EPFL - Fall 2025 Zsolt Patakfalvi
Rings and modules **Exercises**
Sheet 6 - Solutions

Exercise 1. Let $R = k[x,y]$ be the polynomial ring in two variables over an algebraically closed field k. Recall that an ideal $\mathfrak{m}$ in a ring R is maximal if it is not properly contained in any other proper ideal of R. In this exercise you can use freely the Theorem below, which will be proven later in the course.

Theorem (The weak Nullstellensatz in two variables)**.** *Let k be an algebraically closed field. Every maximal ideal $\mathfrak{m}$ in the ring $k[x,y]$ is of the form $\mathfrak{m} = (x-a, y-b)$ for some $a, b \in k$.*

Show the following:

(1) If M is a finite length module over R, then the quotients of its composition series are of the form ${}^{R}\!/\!_{(x-a,y-b)}$.
(2) If M is a module such that $\operatorname{Ann}(M) \supseteq (x-a, y-b)$, then $\operatorname{Ann}(\operatorname{Ext}^i(M,N)) \supseteq (x-a, y-b)$ for every R-module N.
[*Hint*: Consider the multiplication by $x-a$ resp. $y-b$ on M and the induced maps on $\operatorname{Ext}^i_R(M,N)$. Recall also Exercise 7 of Sheet 4.]
(3) If N is any finitely generated module over R, then $\operatorname{Ext}^i\left({}^{R}\!/\!_{(x-a,y-b)}, N\right)$ has finite length.
[*Hint*: Use the previous point.]
(4) For every finite length module M and for every finitely generated module N over R, $\operatorname{Ext}^i_R(M,N)$ has finite length.
[*Hint*: Use the long exact sequence for a compostion series.]

Proof. (1) Let $0 = M_0 < M_1 < \cdots < M_n = M$ be a composition series. Since $Q_i := {}^{M_i}\!/\!_{M_{i-1}}$ is simple we have $Q_i \cong {}^{R}\!/\!_{\operatorname{Ann}(Q_i)}$ by Exercise 1 on Sheet 1. As thus R-submodules of Q_i correspond to ideals of R containing $\operatorname{Ann}(Q_i)$, we obtain that $\operatorname{Ann}(Q_i)$ is maximal. Hence, we conclude by the weak Nullstellensatz.

(2) By Exercise 7.4 of Sheet 4, multiplication by $r \in R$ on M induces multiplication by r on $\operatorname{Ext}^i(M,N)$. Hence if $r \in \operatorname{Ann}(M)$, multiplication by r is equal to multiplication by 0 on M, and hence multiplication by r is equal to multiplication by 0 on $\operatorname{Ext}^i(M,N)$, and thus $r \in \operatorname{Ann}(\operatorname{Ext}^i(M,N))$. Hence we obtain $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(\operatorname{Ext}^i(M,N))$ which is enough to conclude.

(3) By the previous point $\operatorname{Ext}^i\left({}^{R}\!/\!_{(x-a,y-b)}, N\right)$ has a natural structure as ${}^{R}\!/\!_{(x-a,y-b)} \cong k$ module, and the R-submodules are precisely the k-submodules. It is therefore sufficient to prove that $\operatorname{Ext}^i\left({}^{R}\!/\!_{(x-a,y-b)}, N\right)$ has finite length over k, i.e. is a finite dimensional k-vectorspace. To achieve this, we will show that $\operatorname{Ext}^i\left({}^{R}\!/\!_{(x-a,y-b)}, N\right)$ is a finitely generated R-module. Let $P_\bullet \to {}^{R}\!/\!_{(x-a,y-b)}$ be a free resolution. Since R is a Noetherian ring every submodule of R^n is finitely generated, hence we may assume each P_i is finitely generated. Observe that $\operatorname{Hom}_R(R^n, N) \cong N^n$ is finitely generated for every $n \geq 0$. Again using that R is Noetherian any submodule or quotient of a finitely generated module is finitely generated, therefore we

conclude that $\mathrm{Ext}^i\left({}^{R}\!/_{(x-a,y-b)}, N\right)$ is a finitely generated R-module. This implies that $\mathrm{Ext}^i\left({}^{R}\!/_{(x-a,y-b)}, N\right)$ is a finitely generated ${}^{R}\!/_{(x-a,y-b)}$-module and hence a finite dimensional k-vectorspace.

