 0, -1, 1, 0, -1, 0\}$.

$$\begin{aligned}
(t^3_{11}, t^4_{12}) &= \frac{1}{8}\sum_{g\in D_4} t^3_{11}(g)\overline{t^4_{12}(g)} \\
&= \frac{1}{8}(1\cdot 0 + (-1)\cdot 1 + 1\cdot 0 + (-1)\cdot(-1) + 1\cdot 1 + (-1)\cdot 0 + 1\cdot(-1) + (-1)\cdot 0) \\
&= \frac{1}{8}(0 - 1 + 0 + 1 + 1 + 0 - 1 + 0) = 0.
\end{aligned}$$

This also matches the formula.

13 Example of a character table

Let us construct the character table of the group A_4

Let $G = A_4$. The conjugacy classes are:

$$\overset{1}{\{1\}}, \quad \overset{3}{\{(12)(34)\}}, \quad \overset{4}{\{(123)\}}, \quad \overset{4}{\{(132)\}}.$$

We find a normal subgroup, the Klein four-group:

$$K = \{1, (12)(34), (13)(24), (14)(32)\}.$$

Since $K \lhd A_4$, we have the quotient group $A_4/K \cong C_3$.

Suppose (U, ρ) is an irreducible representation of $H = G/K$. Let $\varphi : G \to H$ be the quotient homomorphism. Then $\tilde{\rho}(g) = \rho(\varphi(g))$ is a representation of G. This representation is irreducible because any G-subrepresentation of U would also be an H-subrepresentation.

The irreducible representations of $C_3 = \langle t : t^3 = 1\rangle$ are one-dimensional. Let $\zeta = e^{2\pi i/3}$.

- V_0 (trivial): $\rho(t) = 1$
- V_1: $\rho(t) = \zeta$
- V_2: $\rho(t) = \zeta^2$

These give three 1-dimensional representations of A_4. The sum of squares of dimensions is $|A_4| = 12 = \sum_{i=0}^{3}(\dim V_i)^2 = 1^2 + 1^2 + 1^2 + (\dim V_3)^2$, which implies $\dim V_3 = 3$.

Let the characters of V_3 be $(3, a, b, c)$ on the conjugacy classes $\{1\}, \{(12)(34)\}, \{(123)\}, \{(132)\}$ respectively. We can use the second orthogonality relation to compute χ_{V_3}. Namely,

$$\sum_i \chi_i(1)\overline{\chi_i((12)(34))} = 1 + 1 + 1 + 3a = 0,$$

$$\sum_i \chi_i(1)\overline{\chi_i((123))} = 1 + \xi + \xi^2 + b = 0,$$

$$\sum_i \chi_i(1)\overline{\chi_i((132))} = 1 + \xi^2 + \xi + c = 0,$$

which implies $a = -1$, $b = c = 0$.

We can also obtain the same result using the first orthogonality with the first three characters:

$$\begin{aligned}
(\chi_{V_3}, \chi_{V_0}) &= \frac{1}{12}(3\cdot 1 + 3a\cdot 1 + 4b\cdot 1 + 4c\cdot 1) = 0 \implies 3 + 3a + 4b + 4c = 0 \\
(\chi_{V_3}, \chi_{V_1}) &= \frac{1}{12}(3\cdot 1 + 3a\cdot 1 + 4b\cdot \bar{\zeta} + 4c\cdot \bar{\zeta}^2) = 0 \implies 3 + 3a + 4b\bar{\zeta} + 4c\bar{\zeta}^2 = 0 \\
(\chi_{V_3}, \chi_{V_2}) &= \frac{1}{12}(3\cdot 1 + 3a\cdot 1 + 4b\cdot \bar{\zeta}^2 + 4c\cdot \bar{\zeta}) = 0 \implies 3 + 3a + 4b\bar{\zeta}^2 + 4c\bar{\zeta} = 0
\end{aligned}$$

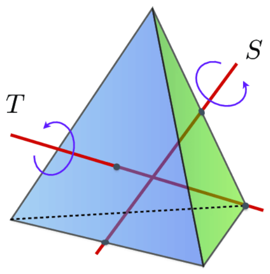
Adding the last two equations gives $2(3+3a)+4b(\bar{\zeta}+\bar{\zeta}^2)+4c(\bar{\zeta}^2+\bar{\zeta}) = 0$. Since $\bar{\zeta}+\bar{\zeta}^2 = -1$, this is $6+6a-4b-4c = 0$. With the first equation, $3+3a+4b+4c = 0$, adding them gives $9+9a = 0 \implies a = -1$. Substituting $a = -1$ into the first equation gives $4b+4c = 0 \implies b = -c$. Substituting into the second equation gives $4b(\bar{\zeta}-\bar{\zeta}^2) = 0$. Since $\bar{\zeta} \neq \bar{\zeta}^2$, we must have $b = 0$, and thus $c = 0$. The character of V_3 is $(3, -1, 0, 0)$.

The character table for A_4 is:

	1	3	4	4
	1	(12)(34)	(123)	(132)
V_0	1	1	1	1
V_1	1	1	ζ	ζ^2
V_2	1	1	ζ^2	ζ
V_3	3	-1	0	0

real type, $\mathrm{FS}(V_0) = 1$
complex type, $\mathrm{FS}(V_1) = 0$
complex type, $\mathrm{FS}(V_2) = 0$
real type, $\mathrm{FS}(V_3) = 1$

The group A_4 is the group of rotational symmetries of a tetrahedron. The representation V_3 corresponds to this 3D action.



The element (12)(34) corresponds to the rotation S by π about an axis through the midpoints of edges (12) and (34). In a suitable basis, this rotation is

$$r_\pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathrm{tr}(r_\pi) = -1 = \chi_{V_3}((12)(34)).$$

The element (123) is the rotation T by $2\pi/3$ about an axis through vertex 4 and the center of the opposite face.

$$r_{2\pi/3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{pmatrix}.$$

The trace is $\mathrm{tr}(r_{2\pi/3}) = 1 - 1/2 - 1/2 = 0 = \chi_{V_3}((123))$. Similarly for $\chi_{V_3}((132))$.

Let's decompose $V_3 \otimes V_3$. The character is $\chi_{V_3^{\otimes 2}} = (\chi_{V_3})^2$.

	1	(12)(34)	(123)	(132)
$\chi_{V_3^{\otimes 2}}$	9	1	0	0

We compute the multiplicities of the irreducibles in $V_3 \otimes V_3$:

$$\begin{aligned}
\langle \chi_{V_3^{\otimes 2}}, \chi_{V_0} \rangle &= \frac{1}{12}(9 \cdot 1 + 3 \cdot 1 \cdot 1) = \frac{12}{12} = 1 \\
\langle \chi_{V_3^{\otimes 2}}, \chi_{V_1} \rangle &= \frac{1}{12}(9 \cdot 1 + 3 \cdot 1 \cdot 1) = \frac{12}{12} = 1 \\
\langle \chi_{V_3^{\otimes 2}}, \chi_{V_2} \rangle &= \frac{1}{12}(9 \cdot 1 + 3 \cdot 1 \cdot 1) = \frac{12}{12} = 1 \\
\langle \chi_{V_3^{\otimes 2}}, \chi_{V_3} \rangle &= \frac{1}{12}(9 \cdot 3 + 3 \cdot 1 \cdot (-1)) = \frac{24}{12} = 2
\end{aligned}$$

Thus, the decomposition is $V_3 \otimes V_3 \cong V_0 \oplus V_1 \oplus V_2 \oplus V_3^{\oplus 2}$.

14 Tensor, Symmetric, and Exterior Power of a Representation

14.1 Tensor Powers of Representations

If V, W are G-representations, then $V \otimes W$ is a representation via $g \cdot (v \otimes w) = (\rho_V(g)v) \otimes (\rho_W(g)w)$. We can consider the n-th tensor power $V^{\otimes n} = V \otimes \cdots \otimes V$.

14.2 Symmetric Powers

Definition 14.1. Let V be a finite-dimensional vector space. The n-th *symmetric power* of V is

$$S^n V = V^{\otimes n} / \mathrm{Span}(\mathcal{T} - \sigma(\mathcal{T})),$$

where $\mathcal{T} = v_1 \otimes v_2 \otimes \cdots \otimes v_n$ and $\sigma(\mathcal{T}) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ for any permutation $\sigma \in S_n$.

Proposition 14.2. *Let $\{e_i\}_{i=1}^k$ be a basis for V, where $\dim V = k$. Then $S^n V$ has a basis consisting of monomials $\{e_{i_1} e_{i_2} \dots e_{i_n} \mid 1 \le i_1 \le i_2 \le \cdots \le i_n \le k\}$, and its dimension is*

$$\dim S^n V = \binom{n+k-1}{n}.$$

Proof sketch. Any tensor $e_{j_1} \otimes \cdots \otimes e_{j_n}$ is identified with $e_{\sigma^{-1}(j_1)} \otimes \cdots \otimes e_{\sigma^{-1}(j_n)}$ in the quotient, so we can reorder the basis vectors to be in non-decreasing order of indices. The number of such monomials is in bijection with the number of choices of $k-1$ dots (shown as stars) out of $(n+k-1)$ dots, where $n+k-1$ is the sum of the power of the monomial n and the $k-1$ separating lines between different $\{e_i\}_{i=1}^k$. Here is an example for $n = 5, k = 4$:

$$\begin{array}{cccccccc}
\bullet & \bullet & \star & \bullet & \star & \bullet & \star & \bullet \\
 & & | & & | & & | & \\
e_1 & e_1 & | & e_2 & | & e_3 & | & e_4
\end{array}$$

This corresponds to the basis element $e_1^2 e_2 e_3 e_4$.
The number of ways to choose $k-1$ dots out of $n+k-1$ dots is $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$. □

Example 14.3. If $\dim V = 3, n = 2$, with basis $\{e_1, e_2, e_3\}$, then $S^2 V$ has basis $\{e_1 e_1,\ e_1 e_2,\ e_1 e_3,\ e_2 e_2,\ e_2 e_3,\ e_3 e_3\}$.

$$\dim S^2 V = \binom{2+3-1}{2} = \binom{4}{2} = 6.$$

Example 14.4. If $\dim V = 2, n = 3$, with basis $\{e_1, e_2\}$, then $S^3 V$ has basis $\{e_1 e_1 e_1,\ e_1 e_1 e_2,\ e_1 e_2 e_2,\ e_2 e_2 e_2\}$.

$$\dim S^3 V = \binom{3+2-1}{3} = \binom{4}{3} = 4.$$

14.3 Exterior Powers

Definition 14.5. Let V be a vector space with $\dim V = k$. The n-th **exterior power** of V is

$$\Lambda^n V = V^{\otimes n} / \mathrm{Span}\{P + s(P) \mid P = v_1 \otimes \cdots \otimes v_n, s \in S_n \text{ is a transposition}\}.$$

This is equivalent to quotienting by $v_1 \otimes \cdots \otimes v_n$ where $v_i = v_j$ for some $i \neq j$.

Proposition 14.6. *For $n \leq k$, $\Lambda^n V$ has a basis given by $\{e_{i_1} \wedge \cdots \wedge e_{i_n} \mid 1 \leq i_1 < i_2 < \cdots < i_n \leq k\}$. The dimension is $\dim \Lambda^n V = \binom{k}{n}$. If $n > k$, $\Lambda^n V = \{0\}$.*

Example 14.7. Let $k = 3, n = 2$. A basis for V is $\{e_1, e_2, e_3\}$. Then $\Lambda^2 V$ has a basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$. Thus, $\dim \Lambda^2 V = \binom{3}{2} = 3$.

Remark 14.8. For any vector space V, we have the decomposition $V^{\otimes 2} \cong S^2 V \oplus \Lambda^2 V$. The dimension check is:

$$\dim S^2 V + \dim \Lambda^2 V = \binom{k+1}{2} + \binom{k}{2} = \frac{k(k+1)}{2} + \frac{k(k-1)}{2} = \frac{k^2 + k + k^2 - k}{2} = k^2 = \dim V^{\otimes 2}.$$

14.4 Algebraic structure on TV, SV, $\wedge V$

Definition 14.9. The *tensor algebra* of V is the graded algebra $TV = \bigoplus_{n \geq 0} V^{\otimes n}$, where $V^{\otimes 0} = \mathbb{C}$. Multiplication is given by concatenation: $a \cdot b = a \otimes b$. If $\{x_1, \ldots, x_k\}$ is a basis for V, then $TV \cong \mathbb{C}\langle x_1, \ldots, x_k \rangle$, the free algebra.

Definition 14.10. The *symmetric algebra* of V is $SV = TV/\langle v \otimes w - w \otimes v \mid v, w \in V \rangle$. It is a graded commutative algebra, $SV = \bigoplus_{n \geq 0} S^n V$. If $\{x_1, \ldots, x_k\}$ is a basis for V, then $SV \cong \mathbb{C}[x_1, \ldots, x_k]$, the polynomial algebra.

Definition 14.11. The *exterior algebra* of V is $\Lambda V = TV/\langle v \otimes v \mid v \in V \rangle$. This is equivalent to the ideal generated by $\{v \otimes w + w \otimes v \mid v, w \in V\}$. It is a graded anti-commutative algebra, $\Lambda V = \bigoplus_{n=0}^{k} \Lambda^n V$. Its dimension is $\dim \Lambda V = \sum_{i=0}^{k} \binom{k}{i} = 2^k$, where $k = \dim V$.

14.5 Symmetric Group Action on $V^{\otimes n}$

Definition 14.12. Let V be a finite-dimensional representation of G. The symmetric group S_n acts on $V^{\otimes n}$ by permutation of factors. For $\sigma \in S_n$, the action is

$$\varphi(\sigma)(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = u_{\sigma^{-1}(1)} \otimes u_{\sigma^{-1}(2)} \otimes \cdots \otimes u_{\sigma^{-1}(n)}.$$

(Note: permuting positions is equivalent to permuting content with the inverse permutation. In particular, the above definition means that whatever is in the i-th position in the pure tensor is sent to the $\sigma(i)$-th position). Let's check that $\varphi(\sigma\mu) = \varphi(\sigma)\varphi(mu)$:

$$\varphi(\sigma\mu)(u_1 \otimes \cdots \otimes u_n) = u_{(\sigma\mu)^{-1}(1)} \otimes \cdots \otimes u_{(\sigma\mu)^{-1}(n)} = u_{\mu^{-1}\sigma^{-1}(1)} \otimes \cdots \otimes u_{\mu^{-1}\sigma^{-1}(n)}.$$

On the other hand $\varphi(\sigma)\varphi(\mu)(u_1 \otimes \cdots \otimes u_n) = \varphi(\sigma)(u_{\mu^{-1}(1)} \otimes \cdots \otimes u_{\mu^{-1}(n)})$. Let $v_i = u_{\mu^{-1}(i)}$. Then this we have

$$\varphi(\sigma)(u_{\mu^{-1}(1)} \otimes \cdots \otimes u_{\mu^{-1}(n)}) = \varphi(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} = u_{\mu^{-1}(\sigma^{-1}(1))} \otimes \ldots u_{\mu^{-1}(\sigma^{-1}(n))} =$$

$$= u_{(\sigma\mu)^{-1}(1)} \otimes \cdots \otimes u_{(\sigma\mu)^{-1}(n)}.$$

This is a left action.

Proposition 14.13. *The action of G on $V^{\otimes n}$ commutes with the action of S_n.*

Proof. Let ρ be the representation of G and φ be the representation of S_n.

$$\begin{aligned} \rho(g)\varphi(\sigma)(u_1 \otimes \cdots \otimes u_n) &= \rho(g)(u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(n)}) \\ &= (\rho(g)u_{\sigma^{-1}(1)}) \otimes \cdots \otimes (\rho(g)u_{\sigma^{-1}(n)}). \end{aligned}$$

Let $v_j = \rho(g)u_j$. Then the above is $v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. On the other hand,

$$\begin{aligned} \varphi(\sigma)\rho(g)(u_1 \otimes \cdots \otimes u_n) &= \varphi(\sigma)(\rho(g)u_1 \otimes \cdots \otimes \rho(g)u_n) \\ &= \varphi(\sigma)(v_1 \otimes \cdots \otimes v_n) \\ &= v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}. \end{aligned}$$

The two expressions are equal. □

Corollary 14.14. *Every S_n-isotypical component in $V^{\otimes n}$ is a G-subrepresentation of $V^{\otimes n}$.*

Definition 14.15. If a representation W of a group H decomposes as $W = \bigoplus_k L_k^{\oplus n_k}$, where $\{L_k\}$ are inequivalent irreducible representations of H, then $L_k^{\oplus n_k}$ is called the **isotypical component** of W corresponding to L_k.

Proof of Corollary. The action of any $g \in G$ provides a linear map $\rho(g) : V^{\otimes n} \to V^{\otimes n}$. Since the actions of G and S_n commute, $\rho(g)$ is an S_n-intertwiner (i.e., a homomorphism of S_n-representations). By Schur's Lemma, any such map must preserve the isotypical components. That is, if $W_k = L_k^{\oplus n_k}$ is an isotypical component, then $\rho(g)(W_k) \subseteq W_k$. This holds for all $g \in G$, so each W_k is a G-subrepresentation. $\square$

In particular, $S^n V$ and $\Lambda^n V$ are G-subrepresentations of $V^{\otimes n}$ (see PS 9).

- $S^n V \subset V^{\otimes n}$ is the isotypical component corresponding to the trivial representation of S_n.
- $\Lambda^n V \subset V^{\otimes n}$ is the isotypical component corresponding to the sign representation of S_n.

14.6 Frobenius-Schur Indicator

Definition 14.16. Let V be an irreducible complex representation of a finite group G.

