

Exercise 1. Show that the following holds for an R -module M of finite length $l(M)$ (i.e., an R -module M that admits a composition series of finite length).

- (1) If there is a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R -modules, then $l(M) = l(M') + l(M'')$.

- (2) If $N <_R M$ is a proper submodule then $l(N) < l(M)$.
 (3) Use (2) to show that any strict chain of submodules in M (not necessary a maximal chain, i.e. not necessarily a composition series) has length smaller than or equal to $l(M)$. Conclude that a module M is of finite length if and only if M is both Noetherian and Artinian.

Proof. (1) The solution has two steps: first we prove that both M' and M'' have finite length, and then we prove the formula.

For the first step, let $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_t = M$ be a composition series of M (in particular $t = l(M)$). Up to isomorphism (of short exact sequences), we can view M' as an actual submodule of M and $M'' = M/M'$ as the actual quotient of M by M' . Now for $0 \leq i \leq t$, define $M'_i = M' \cap M_i$ and $M''_i = (M_i + M')/M'$; we would like to understand the quotients of consecutive terms.

On the one hand, we have a natural map $M'_{i+1} \hookrightarrow M_{i+1} \twoheadrightarrow M_{i+1}/M_i$, and the kernel of this composition is exactly M'_i . Hence we obtain an induced inclusion $M'_{i+1}/M'_i \hookrightarrow M_{i+1}/M_i$. As the latter is simple, we obtain that M'_{i+1}/M'_i is either trivial or simple.

On the other hand, we have by the third isomorphism theorem that $M''_{i+1}/M''_i \cong (M_{i+1} + M')/(M_i + M')$. Then, we have a natural map

$M_{i+1} \hookrightarrow M_{i+1} + M \twoheadrightarrow (M_{i+1} + M)/(M_i + M)$, and the composed arrow is easily seen to be surjective. Also, M'_i is included in the kernel of the composition, so we obtain an induced surjective map $M_{i+1}/M_i \twoheadrightarrow (M_{i+1} + M)/(M_i + M) \cong M''_{i+1}/M''_i$. As M_{i+1}/M_i is simple, we obtain that M''_{i+1}/M''_i is either trivial or simple.

In conclusion, the quotients of consecutive terms both in $M'_0 \subseteq \dots \subseteq M'_t$ and $M''_0 \subseteq \dots \subseteq M''_t$ are all either simple or trivial. So by deleting some of the modules in the sequence, we will obtain composition series both for M' and M'' . Hence M' and M'' have finite length (and length smaller than or equal to t).

Now for the second step, by the one-to-one correspondence of submodules of M' and submodules of M containing M' it is clear that a composition series for M' can be extended to a composition series for M by adding the preimage of a composition series of M'' . This gives a composition series for M of length $l(M') + l(M'')$. Therefore, since by the Jordan Holder Theorem $l(M)$ is the length of *any* composition series, we obtain $l(M') + l(M'') = l(M)$.

- (2) Follows directly from the argument above.
- (3) Let $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ be a strict chain of length n . Then by (2) we have $l(M) > l(M_{n-1}) > \cdots > l(M_0) = 0$, hence $l(M) \geq n$. Since every chain of M is of finite length bounded by $l(M)$, M is both Noetherian and Artinian. The implication in the other direction was discussed in Remark 3.2.4 of the lecture notes. \square

Exercise 2. Let R be a ring and let M be a finitely generated module over R . Let $f : M \rightarrow M$ be an R -module homomorphism.

- (1) Suppose that R is a Noetherian ring.
- Does injectivity of f implies surjectivity?
 - Does surjectivity of f implies injectivity?
- (2) Suppose that M is a module of finite length, show that f is injective if and only if f is surjective.

Proof. (1) (i) Let R be a ring with $a \in R$ neither a unit nor a zero divisor, then multiplication by a is an injective but not surjective morphism $m_a : R \rightarrow R$.

(ii) Suppose that M is a finitely generated module over a Noetherian ring, then M is Noetherian. Let $f : M \rightarrow M$ be a surjective morphism. For all k we have containments $\ker(f^k) \subset \ker(f^{k+1})$. Therefore, there exists a positive integer m such that $\ker(f^{m+1}) = \ker(f^m)$. In particular, $f : \text{im}(f^m) \rightarrow M$ is injective, but by surjectivity $\text{im}(f^m) = M$, therefore f is injective.