(4) We prove this by induction following the hint. To this end let $0 = M_0 < M_1 \cdots < M_n = M$ be a composition series, we note that since M_1 is simple we have that $M_1 \cong {}^{R}\!/_{(x-a,y-b)}$ and thus $\mathrm{Ext}^i_R(M_1, N)$ is of finite length by the previous point. We have a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow {}^{M_2}\!/_{M_1} \longrightarrow 0$$

which induces an exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_R\left({}^{M_2}\!/_{M_1}, N\right) \longrightarrow \mathrm{Ext}^i_R(M_2, N) \longrightarrow \mathrm{Ext}^i_R(M_1, N) \longrightarrow \cdots .$$

By passing to the kernel on the left and the image on the right (since being of finite length is stable under quotients and submodules) we can assume that $\mathrm{Ext}^i_R(M_2, N)$ is the middle term in a short exact sequence with kernel and image of finite length, but then it follows that $\mathrm{Ext}^i_R(M_2, N)$ is of finite length. We can now repeat the argument for M_3 and so on and so forth. By induction, this proves that $\mathrm{Ext}^i_R(M, N) = \mathrm{Ext}^i_R(M_n, N)$ has finite length for all $i \geq 0$.

□

Exercise 2. Let $R = k[x,y]$ be as in the previous exercise (k is algebraically closed). We say that a finite length module is supported at $(x-a, y-b)$ if only ${}^{R}\!/_{(x-a,y-b)}$ appears as quotients in the composition series. Show that if M is a finite length module supported at $(x-a, y-b)$, then $\mathrm{Ext}^i_R\left(M, {}^{R}\!/_{(x-a',y-b')}\right) = 0$ for all $(a', b') \neq (a, b)$.

Proof. We first show that $\mathrm{Ext}^i_R\left({}^{R}\!/_{(x-a,y-b)}, {}^{R}\!/_{(x-a',y-b')}\right) = 0$ for all $i \geq 0$. By a similar argument as in Exercise 1 of this sheet (by using points (3) and (4) of Exercise 6 on Sheet 4) we have that both $(x-a, y-b)$ and $(x-a', y-b')$ are included in the annihilator of $\mathrm{Ext}^i_R\left({}^{R}\!/_{(x-a,y-b)}, {}^{R}\!/_{(x-a',y-b')}\right)$. Therefore, the ideal $(x-a, y-b) + (x-a', y-b') = R$ is in the annihilator of $\mathrm{Ext}^i_R\left({}^{R}\!/_{(x-a,y-b)}, {}^{R}\!/_{(x-a',y-b')}\right)$, which implies $\mathrm{Ext}^i_R(R/(x-a,y-b), R/(x-a',y-b')) = 0$.
Let $0 = M_0 < M_1 \cdots < M_n = M$ be a composition series. Denote $N = R/(x-a', y-b')$. We can now conclude by first looking at the short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_2/M_1 \longrightarrow 0$$

which induces an exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_R\left({}^{M_2}\!/_{M_1}, N\right) \longrightarrow \mathrm{Ext}^i_R(M_2, N) \longrightarrow \mathrm{Ext}^i_R(M_1, N) \longrightarrow \cdots .$$

From here we see that $\mathrm{Ext}^i_R(M_2, N) = 0$, since the other two modules are trivial by what has already been proven. We continue, upon replacing M_1 with M_2 and M_2 with M_3, we can conclude in an analog way that $\mathrm{Ext}^i_R(M_3, N) = 0$. We continue step by step, to conclude by induction that $\mathrm{Ext}^i_R\left(M, {}^{R}\!/_{(x-a',y-b')}\right) = \mathrm{Ext}^i_R\left(M_n, {}^{R}\!/_{(x-a',y-b')}\right) = 0$. □

Exercise 3. Show using the long exact sequence of cohomology that if $\operatorname{Ext}^1_R(M,N) = 0$, then every extension $0 \longrightarrow N \longrightarrow K \longrightarrow M \longrightarrow 0$ splits.

Proof. Denote by i the injection $i : N \to K$. By the long exact sequence of Ext-modules, we obtain that

$$0 \longrightarrow \operatorname{Hom}_R(M,N) \longrightarrow \operatorname{Hom}_R(K,N) \xrightarrow{-\circ i} \operatorname{Hom}_R(N,N) \longrightarrow 0$$

is exact. In particular, there exists $q \in \operatorname{Hom}_R(K,N)$ such that $q \circ i = \mathrm{id}_N$. Thus by Exercise 3 on Sheet 4, the sequence splits. ☐

Exercise 4. Let $R = k[x,y]$, and let $M = R/(x,y)$.