1. If $V \ncong V^*$, V is of **complex type**. The Frobenius-Schur indicator is $\mathrm{FS}(V) = 0$.
2. If $V \cong V^*$, this is equivalent to the trivial representation V_0 appearing in $V \otimes V$. Since $V \otimes V \cong S^2 V \oplus \Lambda^2 V$, one of these subspaces must contain V_0.
   - If $V_0 \subset S^2 V$, V is of **real type**, and $\mathrm{FS}(V) = 1$.
   - If $V_0 \subset \Lambda^2 V$, V is of **quaternionic type**, and $\mathrm{FS}(V) = -1$.

Remark 14.17. 1. An irreducible representation V of G is of complex type if and only if $V \ncong V^*$ if and only if the character $\chi_V \neq \chi_{V^*} = \overline{\chi_V}$ if and only if the χ_V is not real.

2. An irreducible representation V of G is of real type if and only if χ_V is real and there exist a G-invariant non-degenerate symmetric bilinear form on V. Indeed, the space of symmetric bilinear forms on V is given by $S^2(V^*)$. We have $V \cong V^*$ and $S^2(V^*) \cong S^2(V) \supset V_0$ if and only if V is of real type.

3. An irreducible representation V of G is of quaternionic type if and only if χ_V is real and there exist a G-invariant non-degenerate skew-symmetric bilinear form on V. Indeed, the space of skew-symmetric bilinear forms on V is given by $\Lambda^2(V^*)$. We have $V \cong V^*$ and $\Lambda^2(V^*) \cong \Lambda^2(V) \supset V_0$ if and only if V is of quaternionic type.

Lemma 14.18. *(PS 8). Let V be a finite-dimensional representation of G. Then for any $g \in G$,*

$$\chi_V(g^2) = \chi_{S^2V}(g) - \chi_{\Lambda^2V}(g).$$

Corollary 14.19. *The Frobenius-Schur indicator can be calculated as follows:*

$$\mathrm{FS}(V) = \frac{1}{|G|}\sum_{g \in G} \chi_V(g^2).$$

Proof. Consider the sum over the group elements of the equality of the Lemma 14.18:

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g^2) = \frac{1}{|G|}\sum_{g\in G}\chi_{S^2V}(g) - \frac{1}{|G|}\sum_{g\in G}\chi_{\Lambda^2V}(g) = \chi_{S^2V}(P) - \chi_{\Lambda^2V}(P),$$

where $P = \frac{1}{|G|}\sum_{g\in G} g$ is the G-invariant projector onto the trivial representation V_0 and the trace of P over any representation computes the multiplicity of V_0 in it.. Since V is irreducible, we have that $V \otimes V^*$ contains exactly one copy of V_0. The multiplicity of V_0 in $V \otimes V$ is zero if and only if $V \ncong V^*$, otherwise the multiplicity of V_0 in $V \otimes V \cong S^2(V) \oplus \wedge^2(V)$ is exactly 1. In particular, $V_0 \subset S^2(V)$ if and only if $\mathrm{FS}(V) = 1$ and $V_0 \subset \wedge^2(V)$ if and only if $\mathrm{FS}(V) = -1$. Therefore,

$$\mathrm{FS}(V) = \chi_{S^2V}(P) - \chi_{\Lambda^2V}(P) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g^2).$$

$\square$

Proposition 14.20. *The number of elements of order 1 or 2 in G is given by*

$$|\{g \in G : g^2 = 1\}| = \sum_{V \in \mathrm{Irr}(G)} \mathrm{FS}(V) \cdot \dim V = \sum_{V \ \textit{real}} \dim V - \sum_{V \ \textit{quat.}} \dim V.$$

Proof. Consider the left regular representation of G on $\mathbb{C}[G]$. The trace of the action of an element $h \in G$ is

$$\operatorname{tr}|_{\mathbb{C}[G]}(h) = \begin{cases} |G|, & \text{if } h = 1 \\ 0, & \text{if } h \neq 1. \end{cases}$$

Therefore, $\operatorname{tr}|_{\mathbb{C}[G]}(g^2)$ is $|G|$ if $g^2 = 1$ and 0 otherwise. The number of elements with $g^2 = 1$ is

$$|\{g \in G : g^2 = 1\}| = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}|_{\mathbb{C}[G]}(g^2).$$

We can also express this trace by decomposing $\mathbb{C}[G]$ into irreducible representations: $\mathbb{C}[G] \cong \bigoplus_{V \in \operatorname{Irr}(G)} V^{\oplus \dim V}$.

$$\begin{aligned} |\{g \in G : g^2 = 1\}| &= \frac{1}{|G|} \sum_{g \in G} \sum_{V \in \operatorname{Irr}(G)} (\dim V) \chi_V(g^2) \\ &= \sum_{V \in \operatorname{Irr}(G)} (\dim V) \left(\frac{1}{|G|} \sum_{g \in G} \chi_V(g^2) \right) \\ &= \sum_{V \in \operatorname{Irr}(G)} (\dim V) \left(\frac{1}{|G|} \sum_{g \in G} (\chi_{S^2 V}(g) - \chi_{\Lambda^2 V}(g)) \right) \qquad \text{(by Lemma 14.18)} \\ &= \sum_{V \in \operatorname{Irr}(G)} (\dim V) \left(\langle \chi_{S^2 V}, \chi_{V_0} \rangle - \langle \chi_{\Lambda^2 V}, \chi_{V_0} \rangle \right) \\ &= \sum_{V \in \operatorname{Irr}(G)} (\dim V) \left(\dim (S^2 V)^G - \dim (\Lambda^2 V)^G \right). \end{aligned}$$

By definition, for an irreducible representation V:

- $\dim(S^2 V)^G = 1$ and $\dim(\Lambda^2 V)^G = 0$ if V is of real type.
- $\dim(S^2 V)^G = 0$ and $\dim(\Lambda^2 V)^G = 1$ if V is of quaternionic type.
- $\dim(S^2 V)^G = 0$ and $\dim(\Lambda^2 V)^G = 0$ if V is of complex type.

The expression is therefore equal to

$$\sum_{V \in \operatorname{Irr}(G)} (\dim V) \operatorname{FS}(V) = \sum_{V \text{ real}} \dim V - \sum_{V \text{ quaternionic}} \dim V.$$

□

15 Algebraic integers in representation theory

Definition 15.1. $z \in \mathbb{C}$ is an **algebraic integer** if z is a root of a monic polynomial with integer coefficients (Ex: $z^k - 1 = 0$, roots of unity).

Lemma 15.2. *z is an algebraic integer if and only if z is an eigenvalue of a matrix with integer coefficients.*

Proof. If z is an eigenvalue of a matrix with integer coefficients, then it is a root of the characteristic polynomial, which is monic with integer coefficients.

If z is a root of $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then it is an eigenvalue of the companion matrix:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}$$

By induction, the characteristic polynomial is $p(\lambda)$. For example, for $n = 3$:

$$\det \begin{pmatrix} \lambda & 0 & a_0 \\ -1 & \lambda & a_1 \\ 0 & -1 & \lambda + a_2 \end{pmatrix} = \lambda^2(\lambda + a_2) + a_0 + a_1 \lambda = \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0$$

□

Corollary 15.3. *The set of algebraic integers $\mathbb{A}$ is a ring.*

Proof. Let $\alpha, \beta \in \mathbb{A}$. Then they are eigenvalues of integer matrices A, B with eigenvectors v, w respectively.

$$Av = \alpha v, \quad Bw = \beta w$$

Then $\alpha + \beta$ is an eigenvalue of $A \otimes \mathrm{Id} + \mathrm{Id} \otimes B$:

$$(A \otimes \mathrm{Id} + \mathrm{Id} \otimes B)(v \otimes w) = \alpha(v \otimes w) + \beta(v \otimes w) = (\alpha + \beta)(v \otimes w)$$

And $\alpha\beta$ is an eigenvalue of $A \otimes B$:

$$(A \otimes B)(v \otimes w) = (\alpha\beta)(v \otimes w)$$

Since $A \otimes \mathrm{Id} + \mathrm{Id} \otimes B$ and $A \otimes B$ are integer matrices, $\alpha + \beta$ and $\alpha\beta$ are algebraic integers. $\square$

Definition 15.4. Let α be an algebraic integer, and $p(x)$ the minimal monic, integer coefficient polynomial such that α is a root of $p(x)$. Then other roots of $p(x)$ are called the **algebraic conjugates** of α.

Lemma 15.5. *Algebraic conjugates of $\alpha_1 + \alpha_2 + \cdots + \alpha_m$, where $\alpha_i \in \mathbb{A}$, are of the form $\alpha_1' + \alpha_2' + \cdots + \alpha_m'$, where α_i' is an algebraic conjugate to α_i.*

Proof. If α_i is an eigenvalue of A_i, then $\alpha_1 + \alpha_2$ is an eigenvalue of $A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2$. The algebraic conjugates to $(\alpha_1 + \alpha_2)$ are the other eigenvalues of $A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2$.

If $\{v_i\}$ are eigenvectors of A_1 and $\{w_j\}$ are eigenvectors of A_2, then $\{v_i \otimes w_j\}$ are the eigenvectors of $A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2$, with the eigenvalues of the form $\alpha_1' + \alpha_2'$, where α_1' is an eigenvalue of A_1 (and thus conjugate to α_1), and α_2' is an eigenvalue of A_2 (and thus conjugate to α_2). $\square$

Proposition 15.6. $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$.

Proof. Let $z \in \mathbb{A} \cap \mathbb{Q}$. Since $z \in \mathbb{A}$, it is a root of a monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with $a_i \in \mathbb{Z}$. Since $z \in \mathbb{Q}$, we can write $z = \frac{p}{q}$ with $\gcd(p, q) = 1$ and $p, q \in \mathbb{Z}$, $q \neq 0$. Substituting into the polynomial equation:

$$\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

Multiply by q^n:

$$p^n + a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0$$

This can be written as $p^n = -q(a_{n-1}p^{n-1} + \cdots + a_0 q^{n-1})$. This shows that q divides p^n. Since $\gcd(p, q) = 1$, this implies $q = \pm 1$. Therefore, $z = \frac{p}{\pm 1} = \pm p$, which means $z \in \mathbb{Z}$. $\square$

Theorem 15.7. *Let G be a finite group and V a complex irreducible representation of G. Then $\dim V$ divides $|G|$.*

Proof. Let C be a conjugacy class in G, and let $R = \sum_{h \in C} h \in \mathbb{Z}[G]$. Then R is central in $\mathbb{C}[G]$. By Schur's Lemma, R acts by a scalar on an irreducible representation V, so $\rho_V(R) = \lambda \, \mathrm{Id}_V$. We claim that this scalar λ is an algebraic integer. Indeed, consider the action of R by multiplication in the finite $\mathbb{Z}$-module $\mathbb{Z}[G]$ It acts in the basis of matrix elements as a matrix with integer coefficients. The characteristic polynomial of this matrix is a monic polynomial with integer coefficients satisfied by R, so its image $\rho_V(R)$ satisfies the same equation. Thus λ is a root of a monic polynomial with integer coefficients.

Taking the trace of $\rho_V(R) = \lambda \, \mathrm{Id}_V$:

$$\mathrm{tr}(\rho_V(R)) = \mathrm{tr}(\lambda \, \mathrm{Id}_V) = \lambda \dim V$$

Also,

$$\mathrm{tr}(\rho_V(R)) = \mathrm{tr}\left(\sum_{h \in C} \rho_V(h)\right) = \sum_{h \in C} \mathrm{tr}(\rho_V(h)) = \sum_{h \in C} \chi_V(h)$$

Since the character is constant on the conjugacy classes, for any $g \in C$, $\chi_V(h) = \chi_V(g)$ for all $h \in C$.

$$\sum_{h \in C} \chi_V(g) = |C|\chi_V(g)$$

So, $|C|\chi_V(g) = \lambda \dim V$, which implies $\lambda = \frac{|C|\chi_V(g)}{\dim V}$. We know $\lambda \in \mathbb{A}$.

Let $\{C_i\}$ be the set of conjugacy classes of G, and let $g_i \in C_i$ be a representative for each class. Let $\lambda_i = \frac{|C_i|\chi_V(g_i)}{\dim V}$. Each λ_i is an algebraic integer. Consider the sum $\sum_i \lambda_i \overline{\chi_V(g_i)}$. This is an algebraic integer because:

1. $\lambda_i \in \mathbb{A}$.
2. $\overline{\chi_V(g_i)}$ is a sum of roots of unity, hence an algebraic integer.
3. $\mathbb{A}$ is a ring.

Let's compute the sum:

$$\begin{aligned}\sum_i \lambda_i \overline{\chi_V(g_i)} &= \sum_i \frac{|C_i|\chi_V(g_i)}{\dim V}\overline{\chi_V(g_i)} \\ &= \frac{1}{\dim V}\sum_i |C_i|\chi_V(g_i)\overline{\chi_V(g_i)} \\ &= \frac{1}{\dim V}\sum_{g\in G} \chi_V(g)\overline{\chi_V(g)} \\ &= \frac{|G|}{\dim V}(\chi_V, \chi_V)\end{aligned}$$

Since V is irreducible, the inner product of its character with itself is $(\chi_V, \chi_V) = 1$. So, we have $\frac{|G|}{\dim V} \in \mathbb{A}$. But $\frac{|G|}{\dim V}$ is also a rational number. Therefore, $\frac{|G|}{\dim V} \in \mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$. This means that $\dim V$ divides $|G|$. □

Example 15.8. Let $G = S_3$. Let $C = \{(123), (132)\}$. Let $R = (123) + (132) \in \mathbb{Z}[S_3]$. We can show R satisfies a monic polynomial with integer coefficients. For instance, $(R)^2 = (123)^2 + (132)^2 + (123)(132) + (132)(123) = (132) + (123) + 2e = R + 2e$. So $R^2 - R - 2e = 0$. The polynomial is $x^2 - x - 2 = 0$.

16 Burnside's Theorem

Definition 16.1. A group G is **solvable** if there exists a series of nested subgroups $\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ such that $G_i \triangleleft G_{i+1}$ is a normal subgroup and the quotient group G_{i+1}/G_i is abelian for all i.

Theorem 16.2 (Burnside)**.** *Any group of order $p^a q^b$, where p, q are primes, is solvable.*

First, a simpler case:

Proposition 16.3. *Any group G of order p^a (a p-group) is solvable.*

Proof. By induction on a. If $a = 0$, $G = \{e\}$, which is solvable. If $a = 1$, $G \cong C_p$, which is abelian and thus solvable. Assume the proposition holds for all groups of order p^k with $k < a$. Consider the class equation for G:

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$$

where $Z(G)$ is the center of G and the sum is over representatives of conjugacy classes with more than one element. $|G| = p^a$ is divisible by p. Each index $[G : C_G(x_i)]$ is greater than 1 and must be a power of p. Thus, p divides each term in the sum. It follows that p must divide $|Z(G)|$. So, the center $Z(G)$ is non-trivial. $Z(G)$ is a normal subgroup of G. Consider the quotient group $G/Z(G)$. Its order is $|G/Z(G)| = p^k$ for some $k < a$. By the induction hypothesis, $G/Z(G)$ is solvable. Since $Z(G)$ is abelian (and thus solvable), and $G/Z(G)$ is solvable, it follows that G is solvable. □

The proof of Burnside's theorem requires the following lemmas.

Proposition 16.4. *Let V be an irreducible representation of a finite group G and C a conjugacy class in G such that* $\gcd(|C|, \dim V) = 1$. *Then for any $g \in C$, either $\chi_V(g) = 0$ or $\rho_V(g) = \lambda \cdot \mathrm{Id}_V$ for some scalar $\lambda \in \mathbb{C}$.*

Proof. As shown before, $\frac{|C|\chi_V(g)}{\dim V}$ is an algebraic integer. Since $\gcd(|C|, \dim V) = 1$, there exist integers a, b such that $a|C| + b\dim V = 1$. Multiplying by $\frac{\chi_V(g)}{\dim V}$:

$$a\frac{|C|\chi_V(g)}{\dim V} + b\chi_V(g) = \frac{\chi_V(g)}{\dim V}$$

The first term $a\frac{|C|\chi_V(g)}{\dim V}$ is an algebraic integer. The second term $b\chi_V(g)$ is also an algebraic integer (since characters are sums of roots of unity). Since the set of algebraic integers $\mathbb{A}$ is a ring, their sum $\frac{\chi_V(g)}{\dim V}$ is an algebraic integer.

Let $n = \dim V$. The eigenvalues of $\rho_V(g)$ are roots of unity, say $\varepsilon_1, \dots, \varepsilon_n$. Then $\chi_V(g) = \varepsilon_1 + \cdots + \varepsilon_n$. Let $t = \frac{\chi_V(g)}{n} = \frac{\varepsilon_1 + \cdots + \varepsilon_n}{n}$. We know $t \in \mathbb{A}$. By the triangle inequality, $|t| = \frac{|\varepsilon_1 + \cdots + \varepsilon_n|}{n} \leq \frac{|\varepsilon_1| + \cdots + |\varepsilon_n|}{n} = \frac{n}{n} = 1$. Equality holds if and only if all ε_i are equal, say $\varepsilon_1 = \cdots = \varepsilon_n = \varepsilon$. In this case, $\rho_V(g)$ is a scalar matrix $\varepsilon \operatorname{Id}_V$.