Remark 0.1. Amazingly, the statement remains true even if R is not Noetherian. Let us prove it now. Let e_i for $1 \leq i \leq n$ be generators of M as an R -module. Let $f(e_i) = \sum_{j=1}^n a_{ij}e_j$ for all i . By surjectivity there exists b_{jk} such that $e_j = \sum_{k=1}^n b_{jk}f(e_k)$ for all j . Suppose that $m \in \ker(f)$ with $m = \sum_i m_i e_i$. Let $\mathbb{Z}[a_{ij}, b_{ij}, m_k] \rightarrow R$ be the natural inclusion morphism, where $\mathbb{Z}[a_{ij}, b_{ij}, m_k]$ is the \mathbb{Z} -subalgebra of R generated by the a_{ij} 's, b_{ij} 's and m_k 's. There is therefore an induced structure of $R^l = \mathbb{Z}[a_{ij}, b_{ij}, m_k]$ -module on M . Let M^l be the R^l -submodule generated by e_i for $1 \leq i \leq n$. By definition of M^l the morphism f induces a morphism $f^l : M^l \rightarrow M^l$, it is surjective since $e_i = f(\sum_k b_{ik}e_k)$. As now R^l is Noetherian (it is a finitely generated \mathbb{Z} -algebra), we obtain by the previous point that the element $m \in \ker(f^l)$ is zero. As $m \in \ker(f)$ was arbitrary, we conclude that f is injective.

Later, we will see a very important statement called *Nakayama's lemma*. This will provide a very easy proof of this fact over any (commutative) ring, without relying on Noetherian approximation as above.

- (2) Consider the short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow M \longrightarrow \text{im}(f) \longrightarrow 0 .$$

By Exercise 1.1, we have $l(M) = l(\ker(f)) + l(\text{im}(f))$. Since the zero module is the only module of length zero, f being surjective implies that $\ker(f) = 0$. Conversely, if f is injective, then $l(M) = l(\text{im}(f))$, hence $l(\text{im}(f))$ can not be a proper submodule of M by the same exercise, i.e. $M = \text{im}(f)$. \square

Exercise 3. (1) Let R be a PID, and let $f \in R$ be a product of $n \geq 0$ prime elements. Prove that the length of $R/(f)$ as an R -module is equal to n .

(2) Let $f \in \mathbb{R}[x]$ be a nonzero polynomial with exactly $n \geq 0$ non-real roots (counted with multiplicity). Prove that

$$\dim_{\mathbb{R}}(\mathbb{R}[x]/(f)) - \text{length}_{\mathbb{R}[x]}(\mathbb{R}[x]/(f)) = n/2$$

(3) Let M be a \mathbb{Z} -module. Prove that M has finite length if and only if it is finite (as a set).

(4) Give an example of a ring and a module over this ring which has finite length but infinitely many submodules.

Proof. (1) We prove the assertion by induction on n ; for $n = 0$ it clearly holds, and for $n = 1$ it holds since primes in a PID are maximal, and thus the quotient of R by a prime is simple.

So assume that we have shown the assertion for some $n \geq 1$, and let f be a product of $n + 1$ primes. Let p be a prime dividing f and write $f = pg$ where g is a product of n primes. Then we have a natural surjection $R/(f) \twoheadrightarrow R/(g)$ of R -modules, and let K be the kernel. It is straightforward to see that $K = R \cdot (g + (f))$. Now we have a short exact sequence

$$0 \rightarrow \text{Ann}_R(g + (f)) \rightarrow R \xrightarrow{\cdot(g+(f))} K \rightarrow 0.$$

Finally, one can easily verify that $\text{Ann}_R(g + (f)) = (p)$, and thus $K \cong R/(p)$. As we then have a short exact sequence

$$0 \rightarrow R/(p) \rightarrow R/(f) \rightarrow R/(g) \rightarrow 0,$$

it follows from Exercise 1.1 and the induction hypothesis that $R/(f)$ has length $n + 1$.