(1) Show that $\operatorname{Ext}^1_R(M,M) \cong M^2$.

Note that there is canonical bijection $k \to M$, sending $\lambda \in k$ to the class of the constant polynomial λ modulo (x,y). In particular, there is also a natural bijection $k^2 \to M^2$.

(2) For a given $(\lambda,\mu) \in k^2 \setminus \{(0,0)\}$, define

$$N_{\lambda,\mu} = R/(x^2, y^2, xy, \lambda y - \mu x),$$

let $\varphi\colon N_{\lambda,\mu} \to M$ be the map induced by the quotient map $R \to M$, and let $\psi\colon M \to N_{\lambda,\mu}$ be the map sending the class of 1 to the class of $-(xa + yb)$, where $a,b \in k$ are any elements such that $\lambda a + \mu b = 1$.

Then show that the Yoneda extension associated to $(\lambda,\mu) \in k^2\setminus\{(0,0)\}$ is isomorphic to the sequence

$$0 \to M \xrightarrow{\psi} N_{\lambda,\mu} \xrightarrow{\varphi} M \to 0.$$

(3) Under what conditions on (λ,μ) and (λ',μ') do we have an isomorphism $N_{\lambda,\mu} \cong N_{\lambda',\mu'}$? *Hint: Think about torsion.*

Proof. (1) This computation was already made in the lecture notes. Let us recall it quickly. Consider the projective resolution

$$0 \to R \xrightarrow{f_2} R^2 \xrightarrow{f_1} R$$

of M, where $f_1(a) = a(y,-x)$ and $f_2(b,c) = bx + cy$. Then this is truly a resolution by the lecture notes, and applying $\operatorname{Hom}(-,M)$ to it gives

$$\operatorname{Hom}(R,M) \to \operatorname{Hom}(R,M^2) \to \operatorname{Hom}(R,M),$$

and a computation which we already did showed that both maps in this sequence were zero. This, $\operatorname{Ext}^1(M,M) \cong \operatorname{Hom}(R^2,M) \cong M^2$, where the second isomorphism is given by sending $f\colon R^2 \to M$ to $(f(1,0), f(0,1)) \in M^2$.

(2) Let $(\lambda,\mu) \in k^2 \setminus \{(0,0)\}$. Then the associated element in $\operatorname{Ext}^1(M,M) = \operatorname{Hom}(R^2,M)$ is the function $\phi\colon R^2 \to M$ is $\theta(a,b) = [a\lambda + b\mu]$ by the previous point. Therefore, we know by the course (c.f. Notation 5.6.5) that the associated exact sequence is

$$0 \to M \xrightarrow{\alpha} \operatorname{coker}\left(R^2 \xrightarrow{(\phi,f_1)} M \oplus R\right) \xrightarrow{\beta} M \to 0,$$

where $\alpha(a) = [(a,0)]$ and $\beta = [f_0 \circ \mathrm{pr}_R]$, where $f_0\colon R \to M$ is the quotient map.

Let S denote the middle term of this sequence. We want to find an explicit isomorphism between the middle term and $N_{\lambda,\mu}$. In fact, there is a natural map $\pi\colon R \to S$, given by the composition

$$R \xrightarrow{(0,\mathrm{id})} M \oplus R \to S.$$

Then

- $x^2 \in \ker(\pi)$, because $(0, x^2) = (\phi, f_1)(x, 0)$;
- $y^2 \in \ker(\pi)$, because $(0, y^2) = (\phi, f_1)(0, y)$;
- $xy \in \ker(\pi)$, because $(0, xy) = (\phi, f_1)(y, 0)$;
- $\lambda y - \mu x \in \ker(\pi)$, because $(0, \lambda y - \mu x) = (\phi, f_1)(-\mu, \lambda)$.

Hence, $(x^2, xy, y^2, \lambda y - \mu x) \subseteq \ker(\pi)$. Let us show the other containment, so suppose that $a \in R$ satisfies $\pi(a) = 0$. Then by definition, there exists $(b, c) \in R^2$ such that

$$(0, a) = ([\lambda b + \mu c], xb + yc) \in M \oplus R.$$

Write $b = b_0 + b'$, where $b_0 \in k$ is the constant term, and $b' \in (x, y)$. Similarly, write $c = c_0 + c'$. Then $[\lambda b + \mu c] = 0$ is the same as saying that $\lambda b_0 + \mu c_0 = 0 \in k$, or equivalently $\lambda b_0 = -\mu c_0$.