If $|t| < 1$, consider the algebraic conjugates of t. If t' is conjugate to t, then nt' is conjugate to nt and therefore any conjugate t' is of the form $\frac{\varepsilon_1' + \cdots + \varepsilon_n'}{n}$, where ε_i' is a conjugate of ε_i (and thus also a root of unity). So $|t'| \leq 1$ for all conjugates. Let $P(x)$ be the minimal monic polynomial of t with integer coefficients. The roots of $P(x)$ are the conjugates $t_1, \dots, t_k$ of t. The constant term is $P(0) = (-1)^k t_1 \dots t_k$, which is an integer. The product of the absolute values of the roots is $|t_1 \dots t_k| \leq 1$. Since one root t has $|t| < 1$, we have $|P(0)| = |t_1 \dots t_k| < 1$. The only integer with absolute value strictly less than 1 is 0. So $P(0) = 0$. This implies that $t = 0$, so $\chi_V(g) = 0$. ☐

Proposition 16.5. *Let G be a finite group and C a conjugacy class of order p^k, where p is a prime and $k > 0$. Then G is not a simple group.*

Proof. Let $g \in C$. Since $k > 0$, $g \neq e$. By the second orthogonality relation for characters, since g is not conjugate to the identity, we have

$$\sum_{V \in \operatorname{Irr}(G)} \overline{\chi_V(e)} \cdot \chi_V(g) = \sum_{V \in \operatorname{Irr}(G)} (\dim V) \chi_V(g) = 0.$$

We partition the irreducible representations $\operatorname{Irr}(G)$ into three sets:

- V_0: the trivial representation.
- $D = \{V \in \operatorname{Irr}(G) \setminus \{V_0\} \mid p \text{ divides } \dim V\}$.
- $N = \{V \in \operatorname{Irr}(G) \setminus \{V_0\} \mid p \text{ does not divide } \dim V\}$.

The sum becomes:

$$\underbrace{(\dim V_0)\chi_{V_0}(g)}_{1 \cdot 1 = 1} + \sum_{V \in D} (\dim V)\chi_V(g) + \sum_{V \in N} (\dim V)\chi_V(g) = 0.$$

The sum $\sum_{V \in D} (\dim V)\chi_V(g)$ is a sum of terms where $\dim V$ is a multiple of p. Since each $\chi_V(g)$ is an algebraic integer, this sum is of the form $p \cdot a$ for some algebraic integer $a \in \mathbb{A}$. So, $1 + p \cdot a + \sum_{V \in N} (\dim V)\chi_V(g) = 0$. This implies $\frac{1 + \sum_{V \in N} (\dim V)\chi_V(g)}{p} = -a \in \mathbb{A}$.

If the set N were empty, we would have $1 + p \cdot a = 0$, so $a = -1/p$. But $a \in \mathbb{A}$ and $-1/p \in \mathbb{Q} \setminus \mathbb{Z}$ (for $p > 1$), which contradicts $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$. Thus, N cannot be empty. Also, $\sum_{V \in N} (\dim V)\chi_V(g) \neq 0$. This implies there must exist some $V \in N$ such that $\chi_V(g) \neq 0$.

For this representation $V \in N$, we have $p \nmid \dim V$. Since $|C| = p^k$, we have $\gcd(|C|, \dim V) = 1$. By Proposition 16.4, since $\chi_V(g) \neq 0$, it must be that $\rho_V(g)$ acts as a scalar matrix, $\rho_V(g) = \lambda \operatorname{Id}_V$.

Now, let $H = \langle ab^{-1} \mid a, b \in C \rangle$ be the subgroup generated by ab^{-1} with $a, b \in C$. This subgroup has the following properties:

- H is normal: For any $x \in G$, $x(ab^{-1})x^{-1} = (xax^{-1})(xbx^{-1})^{-1}$. Since C is a conjugacy class, $xax^{-1} \in C$ and $xbx^{-1} \in C$. So $xHx^{-1} \subseteq H$.
- H is nontrivial: If $H = \{e\}$, then $ab^{-1} = e$ for all $a, b \in C$, which means $a = b$. This would imply $|C| = 1$. But we are given $|C| = p^k$ with $k > 0$. So $H \neq \{e\}$.
- H is proper: For any generator ab^{-1} of H, its image in the representation V is $\rho_V(ab^{-1}) = \rho_V(a)\rho_V(b)^{-1} = (\lambda \operatorname{Id}_V)(\lambda^{-1} \operatorname{Id}_V) = \operatorname{Id}_V$. So H is contained in the kernel of ρ_V. Since V is a non-trivial irreducible representation ($V \in N$), its kernel is a proper normal subgroup of G. Thus $H \subseteq \ker(\rho_V) \neq G$.

We have found a proper nontrivial normal subgroup H. Therefore, G is not simple. ☐

Proof of Burnside's Theorem. Let G be a counter-example of the smallest order. Such a group G must be simple. If not, it has a proper normal subgroup N. Then $|N|$ and $|G/N|$ are of the form $p^x q^y$ and are smaller than $|G|$. By the minimality of G, both N and G/N are solvable. If N and G/N are solvable, then G is solvable, which is a contradiction. So, a minimal counter-example must be simple.

Let G be a simple group of order $p^a q^b$. By Proposition 16.5 it cannot have a conjugacy class of order p^k or q^k with $k \geq 1$. So the order of any conjugacy class in G is either 1 or is divisible by pq. If all conjugacy classes are one-element, the group is abelian and thus solvable. Suppose we have nontrivial conjugacy classes. Then the class equation gives

$$|G| = |Z(G)| + \sum_i |C_i|.$$

Here $|G| = p^a q^b$, and pq divides $|C_i|$ for all i. Also, $|Z(G) \geq 1$ since it contains $g = 1$. Therefore pq must divide $|Z(G)|$, which means that G is not simple, a contradiction.

□

17 Induced Representations

Induction is a method to construct a representation of a group G starting from a representation of a subgroup $H \subset G$.

If V is a complex representation of G, and $H \subset G$ is a subgroup, then V is also a representation of H by restriction, denoted $\operatorname{Res}_H^G V$. If V is irreducible, the restriction $\operatorname{Res}_H^G V$ does not have to be irreducible.

Definition 17.1. Let $H \subset G$ be a subgroup and let V be a complex representation of H with representation map $\rho_V : H \to GL(V)$. The *induced representation* $\operatorname{Ind}_H^G V$ is defined as the complex vector space of functions

$$\operatorname{Ind}_H^G V = \{f : G \to V \mid f(hx) = \rho_V(h) f(x) \text{ for all } x \in G, h \in H\}$$

with the action of G given by $(g \cdot f)(x) = f(xg)$ for all $g, x \in G$.

It is easy to check that $\operatorname{Ind}_H^G V$ is a complex vector space. This action of G is well defined:

$$(g \cdot (g' \cdot f))(x) = (g' \cdot f)(xg) = f(xgg') = ((gg') \cdot f)(x).$$

This action preserves the space $\operatorname{Ind}_H^G V$:

$$(g \cdot f)(hx) = f(hxg) = \rho_V(h) f(xg) = \rho_V(h)(g \cdot f)(x).$$

We have an isomorphism of vector spaces

$$\operatorname{Ind}_H^G V \simeq \operatorname{Hom}_H(\mathbb{C}[G], V)$$

where $\mathbb{C}[G]$ is considered as a left H-module.

Proposition 17.2. *The dimension of the induced representation is given by*

$$\dim \operatorname{Ind}_H^G V = (\dim V) \cdot \frac{|G|}{|H|} = (\dim V) \cdot [G : H].$$

Indeed, we can introduce a basis in Ind_H^G. Let $\{w_i\}_{i=1}^n$ be a basis in V and $\{\mu \in H \backslash G = \{Hg : g \in G\}\}$ representatives of the right H-cosets in G. Then $\{\delta_\sigma^i\}$ such that $\delta_\sigma^i(\mu) = \delta_{\sigma,\mu} w_i$ is a basis in Ind_H^G.

17.1 Example: Induced Representations of A_4

Let's consider the alternating group A_4.

$$A_4 = \left\{ 1, \begin{pmatrix} (12)(34) \\ (13)(24) \\ (14)(23) \end{pmatrix}, \begin{pmatrix} (123) \\ (134) \\ (142) \\ (243) \end{pmatrix}, \begin{pmatrix} (132) \\ (143) \\ (124) \\ (234) \end{pmatrix} \right\}$$

Let $K = \{1, (12)(34), (13)(24), (14)(23)\}$. K is a normal subgroup of A_4 isomorphic to the Klein four-group $C_2 \times C_2$. The index is $[A_4 : K] = 12/4 = 3$. The right cosets of K in A_4 are:

$$\{K \cdot 1, \quad K \cdot (123), \quad K \cdot (132)\}$$

Let $\{w_i\}$ be a basis for a representation V of K. A basis for $\operatorname{Ind}_K^{A_4} V$ can be constructed from functions δ_σ^i supported on a single coset σ. Let the coset representatives be $1, (123), (132)$. We can define a basis vector δ_σ for each coset σ. For a 1-dimensional representation V, the basis for $\operatorname{Ind}_K^{A_4} V$ can be denoted $\{\delta_1, \delta_{123}, \delta_{132}\}$.

Let V_0 be the trivial one-dimensional representation of K, so $\rho_{V_0}(k) = 1$ for all $k \in K$. Let's compute the representation $\rho_{\text{Ind}} = \rho_{\operatorname{Ind}_K^{A_4} V_0}$.

Action of $g = (12)(34) \in K$:

$$\begin{aligned}
(\rho_{\text{Ind}}(g)\delta_1)(x) &= \delta_1(xg) = \delta_1(x) \quad (\text{since } x \in K \Leftrightarrow xg \in K) \\
(\rho_{\text{Ind}}(g)\delta_{123})(x) &= \delta_{123}(xg) = \delta_{123}(xgx^{-1}x) = \delta_{123}(k'x) \quad (\text{for some } k' \in K) \\
&= \rho_{V_0}(k')\delta_{123}(x) = \delta_{123}(x) \\
(\rho_{\text{Ind}}(g)\delta_{132})(x) &= \delta_{132}(xg) = \delta_{132}(x) \quad (\text{similarly })
\end{aligned}$$

The action of any element of K is trivial on the basis vectors. Thus, for any $k \in K$,

$$\rho_{\mathrm{Ind}}(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \implies \quad \chi_{\mathrm{Ind}_K^{A_4} V_0}(k) = 3.$$

Action of $g = (123)$:

$$(\rho_{\mathrm{Ind}}(g)\delta_1)(x) = \delta_1(x(123))$$

Since δ_1 is supported on K, this function is non-zero only if $x(123) \in K$, which means $x \in K(132)$. This corresponds to the basis vector δ_{132}. So (123) maps $\delta_1 \to \delta_{132}$.

$$(\rho_{\mathrm{Ind}}(g)\delta_{123})(x) = \delta_{123}(x(123)) \quad \text{(non-zero for } x(123) \in K(123)) \implies \delta_{123} \to \delta_1$$
$$(\rho_{\mathrm{Ind}}(g)\delta_{132})(x) = \delta_{132}(x(123)) \quad \text{(non-zero for } x(123) \in K(132)) \implies \delta_{132} \to \delta_{123}$$

So the action of (123) permutes the basis vectors: $\delta_1 \to \delta_{132}$, $\delta_{123} \to \delta_1$, $\delta_{132} \to \delta_{123}$.

$$\rho_{\mathrm{Ind}}(123) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \implies \quad \chi_{\mathrm{Ind}_K^{A_4} V_0}(123) = 0.$$

Similarly for $g = (132)$, we find $\chi_{\mathrm{Ind}_K^{A_4} V_0}(132) = 0$.

Now let $V_\pm$ be one of the non-trivial one-dimensional representations of K. For example, let $\chi_{V_\pm}((12)(34)) = 1$, $\chi_{V_\pm}((13)(24)) = -1$, $\chi_{V_\pm}((14)(23)) = -1$. Let's compute the character of $\mathrm{Ind}_K^{A_4} V_\pm$. For $g = (12)(34)$:

- The action on δ_1: $(\rho_{\mathrm{Ind}}(g)\delta_1)(x) = \delta_1(xg) = \delta_1(gx) = \rho(g)\delta_1(x) = \delta_1(x)$, since $x \in K$ commutes with $g \in K$ and $\rho(g) = 1$. Therefore $\rho_{\mathrm{Ind}}(g) : \delta_1 \to \delta_1$.
- The action on δ_{123}: $(\rho_{\mathrm{Ind}}(g)\delta_{123})(x) = \delta_{123}(xg)$. We compute $x(12)(34) = (123)$ implies $x = (134) = (14)(23)(123)$, therefore we get $\rho((14)(23))\delta_{123} = -\delta_{123}$. Therefore $\rho_{\mathrm{Ind}}(g) : \delta_{123} \to -\delta_{123}$.
- The action on δ_{132}: $(\rho_{\mathrm{Ind}}(g)\delta_{132})(x) = \delta_{132}(xg)$. We compute $x(12)(34) = (132)$ implies $x = (234) = (13)(24)(132)$, therefore we get $\rho((13)(24))\delta_{132} = -\delta_{132}$. Therefore $\rho_{\mathrm{Ind}}(g) : \delta_{132} \to -\delta_{132}$.

The calculation shows:

$$\rho_{\mathrm{Ind}_K^{A_4} V_\pm}((12)(34)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \implies \chi_{\mathrm{Ind}_K^{A_4} V_\pm}((12)(34)) = -1.$$

The action of (123) and (132) is similar permutations as for V_0, so their characters are 0.

The character table of A_4 (with $\zeta = e^{2\pi i/3}$) and our computed induced characters:

A_4	1	(12)(34) (3)	(123) (4)	(132) (4)
V_0	1	1	1	1
V_ζ	1	1	ζ	ζ^2
V_{ζ^2}	1	1	ζ^2	ζ
V_3	3	-1	0	0
$\mathrm{Ind}_K^{A_4} V_0$	3	3	0	0
$\mathrm{Ind}_K^{A_4} V_\pm$	3	-1	0	0

Using character theory, we can decompose these induced representations:

$$\langle \chi_{\mathrm{Ind}\,V_0}, \chi_{V_0} \rangle = \frac{1}{12}(3 \cdot 1 + 3 \cdot 3 \cdot 1) = \frac{12}{12} = 1$$
$$\langle \chi_{\mathrm{Ind}\,V_0}, \chi_{V_\zeta} \rangle = \frac{1}{12}(3 \cdot 1 + 3 \cdot 3 \cdot 1) = 1$$
$$\langle \chi_{\mathrm{Ind}\,V_0}, \chi_{V_{\zeta^2}} \rangle = \frac{1}{12}(3 \cdot 1 + 3 \cdot 3 \cdot 1) = 1$$
$$\langle \chi_{\mathrm{Ind}\,V_0}, \chi_{V_3} \rangle = \frac{1}{12}(3 \cdot 3 + 3 \cdot 3 \cdot (-1)) = 0$$

Therefore, $\mathrm{Ind}_K^{A_4} V_0 \cong V_0 \oplus V_\zeta \oplus V_{\zeta^2}$.

For the other induced representation:

$$\chi_{\mathrm{Ind}_K^{A_4} V_\pm} = (3, -1, 0, 0) = \chi_{V_3}.$$

Therefore, $\mathrm{Ind}_K^{A_4} V_\pm \cong V_3$. This shows that the unique 3-dimensional irreducible representation of A_4 can be constructed by inducing a 1-dimensional representation from the subgroup K.

17.2 Character Formula for the Induced Representation

Theorem 17.3 (Frobenius Character Formula)**.** *Let $H \subset G$ be a subgroup, and let V be a representation of H with character χ_V. Let $\{x_\sigma\}_{\sigma \in H\backslash G}$ be a set of representatives for the right cosets of H in G. The character $\chi_{\mathrm{Ind}\,V}$ of the induced representation $\mathrm{Ind}_H^G V$ is given by*

$$\chi_{\mathrm{Ind}\,V}(g) = \sum_{\substack{\sigma \in H\backslash G \\ x_\sigma g x_\sigma^{-1} \in H}} \chi_V(x_\sigma g x_\sigma^{-1}).$$

Proof. Let ρ_{Ind} be the representation $\mathrm{Ind}_H^G V$. We want to compute $\chi_{\mathrm{Ind}\,V}(g) = \mathrm{tr}(\rho_{\mathrm{Ind}}(g))$. The space $\mathrm{Ind}_H^G V$ can be decomposed as a direct sum of subspaces V_σ, where σ runs over the right cosets $H\backslash G$:

$$\mathrm{Ind}_H^G V = \bigoplus_{\sigma \in H\backslash G} V_\sigma, \quad \text{where } V_\sigma = \{f \in \mathrm{Ind}_H^G V : f(x) = 0 \text{ for all } x \notin \sigma\}.$$

Each subspace V_σ has dimension $\dim V$. The action of $g \in G$ on a function $f \in V_\sigma$ is $(g \cdot f)(x) = f(xg)$. If $x \notin \sigma g^{-1}$, then $xg \notin \sigma$, so $f(xg) = 0$. This means that the action of g maps the subspace V_σ to the subspace $V_{\sigma g^{-1}}$. Therefore, the matrix of $\rho_{\mathrm{Ind}}(g)$ in a basis adapted to this decomposition is a block matrix. A diagonal block corresponding to V_σ is non-zero only if $\sigma g^{-1} = \sigma$, which is equivalent to $\sigma g = \sigma$.

$$\chi_{\mathrm{Ind}\,V}(g) = \mathrm{tr}(\rho_{\mathrm{Ind}}(g)) = \sum_{\substack{\sigma \in H\backslash G \\ \sigma g = \sigma}} \mathrm{tr}(\rho_{\mathrm{Ind}}(g)|_{V_\sigma}).$$

Now, assume $\sigma g = \sigma$. Let x_σ be the representative of the coset $\sigma = Hx_\sigma$. The condition $\sigma g = \sigma$ means $x_\sigma g = h x_\sigma$, which implies $h = x_\sigma g x_\sigma^{-1} \in H$.

Consider the isomorphism of vector spaces $\alpha : V_\sigma \to V$ given by $\alpha(f) = f(x_\sigma)$. This is an isomorphism because any function $f \in V_\sigma$ is uniquely determined by its value at x_σ, since for any other $y = hx_\sigma \in \sigma$, $f(y) = f(hx_\sigma) = \rho_V(h)f(x_\sigma)$.