(2) The dimension of $\mathbb{R}[x]/(f)$ as an \mathbb{R} -vector space is $d = \deg f$. Furthermore, as $\mathbb{R} \subseteq \mathbb{C}$ is a field extension of degree 2, the irreducible polynomials of $\mathbb{R}[x]$ are the linear polynomials and the quadratic polynomials having no real roots. Therefore, if m is the number of real roots of f counted with multiplicity, one can see that f is the product of exactly $m + n/2$ irreducible polynomials. Hence by the previous exercise we obtain that the length of $\mathbb{R}[x]/(f)$ is equal to $m + n/2$. As $d = m + n$, we obtain

$$\dim_{\mathbb{R}}(\mathbb{R}[x]/(f)) - \text{length}_{\mathbb{R}[x]}(\mathbb{R}[x]/(f)) = m + n - (m + n/2) = n/2.$$

(3) If M is finite as a set then M has finite length as there are only finitely many submodules. Conversely, if M has finite length, then by Exercise 1 it is in particular Noetherian, so finitely generated. By the classification of finitely generated \mathbb{Z} -modules, we have an isomorphism $M \cong \mathbb{Z}^{\oplus r} \oplus F$ for some finite \mathbb{Z} -module F and $r \geq 0$. If by contradiction $r \geq 1$, then M contains a copy of \mathbb{Z} as a submodule, so again by Exercise 1 we obtain that \mathbb{Z} has finite length. This is not true, e.g. as \mathbb{Z} is not Artinian. Hence $r = 0$ and $M \cong F$ is finite.

(4) It suffices to take an infinite field k and a finite dimensional k -vector space V of dimension greater than or equal to 2. It is clearly of finite length, and if $v_1, v_2 \in V$ are linearly independent, then $\{k \cdot (v_1 + \lambda v_2)\}_{\lambda \in k}$ is an infinite family of distinct subspaces. \square

Exercise 4. (1) Let $n, m > 0$ be integers, let k be a field and let $R := k[x, y]$. Show that the R -module

$$M := k[x, y] / (x^n, y^m)$$

has length nm .

Hint: Exercise 1 can be useful to decompose this computation into easier ones, allowing some induction argument. The same applies for the next point.

(2) Let $p > 0$ be a prime number. Compute the length of

$$\mathbb{Z}[x] / (p^2, x^2 - p),$$

as a module over the ring $\mathbb{Z}[x]$.

Proof. (1) First let us show the following: for any $d \geq 0$, the module

$$N_d := k[x, y] / (x, y^d)$$

has length d . Set

$$S := k[x, y] / (x)$$

and $\pi : R \rightarrow S$ the quotient map. By Exercise 2.3 on sheet 1, we can define an S -module structure on N_d such that for all $r \in R$ and $n \in N_d$, $r \cdot n = \pi(r) \cdot n$.

With this in mind, it is immediate that S -submodules of N_d are the same as R -submodules of N_d , so in particular its length is unchanged.

Now, $S \cong k[y]$ by setting $x = 0$, and through this isomorphism we see that N_d corresponds to

$$k[y] / (y^d)$$

so we know by Exercise 3.1 that its length is d .

Now, let us compute the length of

$$N_{n,m} := k[x, y] / (x^n, y^m)$$

is nm . If $n = 1$, this was already worked out before, so assume $n \geq 2$. Consider the morphism $\phi : k[x, y] \rightarrow N_{n,m}$ given by sending 1 to $x^{n-1} + (x^n, y^m)$. Note that the sequence

$$k[x, y] \xrightarrow{\phi} N_{n,m} \rightarrow N_{n-1,m} \rightarrow 0$$

is exact where $N_{n,m} \rightarrow N_{n-1,m}$ is the usual quotient map, so we obtain a short exact sequence

$$0 \rightarrow k[x, y] / \ker(\phi) \rightarrow N_{n,m} \rightarrow N_{n-1,m} \rightarrow 0$$

Let us understand $\ker(\phi)$. Clearly, $(x, y^m) \subseteq \ker(\phi)$, and given $a \in \ker(\phi)$, we get that by definition there exists $b, c \in k[x, y]$ such that

$$x^{n-1}a = x^n b + y^m c$$

In particular x^{n-1} divides $y^m c$, so since x and y are coprime ($k[x, y]$ is a UFD) we get that x^{n-1} divides c (write $c = x^{n-1}c'$). Thus,

$$a = xb + y^m c'$$

or in other words $a \in (x, y^m)$.