Then we can write

$$a = bx + cy = b_0 x + c_0 y + d,$$

where $d \in (x^2, xy, y^2)$. Hence, to conclude, we must show that $b_0 x + c_0 y \in (-\mu x + \lambda y)$. Assume without loss of generality that $\lambda \neq 0$. Then $b_0 = \frac{-\mu}{\lambda} c_0$, so

$$b_0 x + c_0 y = c_0 \left(\frac{-\mu}{\lambda} x + y \right) = \frac{c_0}{\lambda}(-\mu x + \lambda y) \in (-\mu x + \lambda y).$$

The computation is exactly the same when $\mu \neq 0$, so we conclude the proof that $\ker(\pi) = (x^2, xy, y^2, \lambda y - \mu x)$.

Finally, let us show that π is surjective. Fix elements $\delta, \epsilon \in k$ such that $\lambda\delta + \mu\epsilon = 1$ (this is possible since either $\lambda \neq 0$ or $\mu \neq 0$). Then given any $([m], r) \in M \oplus R$, we have that

$$([m], r) = ([m](\lambda\delta + \epsilon\mu), r) = (\phi, f_1)((m\delta, m\mu)) + (0, r - (m\delta x + m\epsilon y)),$$

so $[([m], r)] = \pi(r - (m\delta x + m\epsilon y)$, proving the surjectivity of π.

Thus, the first isomorphism theorem gives us that $N_{\lambda,\mu} \cong S$. We have a diagram as follows:

$$\begin{tikzcd}
& & N_{\lambda,\mu} \arrow[d, "\overline{\pi}"] & & \\
0 \arrow[r] & M \arrow[r, "\alpha"] & S \arrow[r, "\beta"] & M \arrow[r] & 0,
\end{tikzcd}$$

so we automatically have an isomorphism of complexes

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\overline{\pi}^{-1}\circ\alpha"] \arrow[d, "\mathrm{id}"] & N_{\lambda,\mu} \arrow[r, "\beta\circ\overline{\pi}"] \arrow[d, "\overline{\pi}"] & M \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha"] & S \arrow[r, "\beta"] & M \arrow[r] & 0,
\end{tikzcd}$$

which in particular shows that the top row is a short exact sequence (and hence we have an isomorphism of Yoneda extensions).

Let us compute explicitly the two maps of the top row. It follows directly from the definitions that the composition $\beta \circ \overline{\pi} \colon N_{\lambda,\mu} \to M$ is simply the quotient map. Furthermore, the composition $\overline{\pi}^{-1} \circ \alpha \colon M \to N_{\lambda,\mu}$ is by definition given by sending $[1]$ to $\overline{\pi}^{-1}[(1,0)]$. However, it follows from our computation above that $[(1,0)] = \pi(-(\delta x + \epsilon y))$, so we are done.

(3) We are going to show that

$$N_{\lambda,\mu} \cong N_{\lambda',\mu'} \iff \text{there exists } s \in k^{\times} \text{s.t. } s(\lambda,\mu) = (\lambda',\mu')$$

(or in other words if they span the same line in k^2).

First of all, note that if there exist $s \in k \setminus \{0\}$, such that $s(\lambda,\mu) = (\lambda',\mu')$, then $(x^2, xy, y^2, \lambda y - \mu x) = (x^2, xy, y^2, s^{-1}(\lambda' y - \mu' x)) = (x^2, xy, y^2, \lambda' y - \mu' x)$, so $N_{\lambda,\mu} = N_{\lambda',\mu'}$.

Our goal is to show that this is the only way that the modules $N_{\lambda,\mu}$ are isomorphic. Hence, assume that there are (λ,μ), $(\lambda',\mu') \in k^2 \setminus \{(0,0)\}$ such that $N_{\lambda,\mu} \cong N_{\lambda',\mu'}$. Then in particular,

$$(x^2, xy, y^2, \lambda y - \mu x) = \operatorname{Ann}(N_{\lambda,\mu}) = \operatorname{Ann}(N_{\lambda',\mu'}) = (x^2, xy, y^2, \lambda' y, -\mu' x).$$