We can see how the action of g on V_σ corresponds to an action on V via this isomorphism. For any $f \in V_\sigma$:

$$\begin{aligned}
\alpha(\rho_{\mathrm{Ind}}(g)f) &= (\rho_{\mathrm{Ind}}(g)f)(x_\sigma) = f(x_\sigma g) \\
&= f(hx_\sigma) \quad (\text{since } x_\sigma g = hx_\sigma) \\
&= \rho_V(h)f(x_\sigma) \quad (\text{by definition of } \mathrm{Ind}_H^G V) \\
&= \rho_V(h)\alpha(f).
\end{aligned}$$

This shows that the action of $\rho_{\mathrm{Ind}}(g)$ on V_σ is conjugate to the action of $\rho_V(h)$ on V:

$$\rho_{\mathrm{Ind}}(g)|_{V_\sigma} = \alpha^{-1} \circ \rho_V(h) \circ \alpha.$$

Therefore, their traces are equal:

$$\mathrm{tr}(\rho_{\mathrm{Ind}}(g)|_{V_\sigma}) = \mathrm{tr}(\rho_V(h)) = \chi_V(h) = \chi_V(x_\sigma g x_\sigma^{-1}).$$

Summing over all cosets σ such that $\sigma g = \sigma$ gives the desired formula. □

An alternative form of the character formula is:

$$\chi_{\mathrm{Ind}\,V}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \chi_V(xgx^{-1}).$$

Example 17.4 (A_4 characters revisited)**.** Let's re-calculate the characters of $\mathrm{Ind}_K^{A_4} V_0$ and $\mathrm{Ind}_K^{A_4} V_\pm$ using Theorem 17.3. The coset representatives are $x_1 = 1$, $x_2 = (123)$, $x_3 = (132)$.

- For $g = 1$: $x_\sigma \cdot 1 \cdot x_\sigma^{-1} = 1 \in K$ for all σ.

  $$\chi_{\mathrm{Ind}\,V}(1) = \chi_V(1) + \chi_V(1) + \chi_V(1) = 3 \dim V.$$

  For V_0 and $V_\pm$, this is 3.

- For $g = (12)(34)$:

(1) $\sigma = K$: $x_1 g x_1^{-1} = (12)(34) \in K$. Term: $\chi_V((12)(34))$.
(2) $\sigma = K(123)$: $x_2 g x_2^{-1} = (123)(12)(34)(132) = (14)(23) \in K$. Term: $\chi_V((14)(23))$.
(3) $\sigma = K(132)$: $x_3 g x_3^{-1} = (132)(12)(34)(123) = (13)(24) \in K$. Term: $\chi_V((13)(24))$.

$$\chi_{\operatorname{Ind} V}((12)(34)) = \chi_V((12)(34)) + \chi_V((14)(23)) + \chi_V((13)(24)).$$

For $V = V_0$, this is $1+1+1 = 3$. For $V = V_\pm$ (any non-trivial 1-dimensional representation of K), the characters for the three non-identity elements are $\{1, -1, -1\}$ in some order. The sum is always -1.

- For $g = (123)$:

(1) $\sigma = K$: $x_1 g x_1^{-1} = (123) \notin K$.
(2) $\sigma = K(123)$: $x_2 g x_2^{-1} = (123)(123)(132) = (132) \notin K$.
(3) $\sigma = K(132)$: $x_3 g x_3^{-1} = (132)(123)(123) = (123) \notin K$.

No terms contribute to the sum. Thus, $\chi_{\operatorname{Ind} V}((123)) = 0$.

This confirms the characters we computed earlier.

17.3 Frobenius Reciprocity

Theorem 17.5 (Frobenius Reciprocity)**.** *Let $H \subset G$ be a subgroup. Let V be a representation of G, and W a representation of H. Then there is a natural isomorphism of vector spaces:*

$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W).$$

Remark 17.6. This theorem states that induction (Ind_H^G) and restriction (Res_H^G) are adjoint functors between the categories of finite dimensional complex representations of H and G.

Proof. We explicitly construct the isomorphism by defining maps in both directions. Let $\alpha : \operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \to \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$ be defined by

$$\alpha(\phi)(v) = \phi(v)(e) \quad \text{for } \phi \in \operatorname{Hom}_G(V, \operatorname{Ind}_H^G W),\ v \in V,$$

where $e \in G$ is the identity element.

Let $\beta : \operatorname{Hom}_H(\operatorname{Res}_H^G V, W) \to \operatorname{Hom}_G(V, \operatorname{Ind}_H^G W)$ be defined by

$$(\beta(\psi)(v))(x) = \psi(\rho_V(x)v) \quad \text{for } \psi \in \operatorname{Hom}_H(\operatorname{Res}_H^G V, W),\ v \in V,\ x \in G.$$

We need to check that these maps are well-defined and are inverses of each other. Let's denote the action $\rho_V(g)v$ by gv and $\rho_W(h)w$ by hw.

1. **$\alpha(\phi)$ is an H-intertwiner:** We need to show $\alpha(\phi)(hv) = h(\alpha(\phi)(v))$ for $h \in H$.

$$\begin{aligned}
\alpha(\phi)(hv) &= \phi(hv)(e) && \text{(by def of } \alpha) \\
&= (h\phi(v))(e) && \text{(since } \phi \text{ is a } G\text{-intertwiner)} \\
&= \phi(v)(eh) = \phi(v)(h) && \text{(by def of action on } \operatorname{Ind}_H^G W) \\
&= h(\phi(v)(e)) && \text{(since } \phi(v) \in \operatorname{Ind}_H^G W) \\
&= h(\alpha(\phi)(v)) && \text{(by def of } \alpha)
\end{aligned}$$

2. **$\beta(\psi)(v)$ is in $\operatorname{Ind}_H^G W$:** We need to show $(\beta(\psi)(v))(hx) = h((\beta(\psi)(v))(x))$ for $h \in H$.

$$\begin{aligned}
(\beta(\psi)(v))(hx) &= \psi((hx)v) && \text{(by def of } \beta) \\
&= \psi(h(xv)) \\
&= h(\psi(xv)) && \text{(since } \psi \text{ is an } H\text{-intertwiner)} \\
&= h((\beta(\psi)(v))(x)) && \text{(by def of } \beta)
\end{aligned}$$

3. **$\beta(\psi)$ is a G-intertwiner:** We need to show $\beta(\psi)(gv) = g(\beta(\psi)(v))$ for $g \in G$.

$$(\beta(\psi)(gv))(x) = \psi(x(gv)) = \psi((xg)v)$$
$$(g(\beta(\psi)(v)))(x) = (\beta(\psi)(v))(xg) = \psi((xg)v)$$

The two are equal, so $\beta(\psi)$ is a G-intertwiner.

4. **$\alpha \circ \beta = \mathrm{Id}$:** For any $\psi \in \mathrm{Hom}_H(\mathrm{Res}^G_H V, W)$, we check $\alpha(\beta(\psi)) = \psi$.

$$(\alpha(\beta(\psi)))(v) = (\beta(\psi)(v))(e) = \psi(ev) = \psi(v).$$

5. **$\beta \circ \alpha = \mathrm{Id}$:** For any $\phi \in \mathrm{Hom}_G(V, \mathrm{Ind}^G_H W)$, we check $\beta(\alpha(\phi)) = \phi$.

$$\begin{aligned}(\beta(\alpha(\phi))(v))(x) &= (\alpha(\phi))(xv) && \text{(by def of } \beta) \\ &= \phi(xv)(e) && \text{(by def of } \alpha) \\ &= (x\phi(v))(e) && \text{(since } \phi \text{ is a } G\text{-intertwiner)} \\ &= \phi(v)(ex) = \phi(v)(x) && \text{(by def of action on } \mathrm{Ind}^G_H W)\end{aligned}$$

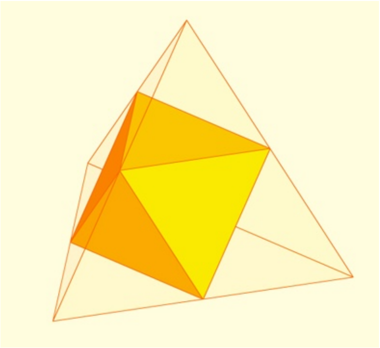
Thus, $\beta(\alpha(\phi)) = \phi$.

Since α and β are inverse linear maps, they are isomorphisms. $\square$ $\square$

Example 17.7 (A_4 Reciprocity)**.** For irreducible representations, $\dim \mathrm{Hom}(V_1, V_2)$ is the multiplicity of V_1 in V_2 (if V_1 is irrep). Frobenius reciprocity implies $\langle V, \mathrm{Ind}^G_H W\rangle_G = \langle \mathrm{Res}^G_H V, W\rangle_H$.

- $\dim \mathrm{Hom}_{A_4}(V_0, \mathrm{Ind}^{A_4}_K V_0) = \dim \mathrm{Hom}_K(\mathrm{Res}^{A_4}_K V_0, V_0)$. $\mathrm{Res}^{A_4}_K V_0$ is the trivial representation of K, which is V_0. So $\dim \mathrm{Hom}_K(V_0, V_0) = 1$. This matches $\mathrm{Ind}^{A_4}_K V_0 \cong V_0 \oplus V_\zeta \oplus V_{\zeta^2}$, which contains one copy of V_0.
- $\dim \mathrm{Hom}_{A_4}(V_3, \mathrm{Ind}^{A_4}_K V_0) = \dim \mathrm{Hom}_K(\mathrm{Res}^{A_4}_K V_3, V_0)$. The character of $\mathrm{Res}^{A_4}_K V_3$ is $(3, -1, -1, -1)$, that can be read off he character table of A_4 if we take into account that each nontrivial element of K is in its own conjugacy class in the abelian group K. The inner product with $\chi_{V_0} = (1, 1, 1, 1)$ is $\frac{1}{4}(3 - 1 - 1 - 1) = 0$. So the dimension is 0. This matches the fact that V_3 is not a component of $\mathrm{Ind}^{A_4}_K V_0$.

  We can also understand the restriction of V_3 to K geometrically. Recall that V_3 is the representation of A_4 acting by rotational symmetries on the regular tetrahedron. In this representation the elements of K act as rotations by π about the axes connecting opposite mid-sides of the regular tetrahedron. These axes form are mutually orthogonal in $\mathbb{R}^3$, which can be seen by inscribing a regular octahedron in the regular tetrahedron as shown in the picture. Then the lines connecting mid-sides of the tetrahedron are the big diagonals of the octahedron, which are mutually orthogonal. Each of the axes spans a one-dimensional representation of K obtained by restriction from V_3, where one of the three nontrivial elements of K acts by 1 and the other two by -1. Therefore, $\mathrm{Res}^{A_4}_K$ is isomorphic to a direct sum of three inequivalent nontrivial one-dimensional representations of K.



- $\dim \mathrm{Hom}_{A_4}(V_3, \mathrm{Ind}^{A_4}_K V_\pm) = \dim \mathrm{Hom}_K(\mathrm{Res}^{A_4}_K V_3, V_\pm)$. We know $\mathrm{Ind}^{A_4}_K V_\pm \cong V_3$, so the LHS dimension is 1. Let's check the RHS. Let $V_\pm$ be the representation of K with character $(1, 1, -1, -1)$.

$$\langle \mathrm{Res}^{A_4}_K V_3, V_\pm\rangle_K = \frac{1}{4}(3 \cdot 1 + (-1) \cdot 1 + (-1) \cdot (-1) + (-1) \cdot (-1)) = \frac{1}{4}(3 - 1 + 1 + 1) = 1.$$

  The dimensions match.

17.4 Induction and Restriction in Tensor Product Form

We can view induced and restricted representations through the lens of bimodules and tensor products. Let k be the underlying field. The group algebra $k[G]$ can be viewed as a bimodule in two ways:

$$\begin{array}{cc} k[H]\text{-}k[G]\text{-bimodule} & k[G]\text{-}k[H]\text{-bimodule} \\ k[H] \curvearrowright k[G] \curvearrowleft k[G] & k[G] \curvearrowright k[G] \curvearrowleft k[H] \end{array}$$

We denote these as $k[G]_1$ and $k[G]_2$ respectively.

If V is a right A-module and W is a left A-module, their tensor product over A is

$$V \otimes_A W \cong (V \otimes_k W)/\langle v \cdot a \otimes w - v \otimes a \cdot w \mid v \in V, w \in W, a \in A\rangle.$$

Lemma 17.8. *The restriction functor is given by tensoring with $k[G]_1$.*

$$\operatorname{Res}^G_H V \cong k[G]_1 \otimes_{k[G]} V.$$

Proof. Consider the map $F : \operatorname{Res}^G_H V \to k[G]_1 \otimes_{k[G]} V$ defined by $F(v) = 1 \otimes_{k[G]} v$. This map is injective and surjective, hence an isomorphism of vector spaces. We check that it is an H-homomorphism. The left H-action on $k[G]_1$ is multiplication, and the action on $\operatorname{Res}^G_H V$ is the restriction of the G-action. For $h \in H$,

$$\begin{aligned} F(h \cdot v) &= 1 \otimes_{k[G]} (h \cdot v) \\ &= h \otimes_{k[G]} v \quad (\text{since } V \text{ is a left } k[G]\text{-module}) \\ &= h \cdot (1 \otimes_{k[G]} v) \quad (\text{by definition of the left } H\text{-action}) \\ &= h \cdot F(v). \end{aligned}$$

□

Lemma 17.9. *The induction functor is given by tensoring with $k[G]_2$.*

$$\operatorname{Ind}^G_H W \cong k[G]_2 \otimes_{k[H]} W.$$

The dimension is $\dim(\operatorname{Ind}^G_H W) = \frac{|G|}{|H|} \dim W$.

Proof. Let $\{x_\sigma\}$ be a set of right coset representatives for H in G. Define a map $\mathcal{T} : \operatorname{Ind}^G_H W \to k[G]_2 \otimes_{k[H]} W$ by

$$\mathcal{T}(f) = \sum_{\sigma \in H\backslash G} x_\sigma^{-1} \otimes_{k[H]} f(x_\sigma).$$

We show this map is a well-defined isomorphism of G-modules.

1. **$\mathcal{T}(f)$ is well-defined.** Suppose we choose different representatives $y_\sigma = h_\sigma x_\sigma$ for some $h_\sigma \in H$. Then

$$\begin{aligned} \sum_\sigma y_\sigma^{-1} \otimes_{k[H]} f(y_\sigma) &= \sum_\sigma (x_\sigma^{-1} h_\sigma^{-1}) \otimes_{k[H]} f(h_\sigma x_\sigma) \\ &= \sum_\sigma x_\sigma^{-1} \otimes_{k[H]} (h_\sigma^{-1} f(h_\sigma x_\sigma)) \\ &= \sum_\sigma x_\sigma^{-1} \otimes_{k[H]} (h_\sigma^{-1}(h_\sigma f(x_\sigma))) \quad (\text{by def of } \operatorname{Ind}^G_H W) \\ &= \sum_\sigma x_\sigma^{-1} \otimes_{k[H]} f(x_\sigma) = \mathcal{T}(f). \end{aligned}$$

2. **$\mathcal{T}$ is a bijection.** Let $\{w^i\}$ be a basis for W. A basis for $\operatorname{Ind}^G_H W$ is given by functions $\{\delta^i_\sigma\}$ where $\delta^i_\sigma(x_\mu) = \delta_{\sigma\mu} w^i$. A basis for $k[G]_2 \otimes_{k[H]} W$ is given by $\{x_\sigma^{-1} \otimes w^i\}$. We see that $\mathcal{T}$ maps basis to basis:

$$\mathcal{T}(\delta^i_\sigma) = \sum_\mu x_\mu^{-1} \otimes_{k[H]} \delta^i_\sigma(x_\mu) = \sum_\mu x_\mu^{-1} \otimes_{k[H]} \delta_{\sigma\mu} w^i = x_\sigma^{-1} \otimes_{k[H]} w^i.$$

3. **$\mathcal{T}$ is a G-homomorphism.** The left G-action on $k[G]_2 \otimes_{k[H]} W$ is by left multiplication on the first factor.

$$\mathcal{T}(g \cdot f) = \sum_\sigma x_\sigma^{-1} \otimes_{k[H]} (g \cdot f)(x_\sigma) = \sum_\sigma x_\sigma^{-1} \otimes_{k[H]} f(x_\sigma g).$$

   On the other hand,

$$g \cdot \mathcal{T}(f) = g \cdot \left(\sum_\sigma x_\sigma^{-1} \otimes_{k[H]} f(x_\sigma) \right) = \sum_\sigma (g x_\sigma^{-1}) \otimes_{k[H]} f(x_\sigma).$$

We have $x_\sigma g^{-1} = hx_\mu$ for some $h \in H$ and a unique coset μ. This implies $gx_\sigma^{-1} = x_\mu^{-1}h^{-1}$. As σ runs over the coset representatives, so does μ. We can re-index the sum.

$$\begin{aligned}
g \cdot \mathcal{T}(f) &= \sum_\sigma (gx_\sigma^{-1}) \otimes_{k[H]} f(x_\sigma) \\
&= \sum_\sigma (x_\mu^{-1}h^{-1}) \otimes_{k[H]} f(x_\sigma) \quad (\text{where } x_\sigma g^{-1} = hx_\mu) \\
&= \sum_\sigma x_\mu^{-1} \otimes_{k[H]} h^{-1} f(x_\sigma) \\
&= \sum_\sigma x_\mu^{-1} \otimes_{k[H]} f(h^{-1}x_\sigma) \\
&= \sum_\sigma x_\mu^{-1} \otimes_{k[H]} f(x_\mu g) = \mathcal{T}(g \cdot f).
\end{aligned}$$

□

This gives the following parallel formulations:

Induction	**Restriction**
$\operatorname{Ind}_H^G W$	$\operatorname{Res}_H^G V$
$\operatorname{Hom}_{k[H]}(k[G]_1, W)$	$\operatorname{Hom}_{k[G]}(k[G]_2, V)$ (exercise)
$k[G]_2 \otimes_{k[H]} W$	$k[G]_1 \otimes_{k[G]} V$

Frobenius reciprocity can be seen as an instance of tensor-hom adjunction:

$$\operatorname{Hom}_{k[G]}(V, k[G]_2 \otimes_{k[H]} W) \cong \operatorname{Hom}_{k[H]}(k[G]_1 \otimes_{k[G]} V, W).$$

18 Representations of the Symmetric Group S_n

18.1 Partitions and Young Tableaux

Definition 18.1. A *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers such that $n = \lambda_1 + \cdots + \lambda_r$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$.