Hence we have proven that $\ker(\phi) = (x, y^m)$, so we finally have a short exact sequence

$$0 \rightarrow N_{1,m} \rightarrow N_{n,m} \rightarrow N_{n-1,m} \rightarrow 0$$

which by induction on n gives us

$$l(N_{n,m}) = l(N_{n-1,m}) + l(N_{1,m}) = (n-1)m + m = nm.$$

(2) Let $M := \mathbb{Z}[x]/(p^2, x^2 - p)$, and consider the quotient map

$$\pi: \mathbb{Z}[x]/(p^2, x^2 - p) \rightarrow \mathbb{Z}[x]/(p, x^2 - p).$$

Note that the latter module is isomorphic to

$$N := (\mathbb{Z}/p\mathbb{Z}[x])/(x^2),$$

and since the $\mathbb{Z}[x]$ -action this module factors through an action of $\mathbb{Z}/p\mathbb{Z}[x]$, let us compute the length of N as a $\mathbb{Z}/p\mathbb{Z}[x]$ -module. Since this ring is a PID, we deduce by Exercise 3.(1) that the length of N is 2.

Let us compute $\ker(\pi)$. By the third isomorphism theorem,

$$\ker(\pi) = (p, x^2 - p)/(p^2, (x^2 - p)),$$

so in particular it is generated by \bar{p} (i.e. the class of p in the quotient). Hence, we have a surjection

$$\theta: \mathbb{Z}[x] \rightarrow \ker(\pi),$$

sending 1 to \bar{p} .

Let us understand $\ker(\theta)$. It is immediate to see that $(p, x^2 - p) \subseteq \ker(\theta)$. On the other hand, if $f(x) \in \ker(\theta)$, then $pf(x) = p^2a(x) + (x^2 - p)b(x)$ for some $a(x), b(x) \in \mathbb{Z}[x]$. In particular, p divides $(x^2 - p)b(x)$, so since p is prime, p divides $b(x)$. Write $b(x) = pb'(x)$. Then

$$f(x) = pa(x) + (x^2 - p)b'(x) \in (p, x^2 - p).$$

Thus, we have proven that

$$(p, x^2 - p) = \ker(\theta).$$

In other words,

$$\ker(\pi) \cong \mathbb{Z}[x]/(p, x^2 - p) \cong N.$$

Equivalently, we have a short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow N \rightarrow 0$$

so by additivity of the length,

$$\text{length}(M) = 2\text{length}(N) = 4.$$

□

Exercise 5. Let R be a Noetherian ring. Are the following rings Noetherian? Are they Artinian?

- (1) $R[x, \frac{1}{x}] := \{\sum_{i=-m}^n a_i x^i : a_i \in R, m, n \in \mathbb{N}\}$.
- (2) $R[x_1, x_2, x_3, \dots]$.

- (3) $R[[x]]$, the ring of formal power series¹ with coefficients in R .
Hint: For an ideal I and each $n \in \mathbb{N}$, let $I_n := \{a_n : \exists \sum_{i=n}^{\infty} a_i x^i \in I\}$. Then adapt the proof of the Hilbert basis theorem.
- (4) $C^0(\mathbb{R})$, the ring of continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ with pointwise operations.
- (5) $\mathbb{R}[x]/((x-1)^2x)$.

Proof. (1) We will show that $R[x, \frac{1}{x}]$ is isomorphic to a quotient of a polynomial ring. It then follows that it is Noetherian by the Hilbert basis theorem (as Noetherianity is preserved under quotients).

The isomorphism in question comes from the R -algebra homomorphism

$$\begin{aligned} \phi : R[u, v] &\rightarrow R[x, \frac{1}{x}] \\ u &\mapsto x, \quad v \mapsto \frac{1}{x}, \end{aligned}$$

which exists by the universal property of $R[u, v]$. This is surjective as any element of $R[x, 1/x]$ can be written as some polynomial in x and $\frac{1}{x}$ by definition. Thus it has some kernel I , and hence $R[x, \frac{1}{x}] \cong R[u, v]/I$ is Noetherian.

As a side note, we can go further, and identify the kernel $\ker \phi = I$ to be the ideal $(uv - 1)$. For it is clear that $uv - 1 \in I$, and suppose that $g \in \ker \phi$. Then we can use elements of $(uv - 1)$ to cancel mixed terms, and so write $g = g_1 + g_2$ where $g_1 \in (uv - 1)$ and $g_2 = \sum_{i \geq 0} a_i u^i + \sum_{j > 0} b_j v^j$ for some $a_i, b_j \in R$. But it is clear that g_2 cannot be in $\ker \phi$ unless all of its coefficients are zero. So $g = g_1 \in (uv - 1)$.