Assume that (λ,μ) and (λ',μ') are not collinear in k^2. In particular, they span k^2 as a vector space, so there exist $s, t \in k$ such that

$$s(\lambda,\mu) = t(\lambda',\mu') = (1,0).$$

In particular,

$$s(\lambda y - \mu x) + t(\lambda' y - \mu' x) = y,$$

so $y \in (\lambda y - \mu x, \lambda' y - \mu' x)$. Similarly, x lies in this ideal, so given that

$$\lambda' y - \mu' x \in (x^2, xy, y^2, \lambda y - \mu x)$$

by assumption, we deduce that $x, y \in (x^2, xy, y^2, \lambda y - \mu x)$, so

$$(x^2, xy, y^2, \lambda y - \mu x) = (x, y)$$

and $N_{\lambda,\mu} = R/(x,y) = M$. However, this is impossible since $N_{\lambda,\mu}$ has length 2 (by additivity of the length) and M has length 1, so we obtained a contradiction. Thus, we win.

□

Exercise 5. Let $R = k[x,y]$.

(1) Show that $\operatorname{Ext}^1\left((x,y), {}^{R}/_{(x,y)}\right) \neq 0$.

(2) Construct a finitely generated module M such that $\operatorname{Tors}(M) \subseteq M$ is not a direct summand.

[*Note:* For M finitely generated over a PID R, $\operatorname{Tors}(M) \subseteq M$ is always a direct summand by the fundamental theorem for finitely generated modules over PIDs.]

Proof. (1) Identify $k = {}^{R}/_{(x,y)}$ as usual. As seen on several occasions in this course, we have a projective resolution

$$0 \to R \to R \oplus R \to (x,y) \to 0$$

where the morphisms are given by $r \mapsto (-ry, rx)$ and $(r_1, r_2) \mapsto r_1 x + r_2 y$, respectively. To calculate $\operatorname{Ext}^1((x,y), k)$ we apply $\operatorname{Hom}(-, k)$ and calculate the cohomology in degree one of the corresponding complex. That is, the cokernel of $k \oplus k \to k$ given by $(r_1, r_2) \mapsto -r_1 y + r_2 x = 0$. Here we used that multiplication by x and y are zero. In particular we obtain

$$\mathrm{Ext}^1((x,y),k) = k \neq 0.$$

(2) We prove the following more general statement:

Lemma 0.1. *Let R be a domain, N a torsion module (i.e. for all $n \in N$, there exists a non-zero $r \in R$ such that $rn = 0$) and L a torsion-free module. Let*

$$0 \to N \to M \to L \to 0$$

be a non-split short exact sequence. Then $\mathrm{Tors}(M) \subseteq M$ is not a direct summand.

Proof. We may assume $N \subseteq M$ and $L = M/N$ (this is just to make notations simpler). First, note that $\mathrm{Tors}(M) = N$. Indeed, since N is torsion, $N \subseteq \mathrm{Tors}(M)$. Conversely, given $m \in \mathrm{Tors}(M)$, let $r \in R$ be non-zero such that $rm = 0$. Then $r\pi(m) = 0$, where $\pi : M \to M/N$ denotes the quotient map. Since L is torsion-free, $\pi(m) = 0$, so $r \in N$.

Now, assume $N = \mathrm{Tors}(M)$ was a direct summand. Then there would exist a morphism $M \to N$ such that the composition $N \subseteq M \to M$ is the identity, or in other words there exists a section of $N \subseteq M$ (see Exercise 3 on sheet 4). By this same exercise, this implies that the sequence

$$0 \to N \to M \to M/N \to 0$$

is split, which is a contradiction with our hypotheses. □

To conclude, we have found that $\mathrm{Ext}^1((x,y),k) \neq 0$ so there exists a non-split extension

$$0 \to k \to M \to (x,y) \to 0$$

We are done by the previous lemma.

□

Exercise 6. Throughout this exercise, R will be a ring and M, N will be R-modules. We will now see another way to compute the Ext-modules than the one we saw in the lectures (one may say a 'dual' way). To do so, we need the following Lemma, which you may use without proof.

Lemma 1. *For every R-module N there exists an injective R-module homomorphism $N \to I$ where I is an injective R-module.*

(1) Using the above Lemma, show that any R-module N admits an injective resolution. That is, there exists an exact sequence

$$0 \longrightarrow N \xrightarrow{i^{-1}} I^0 \xrightarrow{i^0} I^1 \longrightarrow \cdots$$

where I^b is an injective R-module for all $b \geq 0$ (the numbers in superscript are just indices, *not* exponents of any sort).