Partitions of n are in one-to-one correspondence with *Young diagrams* of size n. A Young diagram Y_λ for a partition λ consists of rows of boxes, with λ_i boxes in the i-th row.

Example 18.2 ($\lambda = (4, 4, 2, 1)$, $n = 11$)**.**

$$Y_\lambda = \ydiagram{4,4,2,1}$$

Definition 18.3. A *Young tableau* $\mathcal{T}_\lambda$ is a Young diagram Y_λ filled with the integers from 1 to n, each used exactly once.

Example 18.4. A Young tableau for $\lambda = (4, 4, 2, 1)$:

$$\mathcal{T}_\lambda = \begin{ytableau} 1 & 6 & 7 & 5 \\ 2 & 9 & 3 & 8 \\ 4 & 11 \\ 10 \end{ytableau}$$

18.2 Young Symmetrizers and Specht Modules

Definition 18.5. Given a partition λ of n and a chosen Young tableau $\mathcal{T}_\lambda$:

- The *row stabilizer* $P_\lambda \subset S_n$ is the subgroup of permutations that preserve the set of numbers in each row of $\mathcal{T}_\lambda$.
- The *column stabilizer* $Q_\lambda \subset S_n$ is the subgroup of permutations that preserve the set of numbers in each column of $\mathcal{T}_\lambda$.

It is a fact that $P_\lambda \cap Q_\lambda = \{1\}$.

Example 18.6. Let $n = 7$ and $\lambda = (3, 2, 2)$. Consider the tableau:

$$\mathcal{T}_\lambda = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}$$

Then the stabilizers are:

$$\begin{aligned} P_\lambda &= S_{\{1,2,3\}} \times S_{\{4,5\}} \times S_{\{6,7\}} \cong S_3 \times S_2 \times S_2 \\ |P_\lambda| &= 3! \cdot 2! \cdot 2! = 24 \\ Q_\lambda &= S_{\{1,4,6\}} \times S_{\{2,5,7\}} \times S_{\{3\}} \cong S_3 \times S_3 \times S_1 \\ |Q_\lambda| &= 3! \cdot 3! \cdot 1! = 36 \end{aligned}$$

Definition 18.7 (Young Symmetrizers)**.** In the group algebra $\mathbb{C}[S_n]$, we define two elements associated with a tableau $\mathcal{T}_\lambda$:

- The *row symmetrizer*: $a_\lambda = \frac{1}{|P_\lambda|} \sum_{p \in P_\lambda} p$.
- The *column anti-symmetrizer*: $b_\lambda = \frac{1}{|Q_\lambda|} \sum_{q \in Q_\lambda} \operatorname{sgn}(q) q$.

These elements are idempotents, often called Young projectors. Specifically, for any $p \in P_\lambda$ and $q \in Q_\lambda$:

$$p a_\lambda = a_\lambda p = a_\lambda \quad \text{and} \quad q b_\lambda = b_\lambda q = \operatorname{sgn}(q) b_\lambda.$$

This implies $a_\lambda^2 = a_\lambda$ and $b_\lambda^2 = b_\lambda$.

Definition 18.8. The *Young symmetrizer* (or Specht element) corresponding to $\mathcal{T}_\lambda$ is the element $c_\lambda = a_\lambda b_\lambda \in \mathbb{C}[S_n]$. The coefficient of the identity element $1 \in S_n$ in c_λ is $\frac{1}{|P_\lambda||Q_\lambda|}$, which is non-zero, so $c_\lambda \neq 0$.

Definition 18.9 (Specht Module)**.** The *Specht module* $\mathcal{V}_\lambda$ associated with the partition λ is the left ideal in the group algebra generated by c_λ:

$$\mathcal{V}_\lambda = \mathbb{C}[S_n] c_\lambda \subseteq \mathbb{C}[S_n].$$

This is a left $\mathbb{C}[S_n]$-module, i.e., a representation of S_n.

Goal: To prove the following classification theorem:

1. For each partition λ of n, the Specht module $\mathcal{V}_\lambda$ is an irreducible representation of S_n.
2. Every irreducible representation of S_n is isomorphic to $\mathcal{V}_\lambda$ for a unique partition λ of n.

Remark 18.10. The number of irreducible representations of S_n is equal to the number of its conjugacy classes. The conjugacy classes of S_n are determined by cycle type, which are in one-to-one correspondence with partitions of n. This matches the number of Specht modules.

18.3 Examples

Example 18.11 ($\lambda = (n)$)**.** The Young diagram is a single row.

$$Y_\lambda = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array}$$

For any tableau, $P_\lambda = S_n$ and $Q_\lambda = \{1\}$.

$$a_\lambda = \frac{1}{n!} \sum_{g \in S_n} g, \quad b_\lambda = 1.$$

So, $c_\lambda = a_\lambda$. The Specht module is $\mathcal{V}_{(n)} = \mathbb{C}[S_n] a_\lambda$. For any $x \in S_n$, $x c_\lambda = c_\lambda$. This is the one-dimensional **trivial representation**.

Example 18.12 **($\lambda = (1^n) = (1, 1, \ldots, 1)$).** The Young diagram is a single column.

$$Y_\lambda = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

For any tableau, $P_\lambda = \{1\}$ and $Q_\lambda = S_n$.

$$a_\lambda = 1, \quad b_\lambda = \frac{1}{n!} \sum_{g \in S_n} \operatorname{sgn}(g) g.$$

So, $c_\lambda = b_\lambda$. The Specht module is $\mathcal{V}_{(1^n)} = \mathbb{C}[S_n] b_\lambda$. For any $x \in S_n$, $x c_\lambda = \operatorname{sgn}(x) c_\lambda$. This is the one-dimensional **sign representation**.

Example 18.13 **(S_3, $\lambda = (2, 1)$).** Consider the tableau $\mathcal{T}_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$. The stabilizers are $P_\lambda = \{1, (12)\}$ and $Q_\lambda = \{1, (13)\}$. The symmetrizers are:

$$a_\lambda = \frac{1}{2}(1 + (12)), \quad b_\lambda = \frac{1}{2}(1 - (13)).$$

The Specht element is:

$$c_\lambda = \frac{1}{4}(1 + (12))(1 - (13)) = \frac{1}{4}(1 + (12) - (13) - (12)(13)) = \frac{1}{4}(1 + (12) - (13) - (132)).$$

Let $v_1 = 4c_\lambda$. The Specht module $\mathcal{V}_\lambda$ is spanned by the action of S_3 on v_1.

$$\begin{aligned}
(12)v_1 &= (12) + 1 - (12)(13) - (12)(132) = (12) + 1 - (132) - (13) = v_1. \\
(13)v_1 &= (13) + (13)(12) - 1 - (13)(132) = (13) + (123) - 1 - (23) = v_2. \\
(123)v_1 &= (123) + (13) - (123)(13) - (123)(132) = (123) + (13) - (23) - 1 =: v_2. \\
(132)v_1 &= (132) + (23) - (132)(13) - (132)(132) = (132) + (23) - (12) - (123) =: v_3.
\end{aligned}$$

We can check that $v_1 + v_2 + v_3 = 0$, so the space is 2-dimensional. The element (12) is the reflection about the axis v_1, and (13) the reflection about the axis v_3. The cycles (123) and (132) act as rotations by $2\pi/3$. This is the unique 2-dimensional irreducible representation of S_3. The action can be visualized as the symmetries of an equilateral triangle in a plane, where vectors corresponding to v_1, v_2, v_3 point to the vertices and sum to zero.

18.4 Properties of the Young Symmetrizer

A key step in proving the irreducibility of Specht modules is the following lemma.

Lemma 18.14. *For any $x \in \mathbb{C}[S_n]$, we have $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$ for some scalar $\ell_\lambda(x) \in \mathbb{C}$. The map $\ell_\lambda : \mathbb{C}[S_n] \to \mathbb{C}$ is linear.*

Proof. By linearity, it suffices to prove the lemma for $x = g \in S_n$.

1. Let $g \in P_\lambda Q_\lambda$. Then $g = pq$ for unique $p \in P_\lambda, q \in Q_\lambda$.

   $$a_\lambda g b_\lambda = a_\lambda (pq) b_\lambda = (a_\lambda p)(q b_\lambda) = a_\lambda(\operatorname{sgn}(q) b_\lambda) = \operatorname{sgn}(q) a_\lambda b_\lambda = \operatorname{sgn}(q) c_\lambda.$$

   So in this case, $\ell_\lambda(g) = \operatorname{sgn}(q)$.

2. Let $g \notin P_\lambda Q_\lambda$. We will show that $a_\lambda g b_\lambda = 0$. The proof relies on showing there exists a transposition $t = (ij)$ such that $t \in P_\lambda$ and also $t \in g Q_\lambda g^{-1}$. The latter condition means $g^{-1} t g \in Q_\lambda$. If such a transposition t exists,

let $t' = g^{-1}tg \in Q_\lambda$. Since t is a transposition, $t' = g^{-1}tg$ is also a transposition, so $\operatorname{sgn}(t') = -1$. Then we can write:

$$\begin{aligned} a_\lambda g b_\lambda &= (a_\lambda t) g b_\lambda \quad (\text{since } t \in P_\lambda \implies a_\lambda t = a_\lambda) \\ &= a_\lambda (tg) b_\lambda \\ &= a_\lambda (gt') b_\lambda \\ &= a_\lambda g (t' b_\lambda) \\ &= a_\lambda g (\operatorname{sgn}(t') b_\lambda) \quad (\text{since } t' \in Q_\lambda) \\ &= -a_\lambda g b_\lambda. \end{aligned}$$

This implies that $a_\lambda g b_\lambda = 0$.

It remains to show that if $g \notin P_\lambda Q_\lambda$, such a transposition $t = (ij)$ must exist. This is equivalent to showing that there exist two numbers i, j that are in the same row of the original tableau $\mathcal{T}_\lambda$ and also in the same column of the permuted tableau $g\mathcal{T}_\lambda$.

Assume no such pair (i, j) exists. This means that any two numbers in the same row of $\mathcal{T}_\lambda$ must be in different columns of $g\mathcal{T}_\lambda$. We will show this implies $g \in P_\lambda Q_\lambda$. Let $\mathcal{T}'_\lambda = g\mathcal{T}_\lambda$. Then the column-permuting group for $\mathcal{T}'{}_\lambda$ is the group $Q'_\lambda = gQ_\lambda g^{-1}$.

Consider the numbers in the first row of $\mathcal{T}_\lambda$. In $g\mathcal{T}_\lambda$, they have to lie in different columns. Therefore, we can find a permutation $q'_1 \in Q'_\lambda$ that moves these numbers into the first row of $q'_1\mathcal{T}'_\lambda$. Then, we can find a permutation $p_1 \in P_\lambda$ that arranges these numbers in the first row of $p_1\mathcal{T}_\lambda$ to match their order in the first row of $q'_1\mathcal{T}'_\lambda$. Therefore we have:

$$(p_1\mathcal{T}_\lambda)_{1st\ row} = (q'_1\mathcal{T}'_\lambda)_{1st\ row}.$$

Note that the property that we are interested in, namely that there are no two numbers (i, j) in the same row of $\mathcal{T}_\lambda$ and in the same column of $\mathcal{T}'_\lambda$, is unchanged by the operation that we have performed, or in fact by any action of elements of the row group on $\mathcal{T}_\lambda$ and column group on $\mathcal{T}'_\lambda$.

Since we assume that there are no two elements are in the 2nd row of $p_1\mathcal{T}_\lambda$ and in the same column of $q'_1\mathcal{T}'_\lambda$, we can repeat the argument for the second row, acting by q'_2 on $q'_1\mathcal{T}'_\lambda$ and by p_2 on $p_1\mathcal{T}_\lambda$ to get

$$(p_2p_1\mathcal{T}_\lambda)_{1st\ and\ 2nd\ row} = (q'_2q'_1\mathcal{T}'_\lambda)_{1st\ and\ 2nd\ row}.$$

and so on, until we get

$$\overline{p}\mathcal{T}_\lambda = \overline{q}'\mathcal{T}'_\lambda = \overline{q}'g\mathcal{T}_\lambda.$$

Since $\overline{q}' \in gQ_\lambda q^{-1}$, we have $\overline{q}' = g\overline{q}g^{-1}$ for some $\overline{q} \in Q_\lambda$. Then we have

$$\overline{p}\mathcal{T}_\lambda = g\overline{q}g^{-1}g\mathcal{T}_\lambda = g\overline{q}\mathcal{T}_\lambda,$$

which implies $g = \overline{pq}^{-1} \in P_\lambda Q_\lambda$.

This contradicts our assumption that $g \notin P_\lambda Q_\lambda$. Therefore, a transposition $t = (ij)$ with the desired properties must exist, which completes the proof. ☐

Definition 18.15. The lexicographic order on partitions of n is defined as follows: $\lambda > \mu$ if the first non-zero entry of $\lambda_i - \mu_i$ is positive.

Example 18.16. For partitions of $n = 5$:

- $(3, 2) > (3, 1, 1)$.

For partitions of $n = 11$:

- $(5, 4, 1, 1) > (5, 3, 2, 1)$.

The partition (n) is greater than any other partition of n.

Lemma 18.17. *If $\lambda > \mu$ are partitions of n, then $a_\lambda \mathbb{C}[S_n] b_\mu = 0$.*

Proof. It suffices to show that for any $g \in S_n$, $a_\lambda g b_\mu = 0$. This will follow if we can find a transposition $t \in P_\lambda$ such that $g^{-1}tg \in Q_\mu$. If such a t exists, let $t' = g^{-1}tg$. Since $t \in P_\lambda$, $a_\lambda t = a_\lambda$. Since $t' \in Q_\mu$, $t'b_\mu = -b_\mu$. Then we have

$$a_\lambda g b_\mu = a_\lambda t g b_\mu = a_\lambda g(g^{-1}tg) b_\mu = a_\lambda g t' b_\mu = -a_\lambda g b_\mu.$$

This implies $2a_\lambda g b_\mu = 0$, and thus $a_\lambda g b_\mu = 0$.

We now show that such a transposition t always exists. Let $t = (i, j)$. We need to find i, j that are in the same row of the tableau T_λ (so that $(i, j) \in P_\lambda$) and are also in the same column of the tableau gT_μ (so that $(i, j) \in gQ_\mu g^{-1}$).

Since $\lambda > \mu$, there must be some index k such that $\lambda_k > \mu_k$ and $\lambda_i = \mu_i$ for all $i < k$. Consider the first row of T_λ. It contains λ_1 numbers. The tableau gT_μ has columns whose lengths are given by the partition μ. If $\lambda_1 > \mu_1$, the first row of T_λ has λ_1 elements. The number of columns in gT_μ is μ_1. Since $\lambda_1 > \mu_1$, by the pigeonhole principle, at least two of the elements from the first row of T_λ must lie in the same column of gT_μ. Let these elements be i and j. Then $t = (i, j) \in P_\lambda$ and $(i, j) \in gQ_\mu g^{-1}$.

If $\lambda_1 = \mu_1$, we can permute the elements of the first row of T_λ using some $p_1 \in P_\lambda$ and the elements of the first row of gT_μ using some $q_1 \in gQ_\mu g^{-1}$ so that they match. This does not change the problem. We proceed to the second row, and so on, until we find an index k where $\lambda_k > \mu_k$. At this stage, we consider the λ_k elements in the k-th row of T_λ. By a similar pigeonhole argument, two elements from the k-th row of T_λ must lie in the same column of gT_μ. $\square$

Example 18.18 (S_3)**.** The partitions of 3 are (3), (2, 1), and (1, 1, 1). In lexicographic order:

$$(3) > (2,1) > (1,1,1)$$

Let's denote these by $\sigma > \lambda > \mu$ respectively. The standard tableaux are:

$$T_\sigma = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \qquad T_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad T_\mu = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

The corresponding group elements are:

$$a_\sigma = \frac{1}{6}\sum_{g\in S_3} g \qquad\qquad b_\sigma = e$$

$$a_\lambda = \frac{1}{2}(e + (12)) \qquad\qquad b_\lambda = \frac{1}{2}(e - (13))$$

$$a_\mu = e \qquad\qquad b_\mu = \frac{1}{6}\sum_{g\in S_3}(-1)^g g$$

By Lemma 18.17, since $(3) > (2, 1)$, we must have $a_\sigma \mathbb{C}[S_3] b_\lambda = 0$. Let's check for $g = e$:

$$a_\sigma b_\lambda = \frac{1}{12}\left(\sum_{g\in S_3} g\right)(e - (13)) = \frac{1}{12}\sum_{g\in S_3} g - \frac{1}{12}\sum_{g\in S_3} g(13) = 0$$

since right multiplication by (13) just permutes the elements of S_3. Similarly, since $(2, 1) > (1, 1, 1)$, we must have $a_\lambda \mathbb{C}[S_3] b_\mu = 0$. For $g = e$:

$$a_\lambda b_\mu = \frac{1}{12}(e + (12))\left(\sum_{g\in S_3}(-1)^g g\right) = \frac{1}{12}\sum_{g\in S_3}(-1)^g g + \frac{1}{12}\sum_{g\in S_3}(-1)^g (12)g = 0.$$

The second sum is $-\sum_{h\in S_3}(-1)^h h$, which cancels the first sum.

