EPFL - Fall 2025 Zsolt Patakfalvi
Rings and modules **Exercises**
Sheet 7

Exercise 1. Let F be an algebraically closed field, and let I, J be ideals of $R = F[x_1, ..., x_n]$. Prove that $\sqrt{I} \subseteq \sqrt{J}$ if and only if $V(J) \subseteq V(I)$.

Proof. Suppose that $\sqrt{I} \subseteq \sqrt{J}$. Note that $V(\sqrt{J}) = V(J)$ because the power of a polynomial and the polynomial itself have the same vanishing locus. Hence, if $P \in V(J)$ then $f(P) = 0$ for all $f \in \sqrt{J}$. But then $f(P) = 0$ for all $f \in I$ because $I \subseteq \sqrt{I} \subseteq \sqrt{J}$, and so $P \in V(I)$. Thus $V(J) \subseteq V(I)$.

Now suppose that $V(J) \subseteq V(I)$. Then $I(V(I)) \subseteq I(V(J))$, since $f \in I(V(I))$ iff f vanishes on $V(I)$, but in particular then f vanishes on $V(J)$. By the Nulstellensatz this implies that $\sqrt{I} \subseteq \sqrt{J}$. □

Exercise 2. Let F be an algebraically closed field, and let I, J be ideals of $R = F[x_1, ..., x_n]$. Show that

(1) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$
(2) $V(I) \cap V(J) = V(I + J)$

Proof. (1) First we show that $V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(IJ)$. As $IJ \subseteq I \cap J \subseteq I$, by the previous question $V(I) \subseteq V(I \cap J) \subseteq V(IJ)$ and so by symmetry $V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(IJ)$.

For the other inclusion, suppose conversely that there exists $P \in V(IJ) \setminus (V(I) \cup V(J))$. As P is not in $V(I) \cup V(J)$ we can find $f \in I$ such that $f(P) \neq 0$ and $g \in J$ such that $g(P) \neq 0$. But then $(fg)(P) \neq 0$ and $fg \in IJ$. This contradicts $P \in V(IJ)$.

(2) As $I \subseteq I + J$ we have $V(I + J) \subseteq V(I)$. So by symmetry $V(I + J) \subseteq V(I) \cap V(J)$.

Conversely suppose $P \in V(I) \cap V(J)$. Then $f(P) = 0$ for every $f \in I$ and $g(P) = 0$ for every $g \in J$, hence $(f + g)(P) = 0$ for every $f + g \in I + J$. Thus $P \in V(I + J)$ and we conclude $V(I + J) = V(I) \cap V(J)$

Remark: Let $(I_i)_{i \in \Sigma}$ be a collection of ideals of $R = F[x_1, \ldots, x_n]$, where Σ is an infinite indexing set. The same argument as in point (2) above shows that $\bigcap_i V(I_i) = V(\sum_i I_i)$. However, it is not true that in general $\bigcup_i V(I_i) = V(\bigcap_i I_i)$. For example, let $R = \mathbb{C}[x]$, $\Sigma = \mathbb{N}$ and $I_n = (x - n)$. Then $\bigcup_n V(x - n) = \mathbb{N}$ and $V(\bigcap_n (x - n)) = V(0) = \mathbb{C}$. □

Exercise 3. Let R be a commutative ring, and let I, J be ideals of R. In both Spec(R) and m-Spec(R), show that

(1) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$
(2) $V(I) \cap V(J) = V(I + J)$

Proof. (1) Again, since $IJ \subseteq I \cap J \subseteq I$, $V(I) \subseteq V(I \cap J) \subseteq V(IJ)$. Doing the same for J, we deduce that

$$V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(IJ)$$

so we are left to show that $V(IJ) \subseteq V(I) \subseteq V(J)$. Let $\mathfrak{p}$ be a prime ideal containing IJ, and assume by contradiction that both $I \nsubseteq \mathfrak{p}$ (let $x \in I \setminus \mathfrak{p}$) and $J \nsubseteq \mathfrak{p}$ (let $y \in I \setminus \mathfrak{p}$). Since $\mathfrak{p}$ is prime, $xy \in IJ \setminus \mathfrak{p}$, which contradicts that $IJ \subseteq \mathfrak{p}$.

(2) Since $I \subseteq I+J$, $V(I+J) \subseteq V(I)$. Doing the same for J gives $V(I+J) \subseteq V(I) \cap V(J)$. On the other hand, if $\mathfrak{p}$ contains both I and J, it contains $I+J$, so $V(I) \cap V(J) \subseteq V(I+J)$.

□

Exercise 4.
- Let R, S be commutative rings, and let $f : R \to S$ be a ring morphism. Show that there is an induced continuous map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$.
- Let R be a ring and I an ideal. Show that the morphism $\operatorname{Spec}(R/I) \to \operatorname{Spec}(R)$ induced by the quotient map corresponds to the inclusion of the closed subset $V(I) \subseteq \operatorname{Spec}(R)$.