(2) Show that an R-module I is injective if and only if $\mathrm{Hom}_R(-, I)$ is exact.
[*Reminder:* By Lemma 5.2.2 of the lecture notes $\mathrm{Hom}_R(-, I)$ is always left exact.]

(3) Fix a projective resolution $P_\bullet \twoheadrightarrow M$ and an injective resolution $N \hookrightarrow I^\bullet$. Consider the commutative diagram

$$\begin{tikzcd}
 & \vdots & & \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}_R(M, I^1) \arrow[u] \arrow[rr, "d_{-1,1}"] & & \mathrm{Hom}_R(P_0, I^1) \arrow[u] \arrow[r, "d_{0,1}"] & \mathrm{Hom}_R(P_1, I^1) \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_R(M, I^0) \arrow[u, "\delta_{-1,0}"'] \arrow[rr, "d_{-1,0}"] & & \mathrm{Hom}_R(P_0, I^0) \arrow[u, "\delta_{0,0}"'] \arrow[r, "d_{0,0}"] & \mathrm{Hom}_R(P_1, I^0) \arrow[u, "\delta_{1,0}"'] \arrow[r] & \cdots \\
 & & \bullet \arrow[uuu, dotted, no head] \arrow[rrr, dotted, no head] & & & \\
 & 0 \arrow[uu] \arrow[rr] & & \mathrm{Hom}_R(P_0, N) \arrow[uu, "\delta_{0,-1}"'] \arrow[r, "d_{0,-1}"] & \mathrm{Hom}_R(P_1, N) \arrow[uu, "\delta_{1,-1}"'] \arrow[r] & \cdots \\
 & & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $d_{a,b} = - \circ p_{a+1}$ and $\delta_{a,b} = i^b \circ -$ for all $a, b \geq -1$. Briefly justify that this is indeed commutative, and that all columns and lines of the diagram which are not blue are exact.

(4) Show that $H^0(\mathrm{Hom}_R(M, I^\bullet)) \cong H^0(\mathrm{Hom}_R(P_\bullet, N))$.
[*Hint:* Show that their images inside $\mathrm{Hom}_R(P_0, I^0)$ coincide.]

(5) Show that $H^1(\mathrm{Hom}_R(M, I^\bullet)) \cong H^1(\mathrm{Hom}_R(P_\bullet, N))$.
[*Hint:* Let $C^0 := \mathrm{Hom}_R(P_0, I^0)$ and $C^1 = \mathrm{Hom}_R(P_1, I^0) \oplus \mathrm{Hom}_R(P_0, I^1)$, and let $\Delta^0 : C^0 \to C^1$ be the map sending $x \in C^0$ to $(d_{0,0}(x), \delta_{0,0}(x)) \in C^1$. Show that the cohomology groups in question both embed into $\mathrm{coker}(\Delta^0)$ and that their images therein coincide.]

[*Remark:* One can generalize the above results and prove that in fact $H^i(\mathrm{Hom}_R(M, I^\bullet)) \cong H^i(\mathrm{Hom}_R(P_\bullet, N))$ for all $i \geq 0$, and thus the Ext-modules may also be computed by using an injective resolution of the second module. To do so, one defines the modules $C^m := \bigoplus_{a+b=m} \mathrm{Hom}_R(P_a, I^b)$ and connecting maps $\Delta^m : C^m \to C^{m+1}$ similar to Δ^0, where one replaces $\delta_{a,b}$ by $(-1)^a \delta_{a,b}$ to ensure $\Delta^{m+1} \circ \Delta^m = 0$. We thus obtain a complex $C^\bullet$, and one can then prove that $H^i(\mathrm{Hom}_R(M, I^\bullet))$ and $H^i(\mathrm{Hom}_R(P_\bullet, N))$ embed into $H^i(C^\bullet)$ with equal image.]

Proof. (1) By the Lemma, there exists an injective map $i^{-1} : N \hookrightarrow I^0$ with I^0 injective. Denote $I^{-1} = N$ for convenience. For $b \geq 1$, let I^b be an injective module such that there exists an injective map $\mathrm{coker}(I^{b-2} \xrightarrow{i^{b-2}} I^{b-1}) \hookrightarrow I^b$, and let i^{b-1} be the composition $I^{b-1} \to \mathrm{coker}(I^{b-2} \xrightarrow{i^{b-2}} I^{b-1}) \hookrightarrow I^b$. Then it is straightforward to verify that

$$0 \longrightarrow N \xrightarrow{i^{-1}} I^0 \xrightarrow{i^0} I^1 \longrightarrow \cdots$$

is an injective resolution.