Remark 18.19. It is easy to see that swapping rows and columns in the proof of Lemma 18.17, we can obtain a dual statement, namely that If $\mu < \lambda$ are partitions of n, then $b_\mu \mathbb{C}[S_n] a_\lambda = 0$. In the example above we find: $b_\mu a_\lambda = \frac{1}{12}\left(\sum_{g\in S_3}(-1)^g g\right)(e + (12)) = 0$ and $b_\lambda a_\sigma = \frac{1}{12}(e - (13))\left(\sum_{g\in S_3} g\right) = 0$.

Lemma 18.20. *For any partition λ, $c_\lambda^2 = \alpha c_\lambda$ for some non-zero scalar $\alpha \in \mathbb{C}$.*

Remark 18.21. Problem Set 12: find α. This implies that $e_\lambda = \frac{1}{\alpha} c_\lambda$ is an idempotent, i.e., $e_\lambda^2 = e_\lambda$.

18.5 Irreducibility of Specht modules and the Classification Theorem

Lemma 18.22. *Let A be an associative algebra and let $e \in A$ be an idempotent ($e^2 = e$). Let M be a left A-module. Then $\mathrm{Hom}_A(Ae, M) \cong eM$ as vector spaces.*

Proof. First, recall the standard isomorphism $\mathrm{Hom}_A(A, M) \cong M$. A homomorphism $f : A \to M$ is determined by $f(1) = v \in M$, since $f(a) = f(a \cdot 1) = af(1) = av$. The linear map $\beta : \mathrm{Hom}_A(A, M) \to M$ given by $\beta(f) = f(1)$ is an isomorphism, with inverse $\alpha : M \to \mathrm{Hom}_A(A, M)$ given by $\alpha(v)(a) = av$. We check this:

$$(\beta \circ \alpha)(v) = \beta(\alpha(v)) = \alpha(v)(1) = 1 \cdot v = v.$$

$$(\alpha \circ \beta)(f)(a) = a \cdot \beta(f) = a \cdot f(1) = f(a \cdot 1) = f(a).$$

So $\mathrm{Hom}_A(A, M) \cong M$.

Now, consider the idempotent e. The element $1 - e$ is also an idempotent. The algebra A decomposes as a direct sum of left ideals $A = Ae \oplus A(1 - e)$. Any module M also decomposes as $M = eM \oplus (1 - e)M$. Then,

$$\mathrm{Hom}_A(A, M) = \mathrm{Hom}_A(Ae \oplus A(1 - e), M) \cong \mathrm{Hom}_A(Ae, M) \oplus \mathrm{Hom}_A(A(1 - e), M).$$

A map $f \in \mathrm{Hom}_A(Ae, M)$ is sent by $\beta : \mathrm{Hom}_A(A, M) \to M$ to an element $v \in M$ such that $v = ev$. Specifically, a map $f : Ae \to M$ is determined by $f(e) \in eM$. Conversely, an element $ev \in eM$ is sent by α to $\alpha(ev)(a) = aeev = aef(e)$. Thus, $\mathrm{Hom}_A(Ae, M) \cong eM$. ☐

Theorem 18.23. *The modules $V_\lambda = \mathbb{C}[S_n]c_\lambda$ are irreducible, and the set $\{V_\lambda\}_{\lambda \vdash n}$ forms a complete set of inequivalent irreducible $\mathbb{C}[S_n]$-modules.*

Proof. Let $A = \mathbb{C}[S_n]$. Let $e_\lambda = \frac{1}{\alpha} c_\lambda$ be the idempotent from Lemma 18.20. Then $V_\lambda = Ac_\lambda = Ae_\lambda$. We compute the homomorphism spaces between these modules. Let λ, μ be partitions of n. Using Lemma 18.22, with $M = V_\mu = Ae_\mu$:

$$\mathrm{Hom}_{S_n}(V_\lambda, V_\mu) = \mathrm{Hom}_A(Ae_\lambda, Ae_\mu) \cong e_\lambda Ae_\mu.$$

This is proportional to $c_\lambda \mathbb{C}[S_n]c_\mu$.

Case 1: $\lambda > \mu$. By Lemma 18.17, $a_\lambda \mathbb{C}[S_n]b_\mu = 0$. Then $c_\lambda \mathbb{C}[S_n]c_\mu = a_\lambda b_\lambda \mathbb{C}[S_n]a_\mu b_\mu \subseteq a_\lambda \mathbb{C}[S_n]b_\mu = 0$. So, if $\lambda > \mu$, $\mathrm{Hom}_{S_n}(V_\lambda, V_\mu) = 0$. By symmetry, if $\lambda \neq \mu$, V_λ and V_μ are not isomorphic. We can also notice here that if $\lambda < \mu$, then by the dual statement to Lemma 18.17, we have $b_\lambda \mathbb{C}[S_n]a_\mu = 0$, which also implies $a_\lambda b_\lambda \mathbb{C}[S_n]a_\mu b_\mu = 0$.

Case 2: $\lambda = \mu$. We need to compute $\dim(c_\lambda \mathbb{C}[S_n]c_\lambda)$.

$$c_\lambda \mathbb{C}[S_n]c_\lambda = a_\lambda b_\lambda \mathbb{C}[S_n]a_\lambda b_\lambda.$$

By Lemma 18.14 this implies that $c_\lambda \mathbb{C}[S_n]c_\lambda$ is a one-dimensional space spanned by c_λ. Therefore, $\dim \mathrm{Hom}_{S_n}(V_\lambda, V_\lambda) = 1$.

By Schur's Lemma, if $V = \bigoplus_i V_i^{n_i}$ is a decomposition into irreducibles, then $\dim \mathrm{Hom}(V, V) = \sum_i n_i^2$. Since $\dim \mathrm{Hom}_{S_n}(V_\lambda, V_\lambda) = 1$, we must have that V_λ is irreducible.

We have constructed a set of pairwise non-isomorphic irreducible representations $\{V_\lambda\}_{\lambda \vdash n}$. The number of partitions of n is equal to the number of conjugacy classes in S_n. Since the number of non-isomorphic irreducible representations of a finite group is equal to its number of conjugacy classes, this set is complete. ☐

Remark 18.24. The module $V_\lambda = \mathbb{C}[S_n]c_\lambda$ is defined using a specific tableau T_λ to construct P_λ and Q_λ. If we choose a different standard tableau T'_λ, we get a different symmetrizer c'_λ. However, the resulting module $\mathbb{C}[S_n]c'_\lambda$ is isomorphic to V_λ.

18.6 Induced Representations of S_n

Let P_λ be the row stabilizer subgroup for a tableau T_λ. Consider the representation of S_n induced from the trivial representation of P_λ. Let $U_\lambda = \mathrm{Ind}_{P_\lambda}^{S_n} \mathbb{C}_{\mathrm{triv}}$. This module can be realized as a left ideal in the group algebra:

$$\mathrm{Ind}_{P_\lambda}^{S_n} \mathbb{C}_{\mathrm{triv}} \cong \mathbb{C}[S_n] \otimes_{\mathbb{C}[P_\lambda]} \mathbb{C}_{\mathrm{triv}} \cong \mathbb{C}[S_n]a_\lambda,$$

where $a_\lambda = \sum_{g \in P_\lambda} g$. The isomorphism maps $x \otimes 1 \mapsto xa_\lambda$.

Proposition 18.25. *For partitions λ, μ of n:*

1. *If $\lambda > \mu$, then* $\mathrm{Hom}_{S_n}(U_\lambda, V_\mu) = 0$.
2. $\dim \mathrm{Hom}_{S_n}(U_\lambda, V_\lambda) = 1$.

Proof. Using the isomorphism $\mathrm{Hom}_A(Aa, M) \cong aM$ for $A = \mathbb{C}[S_n]$, we have:

$$\mathrm{Hom}_{S_n}(U_\lambda, V_\mu) = \mathrm{Hom}_{S_n}(\mathbb{C}[S_n]a_\lambda, \mathbb{C}[S_n]c_\mu) \cong a_\lambda\mathbb{C}[S_n]c_\mu = a_\lambda\mathbb{C}[S_n]a_\mu b_\mu.$$

This space is contained in $a_\lambda\mathbb{C}[S_n]b_\mu$. By Lemma 18.17, if $\lambda > \mu$, then $a_\lambda\mathbb{C}[S_n]b_\mu = 0$, so $\mathrm{Hom}_{S_n}(U_\lambda, V_\mu) = 0$.

For $\lambda = \mu$, we have $\mathrm{Hom}_{S_n}(U_\lambda, V_\lambda) \cong a_\lambda\mathbb{C}[S_n]c_\lambda = a_\lambda\mathbb{C}[S_n]a_\lambda b_\lambda$. This space is spanned by $a_\lambda b_\lambda = c_\lambda$, so it is 1-dimensional. To confirm that it is nonzero, we can apply the Frobenius reciprocity:

$$\mathrm{Hom}_{S_n}(U_\lambda, V_\lambda) = \mathrm{Hom}_{S_n}(\mathrm{Ind}_{P_\lambda}^{S_n}\mathbb{C}_{\mathrm{triv}}, V_\lambda) = \mathrm{Hom}_{P_\lambda}(\mathbb{C}_{\mathrm{triv}}, \mathrm{Res}_{P_\lambda}^{S_n} V_\lambda).$$

Since we have

$$\mathrm{Res}_{P_\lambda}^{S_n} V_\lambda = \mathrm{Res}_{P_\lambda}^{S_n} \mathbb{C}[S_n]a_\lambda b_\lambda \supset \mathrm{Res}_{P_\lambda}^{S_n} 1 \cdot a_\lambda b_\lambda \cong \mathbb{C}_{\mathrm{triv}},$$

this implies that $\dim \mathrm{Hom}_{S_n}(U_\lambda, V_\lambda) \neq 0$. $\square$

This proposition implies that the decomposition of U_λ into irreducibles V_μ is of the form:

$$U_\lambda = \bigoplus_{\mu \geq \lambda} K_{\mu\lambda} V_\mu \quad \text{and} \quad K_{\lambda\lambda} = 1.$$

The numbers $\{K_{\mu\lambda}\}$ are the Kostka numbers.

18.7 Character of the Induced Representation U_λ

We now compute the character χ_{U_λ} of the representation $U_\lambda = \mathrm{Ind}_{P_\lambda}^{S_n}\mathbb{C}_{\mathrm{triv}}$.

Theorem 18.26. *Let C_τ be the conjugacy class in S_n corresponding to the cycle type $\tau = (1^{i_1}, 2^{i_2}, \dots)$, where i_k is the number of cycles of length k. Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be a partition of n. Let $H_m(x) = x_1^m + x_2^m + \cdots + x_N^m$ be the power sum symmetric polynomial in $N \geq p$ variables. Then the character value $\chi_{U_\lambda}(g)$ for $g \in C_\tau$ is the coefficient of $x^\lambda = x_1^{\lambda_1} \cdots x_p^{\lambda_p}$ in the polynomial product $\prod_{m \geq 1} H_m(x)^{i_m}$.*

Proof. The character of an induced representation $\mathrm{Ind}_H^G V$ is given by the formula:

$$\chi_{\mathrm{Ind}_H^G V}(g) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} \chi_V(xgx^{-1}).$$

In our case, $G = S_n$, $H = P_\lambda$, and $V = \mathbb{C}_{\mathrm{triv}}$, so χ_V is always 1. For $g \in C_\tau$, the formula simplifies to:

$$\chi_{U_\lambda}(g) = \frac{1}{|P_\lambda|}|\{x \in S_n : xgx^{-1} \in P_\lambda\}| = \frac{|C_\tau \cap P_\lambda| \cdot |Z_g|}{|P_\lambda|},$$

where Z_g is the centralizer of g. The size of the centralizer is $|Z_g| = \prod_{m \geq 1} m^{i_m} i_m!$, where $i_m!$ corresponds to the permutations of the disjoint cycles of length m and m^{i_m} refers to the cyclic permutation inside each of the disjoint cycles of length m. The size of the subgroup $P_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \dots$ is $|P_\lambda| = \prod_j \lambda_j!$.

We need to find $|C_\tau \cap P_\lambda|$. An element in P_λ is a product of permutations, one for each row group S_{λ_j}. Let r_{j_m} be the number of m-cycles in the permutation acting on the j-th row. For an element to be in $C_\tau \cap P_\lambda$, we must have $\sum_j r_{j_m} = i_m$ for each m, and for each row j, $\sum_m m \cdot r_{j_m} = \lambda_j$. The number of elements in S_{λ_j} with cycle type $(1^{r_{j_1}}, 2^{r_{j_2}}, \dots)$ can be computed again as the quotient of the order of the group by the order of the centralizer of the given cycle type, namely

$$\frac{\lambda_j!}{\prod_{m \geq 1} m^{r_{j_m}} r_{j_m}!}.$$

So,

$$|C_\tau \cap P_\lambda| = \sum_r \prod_{j \geq 1} \frac{\lambda_j!}{\prod_{m \geq 1} m^{r_{j_m}} r_{j_m}!},$$

where the sum is over all (r_{j_m}) satisfying the conditions $\sum_j r_{j_m} = i_m$ for each m and $\sum_m m \cdot r_{j_m} = \lambda_j$ for each j.

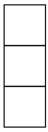
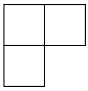
Putting it all together:

$$\begin{aligned}\chi_{U_\lambda}(g) = \frac{|Z_g|}{|P_\lambda|}|C_\tau \cap P_\lambda| &= \frac{\prod_m m^{i_m} i_m!}{\prod_j \lambda_j!} \sum_r \prod_{j \geq 1} \frac{\lambda_j!}{\prod_{m \geq 1} m^{r_{j_m}} r_{j_m}!} \\ &= \left(\prod_m m^{i_m} i_m!\right) \sum_r \frac{1}{\prod_{j,m} m^{r_{j_m}} r_{j_m}!} \\ &= \sum_r \frac{\prod_m m^{i_m} i_m!}{\prod_m m^{\sum_j r_{j_m}} \prod_{j,m} r_{j_m}!} = \sum_r \prod_m \frac{i_m!}{\prod_j r_{j_m}!}\end{aligned}$$

This final expression is the coefficient of $x_1^{\lambda_1} \cdots x_p^{\lambda_p}$ in the expansion of $\prod_{m \geq 1}(x_1^m + \cdots + x_p^m)^{i_m}$. By the multinomial theorem, the coefficient of $\prod_j (x_j^m)^{r_{jm}}$ in $(x_1^m + \cdots + x_p^m)^{i_m}$ is $\frac{i_m!}{\prod_j r_{jm}!}$. The total exponent of x_j is $\sum_m m \cdot r_{jm}$, which must be equal to λ_j. Summing over all possible choices of r_{jm} gives the desired coefficient. $\square$

18.8 Example: S_3

Let's analyze the induced representations for S_3.

- $\lambda = (3)$: . $P_{(3)} = S_3$. $U_{(3)} = \mathrm{Ind}_{S_3}^{S_3} \mathbb{C}_{\text{triv}} = \mathbb{C}_{\text{triv}}$.
- $\lambda = (1,1,1)$: . $P_{(1,1,1)} = \{e\}$. $U_{(1,1,1)} = \mathrm{Ind}_{\{e\}}^{S_3} \mathbb{C}_{\text{triv}} = \mathbb{C}[S_3]$.
- $\lambda = (2,1)$: . $P_{(2,1)} \cong S_2$. $U_{(2,1)} = \mathrm{Ind}_{S_2}^{S_3} \mathbb{C}_{\text{triv}}$.

Let's compute the character of $U_{(2,1)}$ using the theorem. Here $\lambda = (2,1)$. We need two variables, x_1, x_2. We seek the coefficient of $x_1^2 x_2^1$.

- Identity element $g = e$: Cycle type (1^3). $i_1 = 3$. We need the coefficient of $x_1^2 x_2$ in $H_1(x)^3 = (x_1 + x_2)^3 = x_1^3 + 3x_1^2 x_2 + 3x_1 x_2^2 + x_2^3$. The coefficient is 3. So $\chi_{U_{(2,1)}}(e) = 3$.
- Transposition $g = (12)$: Cycle type $(1^1, 2^1)$. $i_1 = 1, i_2 = 1$. We need the coefficient of $x_1^2 x_2$ in $H_1(x) H_2(x) = (x_1 + x_2)(x_1^2 + x_2^2) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$. The coefficient is 1. So $\chi_{U_{(2,1)}}((12)) = 1$.
- 3-cycle $g = (123)$: Cycle type (3^1). $i_3 = 1$. We need the coefficient of $x_1^2 x_2$ in $H_3(x) = x_1^3 + x_2^3$. The coefficient is 0. So $\chi_{U_{(2,1)}}((123)) = 0$.

The character table for S_3 is:

	e	(12)	(123)
$V_{(3)}$	1	1	1
$V_{(1,1,1)}$	1	-1	1
$V_{(2,1)}$	2	0	-1
$U_{(2,1)}$	3	1	0

By inspecting the characters, we can see that $\chi_{U_{(2,1)}} = \chi_{V_{(3)}} + \chi_{V_{(2,1)}}$. Therefore, $U_{(2,1)} \cong V_{(3)} \oplus V_{(2,1)}$.

Conclusions: Representations of S_n

1. The irreducible representations of S_n are indexed by partitions λ of n.

$$V_\lambda = \mathbb{C}[S_n] c_\lambda$$

   We have $V_\lambda \cong V_\mu$ if and only if $\lambda = \mu$.