Take $R \neq 0$ to be any Noetherian ring. There is an infinite descending chain of ideals in $R[x, x^{-1}]$ given by $(x + 1) \supsetneq ((x + 1)^2) \supsetneq ((x + 1)^3) \supsetneq \dots$. We need to prove that the containment is strict. To this end suppose that there exists a $k > 0$ such that $((x + 1)^k) = ((x + 1)^{k+1})$. Then there exists $f \in k[x, x^{-1}]$ such that $(x + 1)^k = f(x, x^{-1})(x + 1)^{k+1}$. Write $f(x, x^{-1}) = \sum_{m \leq i \leq n} a_i x^i$ with $m \leq n$ integers and $a_m, a_n \neq 0$. Then there is a term of degree $k + n + 1$ with coefficient $a_n \neq 0$ on the right-hand side, and thus $m \leq n < 0$ as the left-hand side has only terms of degree less than or equal to k . But then there is a non-zero term of degree $m < 0$ on the right-hand side corresponding to $a_m x^m$. This is not possible, since the left-hand side has no non-zero term with negative degree. We conclude that $f = 0$, but this amounts to a contradiction since $(x + 1)^k \neq 0$ since it has non-zero coefficients corresponding to the terms x^k and 1. Hence $R[x, \frac{1}{x}]$ isn't Artinian.

- (2) $R[x_1, x_2, \dots]$ is not Noetherian, as the ideal (x_1, x_2, \dots) cannot be finitely generated. It is not Artinian (for any choice of $R \neq 0$), since it contains the strictly descending chain $(x_1) \supsetneq (x_1^2) \supsetneq (x_1^3) \supsetneq \dots$.
- (3) $R[[x]]$ is not Artinian (for any choice of $R \neq 0$), since it contains the strictly descending chain $(x) \supsetneq (x^2) \supsetneq (x^3) \supsetneq \dots$.

$R[[x]]$ is Noetherian, and the proof is a variant of the proof of the Hilbert basis theorem.

¹ $R[[x]] = \{\sum_{i=0}^{\infty} a_i x^i : a_i \in R\}$, where multiplication and addition are defined formally, as what you think they should be. These are purely formal objects: there is no requirement for any kind of convergence.

To this end suppose I is an ideal of $R[[x]]$. For each integer $n \geq 0$, let

$$I_n := \{a_n : \exists \sum_{i=n}^{\infty} a_i x^i \in I\}.$$

For each n , this is an ideal of R , and by multiplying each power series by x we see that $I_n \subseteq I_{n+1}$ for each n . So by the ascending chain condition, there is M such that $I_n = I_{n+1}$ for all $n \geq M$.

Also, for each $i \leq M$, I_i is finitely generated, so we may fix a finite set $\{a_{i,j}\}_{0 \leq j \leq N}$ of generators for I_i (we take always the same number N of generators by repeating elements if needed). For each $0 \leq i \leq M$ and $0 \leq j \leq N$, fix $f_{i,j} \in I$ such that

$$f_{i,j} = a_{i,j}x^i + \text{higher order},$$

which exists by construction of I_i .

We claim that the ideal J generated by the set $\{f_{i,j}\}_{\substack{0 \leq i \leq M \\ 0 \leq j \leq N}}$ is equal to I . Let

$g = \sum_{k=0}^{\infty} b_k x^k \in I$. By construction of I_0 , we can find an element $h_0 \in J$ having the same term of order 0 as g : there exists an R -linear combination of $a_{0,0}, \dots, a_{0,N}$ equal to b_0 , and taking h_0 to be the same R -linear combination of $f_{0,0}, \dots, f_{0,N}$ will do. Similarly, we can find an element $h_1 \in J$ having the same term of order 1 as $g - h_0$. Iterating this procedure, we construct an element $h = h_0 + \dots + h_{M-1} \in J$ such that $g - h$ has no terms of degree strictly smaller than M .

Now we proceed similarly, but with a slight modification. As before, we can find coefficients $c_{0,0}, \dots, c_{0,N} \in R$ such that $l_0 = c_{0,0}f_{M,0} + \dots + c_{0,N}f_{M,N}$ has the same term of order M as $g - h$. Then, we can find $c_{1,0}, \dots, c_{1,N} \in R$ such that $l_1 = c_{1,0}xf_{M,0} + \dots + c_{1,N}xf_{M,N}$ (we added a factor x in there to make things of the correct order; in the next step we will need a factor x^2 and so on) has same term of order $M + 1$ as $g - h - l_0$. We iterate this procedure indefinitely, and for $0 \leq j \leq N$ define the power series $c_j = \sum_{k \geq 0} c_{k,j}x^k$, as well as $l = c_0f_{M,0} + \dots + c_Nf_{M,N} \in J$. One can then show by comparing coefficients that $g - h - l = 0$. As $h, l \in J$, we conclude $g \in J$, and as $g \in I$ was arbitrary, we obtain $I = J$. Hence I is finitely generated, and thus $R[[x]]$ is Noetherian.