(2) Suppose that I is injective, and let

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

be an exact sequence of R-modules. To verify that

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(C, I) \arrow[r, "-\circ\beta"] & \mathrm{Hom}_R(B, I) \arrow[r, "-\circ\alpha"] & \mathrm{Hom}_R(A, I) \arrow[r] & 0 \end{tikzcd}$$

is exact, it suffices to verify that $-\circ\alpha$ is surjective, as $\mathrm{Hom}_R(-, I)$ is left exact by Lemma 5.2.2. So let $\phi \in \mathrm{Hom}_R(A, I)$ be arbitrary. Then we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "\alpha"] \arrow[d, "\phi"] & B \\ & I & \end{tikzcd}$$

and thus by definition of I being injective, there exists $\psi : B \to I$ making the diagram commute. This precisely means $(-\circ\alpha)(\psi) = \phi$, so $-\circ\alpha$ is surjective.

Conversely, suppose that $\mathrm{Hom}_R(-, I)$ is exact, and suppose that we have a diagram of R-modules

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, hook, "f"] \arrow[d, "g"] & Y \\ & I. & \end{tikzcd}$$

Then as $\mathrm{Hom}_R(-, I)$ is exact, the map $-\circ f : \mathrm{Hom}_R(Y, I) \to \mathrm{Hom}_R(X, I)$ is surjective. In particular, there exists $h : Y \to I$ such that $h \circ f = g$, and thus a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, hook, "f"] \arrow[d, "g"] & Y \arrow[dl, "h"] \\ & I, & \end{tikzcd}$$

which proves that I is injective.

(3) Exactness of the non-blue rows follows as they are obtained from applying the exact functor $\mathrm{Hom}_R(-, I^b)$ to the exact sequence $P_\bullet \to M \to 0$, and exactness of the non-blue columns follows as they are obtained from applying the exact functor $\mathrm{Hom}_R(P_a, -)$ to the exact sequence $0 \to N \to I^\bullet$. The diagram commutes as vertical arrows are given by post-composition and horizontal arrows are given by pre-composition, and these two operations commute by associativity of composition.

(4) Let $\phi_{0,-1} \in \mathrm{Ker}(d_{0,-1})$ be arbitrary. Then by commutativity we have $d_{0,0} \circ \delta_{0,-1}(\phi_{0,-1}) = 0$, and so by exactness there exists $\phi_{-1,0} \in \mathrm{Hom}_R(M, I^0)$ such that $d_{-1,0}(\phi_{-1,0}) = \delta_{0,-1}(\phi_{0,-1})$. This shows that $d_{-1,0}(H^0(\mathrm{Hom}_R(P_\bullet, N))) \subseteq \delta_{0,-1}(H^0(\mathrm{Hom}_R(M, I_\bullet)))$, and a completely symmetric argument yields also the reverse inclusion. We conclude by injectivity of $d_{-1,0}$ and $\delta_{0,-1}$.

(5) We employ the notations of the Hint. To construct a map $H^1(\mathrm{Hom}_R(P_\bullet, N)) \to \mathrm{coker}(\Delta^0)$, we have to verify that if $\phi_{1,-1} \in \mathrm{im}(d_{0,-1})$, then $(\delta_{1,-1}(\phi_{1,-1}), 0) \in \mathrm{im}\,\Delta^0$. Let $\phi_{0,-1} \in \mathrm{Hom}_R(P_0, N)$ be such that $\phi_{1,-1} = d_{0,-1}(\phi_{0,-1})$. By the commutativity and exactness properties of the diagram, it is straightforward to verify that $\Delta^0(\delta_{0,-1}(\phi_{0,-1})) =$

$(\delta_{1,-1}(\phi_{1,-1}),0)$, and thus the latter is in the image of Δ^0. Therefore, the composition $\mathrm{Ker}(d_{1,-1}) \hookrightarrow \mathrm{Hom}_R(P_1,N) \hookrightarrow C^1 \twoheadrightarrow \mathrm{coker}(\Delta^0)$ factors through $H^1(\mathrm{Hom}_R(P_\bullet,N))$, i.e. we obtain a map $\alpha : H^1(\mathrm{Hom}_R(P_\bullet,N)) \to \mathrm{coker}(\Delta^0)$ given by mapping the class of $\phi_{1,-1} \in \mathrm{Ker}(d_{1,-1})$ to the class of $(\delta_{1,-1}(\phi_{1,-1}),0)$.