Proof.
- Let $\theta : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ be defined by $\theta(\mathfrak{p}) = f^{-1}(\mathfrak{p})$ (recall from basic ring theory that the preimage of a prime ideal is always prime). To show the continuity of θ, we show that the preimage of closed subsets is closed. Let $V(I) \subseteq \operatorname{Spec}(R)$ be a closed subsets: we claim that $\theta^{-1}(V(I)) = V((f(I)))$. If $\mathfrak{p} \in V((f(I)))$, then in particular $\mathfrak{p} \supseteq f(I)$, so $\theta(\mathfrak{p}) = f^{-1}(\mathfrak{p}) \supseteq I$.
  Conversely, if $\mathfrak{p} \in \theta^{-1}(V(I))$, then $I \subseteq \theta(\mathfrak{p}) = f^{-1}(\mathfrak{p})$, and hence $f(I) \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, we deduce that $(f(I)) \subseteq \mathfrak{p}$ so we conclude.
- This is an immediate consequence of the correspondence theorem.

□

Exercise 5. Prove that $Z = \{(u^3, u^2v, uv^2, v^3) : u, v \in \mathbb{C}\} \subset \mathbb{C}^4$ is an algebraic set (i.e. there exists an ideal I of $\mathbb{C}[x_1, x_2, x_3, x_4]$ such that $Z = V(I)$). Find $I(Z)$.
[*Hint:* Make sure you have everything!]

Proof. First we prove that Z is an algebraic set. To start, let $R = \mathbb{C}[w, x, y, z]$; by trying around a bit one finds that the polynomials $x^2 - wy$, $y^2 - xz$ and $wz - xy$ vanish on Z. So if $I := (x^2 - wy, y^2 - xz, wz - xy) \subseteq R$, then $Z \subseteq V(I)$. We are now going to prove that $Z = V(I)$, and hence that Z is algebraic. In order to do so, let $P = (x_0, x_1, x_2, x_3) \in V(I)$ be arbitrary. Now notice that

$$x_1^3 \overset{x^2-wy}{=} x_0x_1x_2 \overset{wz-xy}{=} x_0^2x_3,$$

where the polynomial over the equality sign indicates which equation is used. Similarly, we have

$$x_2^3 \overset{y^2-xz}{=} x_1x_2x_3 \overset{wz-xy}{=} x_0x_3^2.$$

Therefore, if $x_0 = 0$, then $x_1 = x_2 = 0$ as well, and hence by choosing any $v \in \mathbb{C}$ such that $v^3 = x_3$ we see that $P \in Z$. Similarly, if $x_3 = 0$ then $x_1 = x_2 = 0$ and by choosing any $u \in \mathbb{C}$ with $u^3 = x_0$ we obtain $P \in Z$. Hence we may suppose that $x_0x_3 \neq 0$.
Now let $\tilde{u}, \tilde{v} \in \mathbb{C} \setminus \{0\}$ be such that $x_0 = \tilde{u}^3$ and $x_3 = \tilde{v}^3$. By substituting this into the above two equations, we obtain that there exist $\alpha, \beta \in \mathbb{C}$ such that $\alpha^3 = \beta^3 = 1$ and

$$x_1 = \alpha\tilde{u}^2\tilde{v} \quad \text{and} \quad x_2 = \beta\tilde{u}\tilde{v}^2.$$

Now notice that

$$\tilde{u}^3\tilde{v}^3 = x_0x_3 = x_1x_2 = \alpha\beta\tilde{u}^3\tilde{v}^3$$

and so as $\tilde{u}\tilde{v} \neq 0$ we obtain $\alpha\beta = 1$. So by introducing $u = \alpha\tilde{u}$ and $v = \beta\tilde{v}$, we obtain $x_0 = u^3$, $x_1 = u^2v$, $x_2 = uv^2$ and $x_3 = v^3$. Hence $P \in Z$, so we conclude that $Z = V(I)$, and thus Z is algebraic.