2. The induced representations from trivial representations of Young subgroups P_λ:

$$U_\lambda = \mathrm{Ind}_{P_\lambda}^{S_n} \mathbb{C}_{\text{triv}}$$

   where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ is a partition of n.

$$U_\lambda = \mathbb{C}[S_n] a_\lambda$$

The character of U_λ on a conjugacy class $C_{\mathbf{i}}$ of cycle type $\mathbf{i} = (i_1, i_2, \ldots)$ is given by:

$$\chi_{U_\lambda}(C_{\mathbf{i}}) = \text{coefficient of } x^\lambda = \prod_j x_j^{\lambda_j} \text{ in } \prod_{m \geq 1} H_m(x)^{i_m},$$

where $H_m(x) = \sum_{k=1}^N x_k^m$ for $N \geq p$.

18.9 Character of Specht module V_λ

Consider the Vandermonde determinant in N variables:

$$\Delta(x) = \prod_{1\le i<j\le N} (x_i - x_j)$$

Let $\rho = (N-1, N-2, \dots, 1, 0)$. Then $\Delta(x)$ can be expressed as:

$$\Delta(x) = \sum_{\sigma\in S_N} (-1)^\sigma x^{\sigma(\rho)}$$

where $(-1)^\sigma$ is the sign of the permutation σ, and $x^\alpha \overset{\text{def}}{=} \prod_i x_i^{\alpha_i}$. For example, $x^\rho = x_1^{N-1}x_2^{N-2}\cdots x_N^0$. The determinant form is:

$$\Delta(x) = \det \begin{pmatrix} x_1^{N-1} & x_1^{N-2} & \dots & x_1 & 1 \\ x_2^{N-1} & x_2^{N-2} & \dots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_N^{N-1} & x_N^{N-2} & \dots & x_N & 1 \end{pmatrix}$$

For example, with $N = 3$:

$$\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = (x_1-x_2)(x_1-x_3)(x_2-x_3)$$

Theorem 18.27 (Frobenius Character Formula)**.** *Let $N \ge p$, where p is the number of parts in the partition $\lambda = (\lambda_1, \dots, \lambda_p)$. The character of the irreducible representation V_λ on the conjugacy class $C_\mathbf{i}$ of cycle type $\mathbf{i} = (i_1, i_2, \dots)$ is the coefficient Θ_λ of $x^{\lambda+\rho} = \prod_j x_j^{\lambda_j+N-j}$ in the polynomial $\Delta(x)\prod_{m\ge1} H_m(x)^{i_m}$.*

Proof Sketch. Let $\Theta_\lambda(C_\mathbf{i})$ be the specified coefficient.

(1) First, we express Θ_λ in terms of the characters of induced representations χ_{U_μ}.

$$\begin{aligned}\Theta_\lambda(C_\mathbf{i}) &= \text{coeff of } x^{\lambda+\rho} \text{ in } \left(\sum_{\sigma\in S_N} (-1)^\sigma x^{\sigma(\rho)}\right)\left(\prod_{m\ge1} H_m(x)^{i_m}\right) \\ &= \sum_{\sigma\in S_N} (-1)^\sigma \left(\text{coeff of } x^{\lambda+\rho} \text{ in } x^{\sigma(\rho)} \prod_{m\ge1} H_m(x)^{i_m}\right) \\ &= \sum_{\sigma\in S_N} (-1)^\sigma \left(\text{coeff of } x^{\lambda+\rho-\sigma(\rho)} \text{ in } \prod_{m\ge1} H_m(x)^{i_m}\right) \\ &= \sum_{\sigma\in S_N} (-1)^\sigma \chi_{U_{\lambda+\rho-\sigma(\rho)}}(C_\mathbf{i})\end{aligned}$$

Here, if $\lambda+\rho-\sigma(\rho)$ has negative entries, the corresponding character is taken to be zero, and if it is not non-increasing, then the terms are rearranged to make it a partition.

This shows that Θ_λ is a virtual character. Let $\mu = \lambda+\rho-\sigma(\rho)$. The term for $\sigma = 1$ (the identity) gives χ_{U_λ}. For other σ, the partition $\mu > \lambda$ since $\rho > \sigma(\rho)$.

$$\Theta_\lambda = \chi_{U_\lambda} + \sum_{\mu>\lambda} c_{\lambda\mu}\chi_{U_\mu}$$

Substituting $\chi_{U_\lambda} = \sum_{\mu\ge\lambda} K_{\lambda\mu}\chi_{V_\mu}$ where $K_{\lambda\mu}$ are the Kostka numbers and $K_{\lambda\lambda} = 1$, we get

$$\Theta_\lambda = \chi_{V_\lambda} + \sum_{\mu>\lambda} M_{\lambda\mu}\chi_{V_\mu}$$

for some integers $M_{\lambda\mu}$.

(2) To show that $\Theta_\lambda = \chi_{V_\lambda}$, it suffices to show that $(\Theta_\lambda, \Theta_\lambda) = 1$, where $(\cdot,\cdot)$ is the standard inner product on class functions. See a combinatorial argument given see Etingof, *Introduction to Representation Theory.*

□

18.10 Dimension of the Specht module V_λ

The dimension of a representation is its character evaluated at the identity element. The identity element in S_n consists of n 1-cycles, so its cycle type is $\mathbf{i} = (n, 0, 0, \dots)$. For this class, $H_m(x)^0 = 1$ for $m > 1$ and $H_1(x)^n = (x_1 + \dots + x_N)^n$. Thus, from Theorem 18.27:

$$\dim V_\lambda = \chi_{V_\lambda}(C_{(n,0,\dots)}) = \text{coeff of } x^{\lambda+\rho} \text{ in } \Delta(x)(x_1 + \dots + x_N)^n.$$

Theorem 18.28. *The dimension of the irreducible representation V_λ is given by*

$$\dim V_\lambda = \frac{n!}{\prod_{j=1}^N l_j!} \prod_{1 \le i < j \le N} (l_i - l_j)$$

where $l_j = \lambda_j + N - j$.

Proof. We need to find the coefficient of $x^{\lambda+\rho} = \prod_j x_j^{l_j}$ in $\Delta(x)(x_1 + \dots + x_N)^n$.

$$\begin{aligned}
\dim V_\lambda &= \text{coeff of } x^{\lambda+\rho} \text{ in } \left(\sum_{\sigma \in S_N} (-1)^\sigma x^{\sigma(\rho)} \right) (x_1 + \dots + x_N)^n \\
&= \sum_{\sigma \in S_N} (-1)^\sigma \left(\text{coeff of } x^{\lambda+\rho} \text{ in } x^{\sigma(\rho)} (x_1 + \dots + x_N)^n \right)
\end{aligned}$$

For a given σ, we have the term $x^{\sigma(\rho)} = \prod_j x_j^{N-\sigma(j)}$. To get the monomial $x^{\lambda+\rho} = \prod_j x_j^{l_j}$, we need to extract the monomial $x_j^{\lambda_j+N-j} = x_j^{l_j}$ in the expression $x^{\sigma(\rho)}(x_1 + \ldots + x_N)^n$. We already have the factor $x_j^{N-\sigma(j)}$ inside $x^{\sigma(\rho)}$ and it remains to take $x_j^{l_j - N + \sigma(j)}$ from $(x_1 + \ldots + x_N)^n$, if $l_j \ge N - \sigma(j)$. The coefficient of this monomial is given by the multinomial coefficient, provided all exponents are non-negative. This coefficient is $\frac{n!}{\prod_j (l_j - N + \sigma(j))!}$, assuming $l_j \ge N - \sigma(j)$ for all j. If this condition fails for some j, the coefficient is 0.

$$\begin{aligned}
\dim V_\lambda &= \sum_{\sigma \in S_N} (-1)^\sigma \frac{n!}{\prod_i (l_i - N + \sigma(i))!} \\
&= \frac{n!}{\prod_j l_j!} \sum_{\sigma \in S_N} (-1)^\sigma \frac{\prod_j l_j!}{\prod_i (l_i - N + \sigma(i))!} \\
&= \frac{n!}{\prod_j l_j!} \sum_{\sigma \in S_N} (-1)^\sigma \prod_j (l_j(l_j - 1) \ldots (l_j - N + \sigma(j) + 1))
\end{aligned}$$

The condition $l_i \ge N - \sigma(i)$ is implicitly handled because if $l_i < N - \sigma(i)$, then $l_i - N + \sigma(i) < 0$ and $l_i - N + \sigma(i) + 1 \le 0$, and $l_j \ge 0$, so the product contains zero.

$$\begin{aligned}
&= \frac{n!}{\prod_j l_j!} \sum_{\sigma \in S_N} (-1)^\sigma \prod_j a_{j,\sigma(j)} \quad \text{where } a_{ji} = l_j(l_j - 1) \ldots (l_j - N + i + 1) \\
&= \frac{n!}{\prod_j l_j!} \det(a_{ji})
\end{aligned}$$

The $N \times N$ matrix (a_{ji}) has entries $a_{ji} = \prod_{k=0}^{N-i-1} (l_j - k)$.

$$\det(a_{ji}) = \det \begin{pmatrix} l_1(l_1 - 1) \ldots (l_1 - N + 2) & \ldots & l_1 & 1 \\ l_2(l_2 - 1) \ldots (l_2 - N + 2) & \ldots & l_2 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ l_N(l_N - 1) \ldots (l_N - N + 2) & \ldots & l_N & 1 \end{pmatrix}$$

The columns of this matrix are polynomials in l_j. The i-th column (from the right, $i = 1, \dots, N$) is a polynomial of degree $i - 1$. By performing column operations, subtracting the second to the right column from the previous one, then a combination of the second and third column from the previous one, and so on, this determinant can be transformed into the Vandermonde determinant.

$$\det(a_{ji}) = \det \begin{pmatrix} l_1^{N-1} & l_1^{N-2} & \ldots & l_1 & 1 \\ l_2^{N-1} & l_2^{N-2} & \ldots & l_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_N^{N-1} & l_N^{N-2} & \ldots & l_N & 1 \end{pmatrix} = \prod_{1 \le i < j \le N} (l_i - l_j)$$

Substituting this back, we get the desired formula:

$$\dim V_\lambda = \frac{n!}{\prod_j l_j!} \prod_{1 \le i < j \le N} (l_i - l_j) \quad \text{where } l_i = \lambda_i + N - i. \qquad \square$$

18.11 Example: S_3, $\lambda = (2,1)$

Let $n = 3$, $\lambda = (2,1,0)$. We take $N = 3$. The values of l_j are:

- $l_1 = \lambda_1 + N - 1 = 2 + 3 - 1 = 4$
- $l_2 = \lambda_2 + N - 2 = 1 + 3 - 2 = 2$
- $l_3 = \lambda_3 + N - 3 = 0 + 3 - 3 = 0$

Using the formula from Theorem 18.28:

$$\dim V_{(2,1)} = \frac{3!}{4!2!0!}(4-2)(4-0)(2-0) = \frac{6}{24 \cdot 2 \cdot 1}(2)(4)(2) = \frac{6 \cdot 16}{48} = 2.$$

Alternatively, using the determinant form:

$$\begin{aligned}\dim V_{(2,1)} &= \frac{3!}{4!2!0!} \begin{vmatrix} l_1(l_1-1) & l_1 & 1 \\ l_2(l_2-1) & l_2 & 1 \\ l_3(l_3-1) & l_3 & 1 \end{vmatrix} = \frac{6}{48} \begin{vmatrix} 4 \cdot 3 & 4 & 1 \\ 2 \cdot 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \frac{1}{8} \cdot (12 \cdot 2 - 4 \cdot 2) = \frac{1}{8}(24 - 8) = \frac{16}{8} = 2.\end{aligned}$$

As a direct calculation, we need the coefficient of $x^{\lambda+\rho} = x_1^4 x_2^2 x_3^0$ in $\Delta(x)(x_1 + x_2 + x_3)^3$. The condition $l_j \ge N - \sigma(j)$ for $j = 3$ is $0 \ge 3 - \sigma(3)$, which implies $\sigma(3) = 3$. The permutations in S_3 satisfying this are $\sigma = 1$ and $\sigma = (12)$.

- For $\sigma = 1$: $(-1)^1 \frac{3!}{(4-3+1)!(2-3+2)!(0-3+3)!} = \frac{6}{2!1!0!} = 3$.
- For $\sigma = (12)$: $(-1)^{(12)} \frac{3!}{(4-3+2)!(2-3+1)!(0-3+3)!} = -\frac{6}{3!0!0!} = -1$.

Total dimension is $3 - 1 = 2$.

Exercise

Calculate $\dim V_{(2,1)}$ with $N = 2$.

18.12 The Hook Length Formula

Definition 18.29. Let Y_λ be the Young diagram of a partition λ. For a cell (i,j) in the diagram $(i, j \ge 1, i \le \lambda_j)$ the **hook length** $h(i,j)$ is the number of cells in the hook starting at (i,j), which consists of the cell (i,j) itself, all cells to its right in the same row, and all cells below it in the same column.

Example 18.30. For $\lambda = (2,2,1)$, the hook lengths are:

4	2
3	1
1	

Theorem 18.31 (Hook Length Formula)**.** *The dimension of the irreducible representation V_λ is given by*

$$\dim V_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$$

where the product is over all cells (i,j) in the Young diagram of λ.

Proof Sketch. We start with the formula from Theorem 18.28, taking $N = p$ (the number of parts of λ).

$$\dim V_\lambda = \frac{n!}{\prod_{j=1}^{p} l_j!} \prod_{1\le i<j\le p} (l_i - l_j)$$

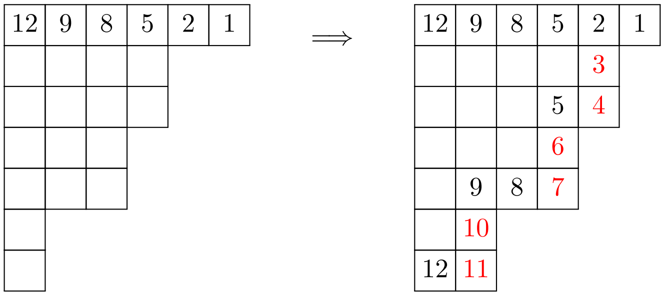
The proof consists of showing that $\frac{\prod_{j=1}^{p} l_j!}{\prod_{1\le i<j\le p}(l_i-l_j)} = \prod_{(i,j)\in\lambda} h(i,j)$. This is done row by row. For the first row ($k = 1$):

$$\frac{l_1!}{\prod_{j=2}^{p}(l_1 - l_j)} = \prod_{i=1}^{\lambda_1} h(i,1)$$

Let's analyze the terms. $l_1 = \lambda_1 + p - 1$. This is the number of cells in the first row plus the number of cells in the first column, minus one, which is exactly the hook length of the top-left cell, $h(1,1)$. The terms in the denominator are $l_1 - l_j = (\lambda_1 + p - 1) - (\lambda_j + p - j) = \lambda_1 - \lambda_j + j - 1$. These values correspond to the "missing" numbers in the product $l_1!$ that one needs to cancel to obtain the product of hook lengths in the first row.

Put $l_1 = h(1,1)$ in the lower left corner of Y_λ. Then go down from l_1 to 1 by putting the number $k-1$ to the right of k if k lies inside Y_λ, or above k if k lies outside of Y_λ. Since an orthogonal north-east walk on checkered paper between two given squares takes the same number of steps, you will finish exactly at the upper right corner of Y_λ with the number 1. Now notice that the numbers inside the diagram Y_λ are exactly the hook lengths in the first row. The numbers outside Y_λ are those that we need to skip from $l_1!$ to get the product $\prod_{i\in\lambda_1} h(i,1)$. In the j-th row we have $l_1 - l_j = \lambda_1 - \lambda_j + j - 1$, where $(\lambda_1 - \lambda_j)$ indicates by how many steps λ_1 is longer than λ_j, plus j indicates the number of squares down from the first row, and -1 stands for not counting the corner twice. So $l_1 - l_j$ is exactly the length of the "missing hook" in each row starting from row 2 down to the last row p.

The figure below shows a Young diagram of type $(6,4,4,3,3,1,1)$ with hook lengths in the first row and the same diagram with all numbers from 1 to 12 listed according to the rule above and the missing hook lengths $l_1 - l_j$ for $j = 2, \dots p$ shown in red outside of the original Young diagram.



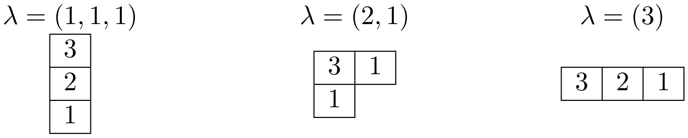
A similar argument holds for each row:

$$\frac{l_i!}{\prod_{j=i+1}^{p}(l_i - l_j)} = \prod_{k=1}^{\lambda_i} h(k,i)$$

Multiplying these expressions for all rows $i = 1, \dots, p$ gives the desired result.

$$\dim V_\lambda = n! \prod_{i=1}^{p} \left(\frac{1}{l_i!} \prod_{j=i+1}^{p} (l_i - l_j) \right)^{-1} = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}. \qquad \square$$

Example 18.32. Hook length formula for the irreducible representations of S_3.



$$\dim V_{(1,1,1)} = \frac{3!}{3\cdot 2\cdot 1} = 1 \qquad \dim V_{(2,1)} = \frac{3!}{3\cdot 1\cdot 1} = 2 \qquad \dim V_{(3)} = \frac{3!}{3\cdot 2\cdot 1} = 1$$

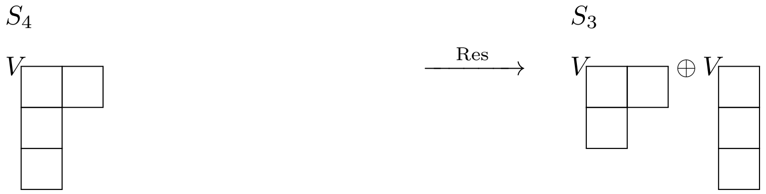
18.13 Induction and restriction between S_{n-1} and S_n.