Now we verify that α is injective. To do so, suppose that $\phi_{1,-1} \in \mathrm{Ker}(d_{1,-1})$ is such that $(\delta_{1,-1}(\phi_{1,-1}),0) \in \mathrm{im}(\Delta^0)$; we have to show that then $\phi_{1,-1} \in \mathrm{im}(d_{0,-1})$. Let $\phi_{0,0} \in \mathrm{Hom}_R(P_0,I^0)$ be such that $\Delta^0(\phi_{0,0}) = (\delta_{1,-1}(\phi_{1,-1}),0)$. In particular, we have $\delta_{0,0}(\phi_{0,0}) = 0$, so by exactness there exists $\phi_{0,-1} \in \mathrm{Hom}_R(P_0,N)$ such that $\phi_{0,0} = \delta_{0,-1}(\phi_{0,-1})$. Hence we obtain

$$\delta_{1,-1}(\phi_{1,-1}) = d_{0,0}(\psi_{0,0}) = d_{0,0}(\delta_{0,-1}(\phi_{0,-1})) = \delta_{1,-1}(d_{0,-1}(\phi_{0,-1})),$$

and so by injectivity of $\delta_{1,-1}$ it follows that $\phi_{1,-1} = d_{0,-1}(\phi_{0,-1})$. So $\phi_{1,-1}$ is in the image of $d_{0,-1}$, and thus α is injective.

Now by a completely symmetrical argument, there exists and injective map $\beta : H^1(\mathrm{Hom}_R(M,I^\bullet)) \to \mathrm{coker}(\Delta^0)$, mapping the class of $\phi_{-1,1} \in \mathrm{Ker}(\delta_{-1,1})$ to the class of $(0,d_{-1,1}(\psi)_{-1,1})$. So what is left to show is that the image of α is the same as the image of β. To this end, let $\phi_{1,-1} \in \mathrm{Ker}(d_{1,-1})$ be arbitrary. Then by commutativity we have $d_{1,0}(\delta_{1,-1}(\phi_{1,-1})) = 0$, and so by exactness there exists $\phi_{0,0} \in \mathrm{Hom}_R(P_0,I^0)$ with $d_{0,0}(\phi_{0,0}) = \delta_{1,-1}(\phi_{1,-1})$. Then notice that

$$d_{0,1}(\delta_{0,0}(\phi_{0,0})) = \delta_{1,0}(d_{0,0}(\phi_{0,0})) = \delta_{1,0}(\delta_{1,-1}(\phi_{1,-1})) = 0$$

and thus by exactness there exists $\phi_{-1,1} \in \mathrm{Hom}_R(M,I^1)$ such that $d_{-1,1}(\phi_{-1,1}) = \delta_{0,0}(\phi_{0,0})$. By a similar string of equations as above, we obtain $d_{-1,2}(\delta_{-1,1}(\phi_{-1,1})) = 0$, which by injectivity of $d_{-1,2}$ gives $\phi_{-1,1} \in \mathrm{Ker}(\delta_{-1,1})$. Now we verify that $\alpha(\phi_{1,-1} + \mathrm{im}(d_{0,-1})) = \beta(-\phi_{-1,1} + \mathrm{im}(\delta_{-1,0}))$. To this end, notice that

$$\Delta^0(\phi_{0,0}) = (d_{0,0}(\phi_{0,0}),\delta_{0,0}(\phi_{0,0})) = (\delta_{1,-1}(\phi_{1,-1}),d_{-1,1}(\phi_{-1,1})) = (\delta_{1,-1}(\phi_{1,-1}),0) - (0,d_{-1,1}(-\phi_{-1,1})).$$

Thus the classes of $(\delta_{1,-1}(\phi_{1,-1}),0)$ and $(0,d_{-1,1}(-\phi_{-1,1}))$ inside $\mathrm{coker}(\Delta^0)$ coincide, which proves $\alpha(\phi_{1,-1} + \mathrm{im}(d_{0,-1})) = \beta(-\phi_{-1,1} + \mathrm{im}(\delta_{-1,0}))$. We hence conclude that $\mathrm{im}\,\alpha \subseteq \mathrm{im}\,\beta$. By a completely symmetrical argument we also obtain the reverse inclusion, and thus we are done.

□