- (4) $C^0(\mathbb{R})$ is neither Artinian nor Noetherian. To this end define $I_n = \{f \in C(\mathbb{R}) : f(x) = 0 \text{ for all } x \geq n\}$, where $n \in \mathbb{Z}$. It is clear that $I_n \subset I_{n+1}$. We need to show that the containment is strict. To this end, define for example the continuous function f by $f(x) = 0$ for all $x \geq n + 1$ and $f(x) = x - (n + 1)$ for all $x \leq n + 1$, this is a well-defined continuous function $f \in I_{n+1} \setminus I_n$. So $(I_n)_{n \in \mathbb{Z}}$ is a strictly increasing sequence of ideals indexed by \mathbb{Z} , showing that $C^0(\mathbb{R})$ is neither Artinian nor Noetherian.
- (5) The most efficient solution is the following: it suffices to notice that the dimension of $\mathbb{R}[x]/((x-1)^2x)$ as an \mathbb{R} -vector space is equal to 3 (the degree of the polynomial), so in particular it is finite. As ideals of $\mathbb{R}[x]/((x-1)^2x)$ are in particular \mathbb{R} -subspaces, and finite dimensional vector spaces obviously satisfy the ascending and descending chain conditions, we obtain that $\mathbb{R}[x]/((x-1)^2x)$ is both Artinian and Noetherian. \square

Exercise 6. Computing a presentation of an R module M means explicitly determining an exact sequence of the form $R^{\oplus t} \xrightarrow{\eta} R^{\oplus s} \xrightarrow{\varepsilon} M \longrightarrow 0$. Do the following computations.

- (1) Compute a presentation of the
- \mathbb{Z}
- module

$$M := \mathbb{Z}(2, 9) + \mathbb{Z}(4, 3) + \mathbb{Z}(6, 8) \subseteq \mathbb{Z} \oplus \mathbb{Z}.$$

- (2) Let
- $R = \text{Mat}_{2 \times 2}(\mathbb{Z})$
- be the ring of
- 2×2
- matrices over
- \mathbb{Z}
- . Compute a presentation of the left
- R
- module

$$M := R \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + R \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \subseteq R.$$

Proof. (1) We define a surjective morphism $\varepsilon : \mathbb{Z}^3 \rightarrow M$ by $e_1 \mapsto (2, 9)$, $e_2 \mapsto (4, 3)$, $e_3 \mapsto (6, 8)$. Then we calculate generators of the kernel:

(a_1, a_2, a_3) is mapped to zero if and only if the following two equations are satisfied:

$$\begin{aligned} 2a_1 + 4a_2 + 6a_3 &= 0 \\ 9a_1 + 3a_2 + 8a_3 &= 0 \end{aligned}$$

From the first equation we find $a_1 = -2a_2 - 3a_3$. Substituting for a_1 in the second equation gives us $15a_2 = -19a_3$. This implies that $a_2 = -19t$, $a_3 = 15t$ for $t \in \mathbb{Z}$. This gives that $a_1 = -2(-19t) - 3(15t) = -7t$. We conclude that a presentation is given by

$$\mathbb{Z} \xrightarrow{\eta} \mathbb{Z}^3 \xrightarrow{\varepsilon} M \rightarrow 0$$

where the first map is $\eta : t \mapsto (-7t, -19t, 15t)$

- (2) We define a surjective morphism
- $\varepsilon : R^2 \rightarrow M$
- by

$$e_1 \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, e_2 \mapsto \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$$

and we are interested in calculating generators of the kernel. I.e., we calculate the solution set of the matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2a + 2\beta & 3\alpha \\ 2c + 2\delta & 3\gamma \end{pmatrix} = 0$$

Hence the kernel consists of the elements $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right)$ such that $a = -\beta$, $c = -\delta$, $\alpha = \gamma = 0$. I.e., the elements of the form

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & -a \\ 0 & -c \end{pmatrix} \right).$$

Thus, the map $\eta : R \rightarrow R^2$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right)$$

gives a presentation $R^{\oplus 2} \xrightarrow{\eta} R^{\oplus 2} \xrightarrow{\varepsilon} M \rightarrow 0$ of M .

□