1. $\mathrm{Res}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\lambda\in R(\mu)} V_\lambda$, where $R(\mu)$ is the set of Young diagrams obtained by removing one square from Y_μ.

2. $\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\lambda \in A(\mu)} V_\lambda$, where $A(\mu)$ is the set of Young diagrams obtained by adding one square to Y_μ.

See PS 13 for the proof.

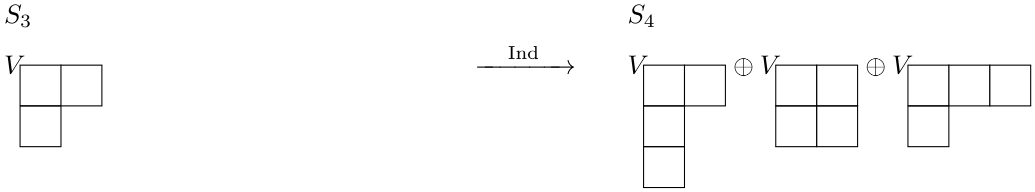
Example 18.33. Restriction from S_4 to S_3. We used the hook length formula to check the dimensions.

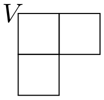


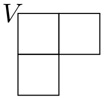
$$\dim = \frac{4!}{4\cdot 2\cdot 1\cdot 1} = 3 \qquad \dim = \frac{3!}{3\cdot 1\cdot 1} = 2 \qquad \dim = \frac{3!}{3\cdot 2\cdot 1} = 1$$

3 $\qquad$ $=2+1$

Induction from S_3 to S_4:



$\dim V$  $= 2$ $\qquad$ $\dim = 3 \quad \dim = \frac{4!}{3\cdot 2\cdot 2\cdot 1} = 2 \quad \dim = 3$

$\dim \operatorname{Ind}_{S_3}^{S_4} V$  $= 4\cdot 2 = 8$ $\qquad$ $=3+2+3$

Conclusions: Representations of S_n

1. Complex irreducible representations of S_n are the Specht modules $\{V_\lambda\}_{\lambda \text{ a partition of } n}$; $V_\lambda = \mathbb{C}[S_n]c_\lambda$.
2. The character is given by the Frobenius character formula: $\chi_{V_\lambda}(C_i) =$ coefficient of $X^{\lambda+\rho} = \prod_j x_j^{\lambda_j+N-j}$ in the polynomial $\Delta(x)\prod_{m\geq 1} H_m(x)^{i_m}$, where $\Delta(x) = \prod_{1\leq i<j\leq N}(x_i - x_j)$.
   Here C_i is the conjugacy class of type $(i_1, i_2, \ldots, i_l, \ldots)$, where $i_l = \#$of cycles of length l.
3. The dimension is given by the hook-length formula: $\dim V_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$, where $h(i,j)$ is the number of squares in the hook starting at square (i,j) in the Young diagram Y_λ.
4. Branching Rules: Induction and restriction between irreducible representations of S_{n-1} and S_n can be encoded by adding and removing a single square from the Young diagram.

19 Schur-Weyl Duality

Theorem 19.1 (Double Centralizer Theorem)**.** *Let A, B be two subalgebras of $\operatorname{End} E$, where E is a finite dimensional complex vector space. Suppose that A is semisimple and $B = \operatorname{End}_A E$. Then*

1. $A = \operatorname{End}_B E = \operatorname{End}_{\operatorname{End}_A E} E$ *(double centralizer property).*
2. *B is semisimple.*
3. *As a representation of $A\otimes B$, E decomposes as $E \cong \bigoplus_{i\in I} V_i \otimes W_i$ where $\{V_i\}_{i\in I}$ are the irreducible representations of A and $\{W_i\}_{i\in I}$ are the irreducible representations of B.*

Proof. Since A is semisimple, the representation E decomposes as $E = \bigoplus_{i \in I} V_i \otimes W_i$, where V_i are the irreducible representations of A and $W_i = \operatorname{Hom}_A(V_i, E)$ are the multiplicity spaces.

Consider the endomorphism algebra B:

$$B = \operatorname{End}_A E = \operatorname{Hom}_A \left(\bigoplus_{j \in I} V_j \otimes W_j, \bigoplus_{k \in I} V_k \otimes W_k \right) \simeq \bigoplus_{i \in I} \operatorname{End}(W_i)$$

This follows from Schur's Lemma, which states that $\operatorname{Hom}_A(V_i, V_j) = \delta_{ij}\mathbb{C}$. The algebra B is a direct sum of matrix algebras, which implies that B is semisimple, and $\{W_i\}_{i \in I}$ is the complete set of irreducible representations of B.

Now consider the centralizer of B, $\operatorname{End}_B E$:

$$\operatorname{End}_B E = \operatorname{Hom}_B \left(\bigoplus_{j \in I} V_j \otimes W_j, \bigoplus_{k \in I} V_k \otimes W_k \right) \simeq \bigoplus_{i \in I} \operatorname{End}(V_i)$$

Since A is semisimple, $A \simeq \bigoplus_{i \in I} \operatorname{End}(V_i)$. Thus, $\operatorname{End}_B E \simeq A$. The irreducible representations of this algebra are $\{V_i\}_{i \in I}$. Therefore, we have the decomposition

$$E = \bigoplus_{i \in I} V_i \otimes W_i \qquad \square$$

The idea is to apply this theorem to $E = V^{\otimes n}$ where $A = \mathbb{C}[S_n]$ is a semisimple algebra acting by permutation of factors, and $B = \operatorname{End}_A(V^{\otimes n})$.

Let $\dim V = k$. The endomorphism algebra $\operatorname{End}(V) = \operatorname{Mat}_k(\mathbb{C})$ can be viewed as the Lie algebra $\mathfrak{gl}(V)$ with the commutator bracket $[A, B] = AB - BA$. The Jacobi identity holds: $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$.

Let $\mathcal{T}(\mathfrak{gl}(V))$ be the tensor algebra of $\mathfrak{gl}(V)$. The universal enveloping algebra $U(\mathfrak{gl}(V))$ is defined as the quotient

$$U(\mathfrak{gl}(V)) \overset{\text{def}}{=} \mathcal{T}(\mathfrak{gl}(V)) / \langle u \otimes v - v \otimes u - [u, v] \rangle_{u,v \in \mathfrak{gl}(V)}$$

This is an infinite-dimensional associative algebra.

Example 19.2. Let $V = \mathbb{C}^2$, so $\mathfrak{gl}(V) = \mathfrak{gl}_2$. A basis is given by:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Lie algebra relations are:

$$\begin{aligned} [e, f] &= ef - fe = h \\ [h, e] &= he - eh = 2e \\ [h, f] &= hf - fh = -2f \end{aligned}$$

and $[x, I] = 0$ for all $x \in \{e, f, h\}$. The universal enveloping algebra is $U(\mathfrak{gl}_2) \simeq \mathbb{C}\langle e, f, h, I \rangle / \langle \text{Relations} \rangle$. Elements of $U(\mathfrak{gl}_2)$ are polynomials in non-commuting variables e, f, h, I. For example, f^2 corresponds to $f \otimes f$ in the tensor algebra.

The algebra $\mathfrak{gl}(V) = \operatorname{End}(V)$ acts on V. This action extends to an action of $U(\mathfrak{gl}(V))$ on tensor powers $V^{\otimes n}$. This is defined via the coproduct $\Delta : U(\mathfrak{gl}(V)) \to \operatorname{End}(V^{\otimes n})$. For $X \in \mathfrak{gl}(V)$, we define recursively:

$$\Delta^n X = \sum_{i=1}^{n} 1^{\otimes(i-1)} \otimes X \otimes 1^{\otimes(n-i)}$$

For $n = 2$, $\Delta X = X \otimes 1 + 1 \otimes X$ for any $X \in \mathfrak{gl}(V)$. For the elements of $U(\mathfrak{gl}(V))$, we define $\Delta(ab) = \Delta(a)\Delta(b)$. Let's check that it preserves the Lie bracket relation:

$$\begin{aligned} \Delta([e, f]) &= \Delta(h) = h \otimes 1 + 1 \otimes h \\ [\Delta(e), \Delta(f)] &= (e \otimes 1 + 1 \otimes e)(f \otimes 1 + 1 \otimes f) - (f \otimes 1 + 1 \otimes f)(e \otimes 1 + 1 \otimes e) \\ &= (ef \otimes 1 + e \otimes f + f \otimes e + 1 \otimes ef) - (fe \otimes 1 + f \otimes e + e \otimes f + 1 \otimes fe) \\ &= (ef - fe) \otimes 1 + 1 \otimes (ef - fe) = h \otimes 1 + 1 \otimes h \end{aligned}$$

This confirms that Δ defines a representation of $U(\mathfrak{gl}(V))$ on $V^{\otimes 2}$. The map Δ^n is coassociative, so we have a well-defined action of $U(\mathfrak{gl}(V))$ on $V^{\otimes n}$ for all $n \in \mathbb{N}_+$.

Theorem 19.3 (Schur-Weyl Duality)**.** *Let $E = V^{\otimes n}$, where V is a finite dimensional vector space over $\mathbb{C}$. Let S_n act by permutation of factors in $V^{\otimes n}$. Let $f : \mathbb{C}[S_n] \to \operatorname{End} E$ be the corresponding representation map, and let $A = \operatorname{Im} f \subset \operatorname{End} E$. Then the centralizer algebra $B = \operatorname{End}_A E$ is the image of $U(\mathfrak{gl}(V))$ in $\operatorname{End} E$ under the map Δ^n.*

Proof. First, we show that the image of $U(\mathfrak{gl}(V))$ is contained in the centralizer. Let $b \in U(\mathfrak{gl}(V))$. The operator $\Delta^n b$ acts on $V^{\otimes n}$. The action of $\Delta^n X$ for $X \in \mathfrak{gl}(V)$ is symmetric with respect to permutation of factors. For example, for $n = 2$ and the transposition (12):

$$(12) \circ (\Delta b)(v \otimes w) = (12)(bv \otimes w + v \otimes bw) = w \otimes bv + bw \otimes v$$
$$(\Delta b) \circ (12)(v \otimes w) = (\Delta b)(w \otimes v) = bw \otimes v + w \otimes bv$$

The actions commute. Since the algebra $U(\mathfrak{gl}(V))$ is generated by elements of $\mathfrak{gl}(V)$, the image of $U(\mathfrak{gl}(V))$ commutes with the action of S_n. Thus, $\operatorname{Im}(U(\mathfrak{gl}(V))) \subset \operatorname{End}_A E = B$.

For the other inclusion, we need to show that any endomorphism that commutes with the S_n action comes from $U(\mathfrak{gl}(V))$. The algebra of endomorphisms commuting with the S_n action is the algebra of symmetric tensors in $\operatorname{End}(V^{\otimes n}) \cong (\operatorname{End} V)^{\otimes n}$, which is $S^n(\operatorname{End} V)$. So we need to show that the image of $U(\mathfrak{gl}(V))$ is precisely $S^n(\operatorname{End} V)$. This follows from the next lemma. ☐

Lemma 19.4.

1. *If U is a $\mathbb{C}$-vector space, then the space of symmetric tensors $S^n U$ is spanned by elements of the form $u^{\otimes n} = u \otimes \cdots \otimes u$, for $u \in U$.*

2. *For a $\mathbb{C}$-algebra A, the algebra of symmetric tensors $S^n A$ is generated by the elements $\Delta^n(a) = \sum_{i=1}^n 1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes(n-i)}$ for $a \in A$.*

Proof. (1) The space $S^n U$ is an irreducible representation of $GL(U)$ under the action $g \cdot (u_1 \otimes \cdots \otimes u_n) = gu_1 \otimes \cdots \otimes gu_n$. The subspace $W = \operatorname{span}\{u^{\otimes n} \mid u \in U\}$ is clearly a subrepresentation, since $g \cdot u^{\otimes n} = (gu)^{\otimes n} \in W$. Since W is non-zero, and $S^n U$ is irreducible, we must have $W = S^n U$.

(2) This is a consequence of the fundamental theorem of symmetric functions (related to Newton's identities). This theorem states that any symmetric polynomial can be expressed as a polynomial in the power sum symmetric polynomials. Let $H_k(x_1, \dots, x_n) = \sum_{i=1}^n x_i^k$. For example, the elementary symmetric polynomial $e_n(x_1, \dots, x_n) = x_1 \dots x_n$ can be written as a polynomial $P(H_1, \dots, H_n)$ with rational coefficients. For $n = 2$, $x_1 x_2 = \frac{1}{2}((x_1 + x_2)^2 - (x_1^2 + x_2^2)) = \frac{1}{2}(H_1^2 - H_2)$.

Now, let $A = \operatorname{End}(V)$. For any $a \in A$, consider the elements $X_i = 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$ (with a in the i-th position). These commute with each other. For $a^k \in A$ we have $\Delta^n(a^k) = \sum_{i=1}^n X_i^k$. By the theorem on symmetric functions, any symmetric polynomial in the X_i can be generated by the power sums $\Delta^n(a^k)$. In particular, we can generate $X_1 X_2 \dots X_n = a \otimes a \otimes \cdots \otimes a$. By part (1) of the lemma, these elements span $S^n A$. Thus, the elements $\Delta^n(a)$ for $a \in A$ generate $S^n A$. Since $A = \mathfrak{gl}(V)$, this shows that the image of $U(\mathfrak{gl}(V))$ is $S^n(\operatorname{End} V)$. ☐

Theorem 19.5 (Schur-Weyl Duality for $U(\mathfrak{gl}(V))$)**.**

1. *The image A of $\mathbb{C}[S_n]$ and the image B of $U(\mathfrak{gl}(V))$ in $\operatorname{End}(V^{\otimes n})$ are centralizers of each other.*

2. *Both A and B are semisimple. $V^{\otimes n}$ is a semisimple representation of both $\mathbb{C}[S_n]$ and $U(\mathfrak{gl}(V))$.*

3. *The space $V^{\otimes n}$ decomposes as a representation of $\mathbb{C}[S_n] \times U(\mathfrak{gl}(V))$ as*

$$V^{\otimes n} = \bigoplus_{\lambda} V_\lambda \otimes L_\lambda$$

where λ runs over partitions of n. Here $\{V_\lambda\}$ are the irreducible Specht modules for S_n, and $\{L_\lambda = \operatorname{Hom}_{S_n}(V_\lambda, V^{\otimes n})\}$ are inequivalent irreducible representations of $U(\mathfrak{gl}(V))$, or zero.

Proof. The algebra $A = \operatorname{Im}(\mathbb{C}[S_n])$ is semisimple (since $\mathbb{C}[S_n]$ is). By Theorem 19.3, $B = \operatorname{End}_A(V^{\otimes n})$ is the image of $U(\mathfrak{gl}(V))$. All statements now follow directly from the Double Centralizer Theorem 19.1. ☐

Example 19.6. Let $\mathcal{V} = \mathbb{C}^2$. For $n = 2$, $\mathcal{V}^{\otimes 2} = S^2(\mathcal{V}) \oplus \Lambda^2(\mathcal{V})$. In terms of the Schur-Weyl decomposition:

$$\mathcal{V}^{\otimes 2} = V_{\ydiagram{2}} \otimes L_{\ydiagram{2}} \oplus V_{\ydiagram{1,1}} \otimes L_{\ydiagram{1,1}}$$

Here $V_{\ydiagram{2}}$ is the trivial representation and $V_{\ydiagram{1,1}}$ is the sign representation. $L_{\ydiagram{2}} = S^2(\mathcal{V})$ is 3-dimensional, and $L_{\ydiagram{1,1}} = \Lambda^2(\mathcal{V})$ is 1-dimensional. If $\{e_1, e_2\}$ is a basis for $\mathcal{V}$, then $L_{\ydiagram{2}}$ is spanned by $\{e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, e_2 \otimes e_2\}$ and $L_{\ydiagram{1,1}}$ is spanned by $\{e_1 \otimes e_2 - e_2 \otimes e_1\}$.

For $n = 3$, $\mathcal{V}^{\otimes 3}$ has dimension $2^3 = 8$. The decomposition is

$$\mathcal{V}^{\otimes 3} = (V_{\ydiagram{3}} \otimes L_{\ydiagram{3}}) \oplus (V_{\ydiagram{2,1}} \otimes L_{\ydiagram{2,1}})$$

The dimensions of the S_3 representations are $\dim V_{\ydiagram{3}} = 1$ and $\dim V_{\ydiagram{2,1}} = 2$. The dimensions of the $\mathfrak{gl}_2$ representations are: $\dim L_{\ydiagram{3}} = \dim S^3(\mathcal{V}) = \binom{3+2-1}{3} = \binom{4}{3} = 4$. $\dim L_{\ydiagram{2,1}}$ can be found by checking dimensions: $8 = (\dim V_{\ydiagram{3}} \cdot \dim L_{\ydiagram{3}}) + (\dim V_{\ydiagram{2,1}} \cdot \dim L_{\ydiagram{2,1}}) = (1 \cdot 4) + (2 \cdot \dim L_{\ydiagram{2,1}})$. This implies $2 \dim L_{\ydiagram{2,1}} = 4$, so $\dim L_{\ydiagram{2,1}} = 2$.

Remark 19.7. Only the partitions λ with a number of rows less than or equal to $\dim V$ will appear in the decomposition (i.e., $L_\lambda \neq 0$ only if $\lambda'_1 \leq \dim V$).

The End