Now to finish the exercise, we are going to prove that $I = I(Z)$; by the above we already know $I \subseteq I(Z)$. Let us investigate the class $f + I$ of a polynomial $f \in R$. By using the equation $xy - wz \in I$, we may suppose that no monomial in f contains both x and y. Then by using the equations $x^3 - w^2z, y^3 - wz^2 \in I$, we may assume that no monomial in f is divisible by x^3 nor by y^3. Finally, by using the equations $x^2 - wy, y^2 - xz \in I$, we may suppose that no monomial in f is divisible by x^2 nor y^2. In conclusion, we have that for every $f \in R$ there exist $p_0, p_1, p_2 \in \mathbb{C}[w, z]$ such that

$$f + I = p_0 + xp_1 + yp_2 + I.$$

Now in order to prove the inclusion of $I(Z)$ inside I, let $f \in I(Z)$ be arbitrary. Consider the $\mathbb{C}$-algebra morphism

$$\begin{gathered} \Phi : \mathbb{C}[w, x, y, z] \to \mathbb{C}[u, v] \\ w \mapsto u^3,\ x \mapsto u^2v,\ y \mapsto uv^2,\ z \mapsto v^3. \end{gathered}$$

Then as $f \in I(Z)$, we have that $\Phi(f)$ vanishes on every point of $\mathbb{C}^2$, and thus $\Phi(f) = 0$. In particular, we have $x^2 - wy, y^2 - xz, wz - xy \in \operatorname{Ker}\Phi$, and so $I \subseteq \operatorname{Ker}\Phi$. Now by the argument in the beginning of this paragraph, there exist $p_0, p_1, p_2 \in \mathbb{C}[w, z]$ and $g \in I$ such that $f = p_0 + xp_1 + yp_2 + g$. Hence, as $\Phi(f) = \Phi(g) = 0$, we obtain

$$0 = \Phi(p_0 + xp_1 + yp_2) = p_0(u^3, v^3) + u^2vp_1(u^3, v^3) + uv^2p_2(u^3, v^3)$$

inside $\mathbb{C}[u, v]$. This then shows that $p_0 = p_1 = p_2 = 0$, and thus $f = g \in I$. As $f \in I(Z)$ was arbitrary, we conclude $I(Z) \subseteq I$, and thus $I(Z) = I$.

It is quite natural to expect the dimension of an algebraic set to be equal to the dimension of the space it is embedded into minus the number of generators of its ideal, as in linear algebra. This example shows that this idea is false in general.

$\square$

Exercise 6. Let F be an algebraically closed field, and $X \subseteq F^m$ an algebraic set with ideal $I = I(X)$. Define the coordinate ring $A(X)$ of X to be $A(X) := F[x_1, \dots, x_m]/I$. Notice that every element of $A(X)$ naturally defines a set-map from X to F, and thus one may think of $A(X)$ as the set of global algebraic functions on X.

(1) If $X = V(I) \subseteq F^m$, and $Y = V(J) \subseteq F^n$ are algebraic sets with ideals $I = I(X)$ and $J = I(Y)$, then a morphism $f : X \to Y$ is defined to be a set-map from the points of X to the points of Y, for which the following holds: there exists a vector $(h_1, \dots, h_n)$ of polynomials $h_i \in F[x_1, \dots, x_m]$, such that for every $\underline{a} \in X$ we have $f(\underline{a}) = (h_1(\underline{a}), h_2(\underline{a}), \dots, h_n(\underline{a})) \in Y$.

Show that whenever there is a morphism $f : X \to Y$ of algebraic sets as defined above, there is a unique homomorphism of F-algebras $\lambda_f : A(Y) \to A(X)$, such that the following diagram commutes.

$$\begin{tikzcd}
F[y_1,\dots,y_n] \arrow[r, "y_i \mapsto h_i"] \arrow[d] & F[x_1,\dots,x_m] \arrow[d] \\
A(Y) \arrow[r, "\lambda_f"] & A(X)
\end{tikzcd}$$

Here the vertical arrows are the quotient maps stemming from the definition of $A(X)$ and $A(Y)$, and the top horizontal map is given by sending y_i to $h_i(x_1,...,x_m)$.

(2) With setup as above, show that if there is a homomorphism of F-algebras $\lambda : A(Y) \to A(X)$, then there is a morphism $f : X \to Y$ such that $\lambda = \lambda_f$. Furthermore, all choices of f are the same (as set-maps from the points of X to the points of Y).

Proof. (1) Let $I = I(X)$ and $J = I(Y)$. Let ϕ be the given F-algebra homomorphism $F[y_1,...,y_n] \to F[x_1,...,x_m]$, sending y_j to h_j.

If the homomorphism $\lambda = \lambda_f$ exists, the diagram implies that for any $p + J \in A(Y)$ we must have $\lambda(p + J) = \phi(p) + I$. So λ is unique if it exists.

In order to show that it exists, let $\pi_X : F[x_1,\dots,x_m] \to A(X)$ and $\pi_Y : F[y_1,\dots,y_n] \to A(Y)$ be the projection maps. We want to show that $\pi_X \circ \phi$ factors through $A(Y)$, and to this end we want to show that $J \subseteq \mathrm{Ker}(\pi_X \circ \phi)$. So let $p \in J$ be arbitrary. Then $\phi(p) = p(h_1(x_1,\dots,x_m),\dots,h_n(x_1,\dots,x_m))$. Hence, if we evaluate $\phi(p)$ at a point $\underline{a} \in X$, we obtain $\phi(p)(\underline{a}) = p(h_1(\underline{a}),\dots,h_n(\underline{a})) = p(f(\underline{a}))$. But then as $f(\underline{a}) \in Y$ and $p \in J$, we obtain $\phi(p)(\underline{a}) = p(f(\underline{a})) = 0$. Hence $\phi(p)$ vanishes on every point of X, and thus $\phi(p) \in I$. Hence $p \in \mathrm{Ker}(\pi_X \circ \phi)$, and thus $J \subseteq \mathrm{Ker}(\pi_X \circ \phi)$. Therefore, there exists a morphism of F-algebras $\lambda : A(Y) \to A(X)$ such that $\pi_X \circ \phi = \lambda \circ \pi_Y$, i.e. the above diagram commutes.

(2) Now suppose we are given a homomorphism $\lambda : A(Y) \to A(X)$. For $j = 1,...,n$, choose $h_j \in F[x_1,\dots,x_m]$ such that $\lambda(y_j + J) = h_j + I$. Let $\phi : F[y_1,\dots,y_n] \to F[x_1,\dots,x_m]$ be defined as before, i.e. y_j is mapped to h_j.
Define the morphism of algebraic sets $f : F^m \to F^n$ by $f(\underline{a}) = (h_1(\underline{a}),...,h_m(\underline{a}))$. We must show that if $\underline{a} \in X$ then $f(\underline{a}) \in Y$. For this it is enough to show that $p(f(\underline{a})) = 0$ for all $p \in J$, by the Nullstellensatz. But as in the previous point, we have $p(f(\underline{a})) = p(h_1(\underline{a}),...,h_m(\underline{a})) = \phi(p)(\underline{a})$. So if we can show that $\phi(p) \in I(X)$ then we are done. But now notice that by definition of $h_1,\dots,h_n$ we have $\phi(p) + I = \lambda(p + J) = 0$, so $\phi(p) \in I$. Hence $f : F^m \to F^n$ restricts and co-restricts to a morphism of algebraic sets $f : X \to Y$. By comparing with the previous point, it is then straightforward to check that $\lambda = \lambda_f$, as both send $y_j + J$ to $h_j + I$.

Now we must show that two choices of lifting h_i and h_i' of $\overline{h}_i$ result in the same map on points of X. This holds because $h_i' = h_i + p_i$ for some $p_i \in I$, as the lifting is well defined up to addition of an element of I, but p_i vanishes on all points of X. Hence $h_i(\underline{a}) = h_i'(\underline{a})$ for all $\underline{a} \in X$, so $(h_1,\dots,h_n)$ and $(h_1',\dots,h_n')$ define the same set-map. □

Exercise 7. Let F be an algebraically closed field. Let X be an algebraic set in F^n with ideal $I(X) = I$. Prove that points of F^n contained in X are naturally in bijection with maximal ideals of the coordinate ring $A(X) = F[x_1,...,x_n]\big/I$.

Proof. Given a point $P = (a_1, ..., a_n) \in X$, let $\mathfrak{m}_P = (x_1 - a_1, ..., x_n - a_n)$. Since $P \in X$, we have $I \subseteq \mathfrak{m}_P$ by Exercise 1. Thus $\mathfrak{m}_P$ is a maximal ideal containing $I(X)$, and hence defines a maximal ideal $\overline{\mathfrak{m}}_P$ of $A(X) = F[x_1, ..., x_n] \big/ I$. Conversely, a maximal ideal $\overline{\mathfrak{m}}$ of $A(X) = F[x_1, ..., x_n] \big/ I$ is equivalent to a maximal ideal $\mathfrak{m}$ of $F[x_1, ..., x_n]$ containing I. By the Weak Nullstellensatz $\mathfrak{m} = (x_1 - a_1, ..., x_n - a_n)$, for some $a_i \in F$. The containment $I \subseteq \mathfrak{m}$ implies that $P = (a_1, ..., a_n) \in X$, and thus $\overline{\mathfrak{m}} = \overline{\mathfrak{m}}_P$. Thus the set of maximal ideals of $A(X)$ is given by $\{\overline{\mathfrak{m}}_P \mid P \in X\}$. Finally, suppose that $\overline{\mathfrak{m}}_P = \overline{\mathfrak{m}}_Q$ for $P, Q \in X$. Then necessarily $\mathfrak{m}_P = \mathfrak{m}_Q$ and thus $\{P\} = V(\mathfrak{m}_P) = V(\mathfrak{m}_Q) = \{Q\}$, and thus $P = Q$. Thus there is a bijection between X and the set of maximal ideals $A(X)$.

□