EPFL - Fall 2025	Zsolt Patakfalvi
Rings and modules	**Exercises**
Sheet 12 - Solutions	

Exercise 1. The goal of this exercise is to see that the statement of Exercise 8 is wrong without the algebraically closed assumption.

(1) Let $R \to S$ be a morphism of commutative rings (thus making S an R-algebra), and let I be an ideal of $R[x_1, \dots, x_n]$. Then we have an isomorphism of S-algebras

$$R[x_1, \dots, x_n] \big/ I \otimes_R S \cong S[x_1, \dots, x_n] \big/ (I)$$

[*Hint: First show it for $I = 0$, and then deduce the general case using right exactness of the tensor product. The case $I = 0$ can be handled by a direct computation, or by showing that both sides satisfy the same universal property.*]

(2) Show that

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$$

and hence it is not a domain (but it is nevertheless reduced!)

(3) Show that

$$\mathbb{F}_p(x) \otimes_{\mathbb{F}_p(x^p)} \mathbb{F}_p(x) \cong \mathbb{F}_p(x)[t] \big/ (t - x)^p$$

which is not even reduced.

Proof. (1) First let us deal with the case $I = 0$.

Hands-on approach: There is a bilinear map

$$R[x_1, \dots, x_n] \times S \to S[x_1, \dots, x_n]$$

given $(p, s) \mapsto sp$, so by definition this induced a morphism

$$R[x_1, \dots, x_n] \otimes_R S \to S[x_1, \dots, S_n]$$

and it is staightforward to see that this is an S-alebra morphism. Thus we are left to show that it is bijective. The point is that $R[x_1, \dots, x_n]$ is free as an R-module, with basis $\{x_1^{i_1} \cdots x_n^{i_n}\}_{i_1, \dots, i_n \geq 0}$. Therefore, as an S-module,

$$R[x_1, \dots, x_n] \otimes_R S$$

is also free with basis $\mathcal{B}_1 = \{x_1^{i_1} \cdots x_n^{i_n} \otimes 1\}_{i_1, \dots, i_n \geq 0}$ (we are using that $R \otimes_R S \cong S$ and that tensor products commute with direct sums). On the other hand, $S[x_1, \dots, x_n]$ is free with basis $\mathcal{B}_2 = \{x_1^{i_1} \cdots x_n^{i_n}\}_{i_1, \dots, i_n \geq 0}$, so since the maps $R[x_1, \dots, x_n] \otimes_R S \to S[x_1, \dots, x_n]$ described before maps bijectively $\mathcal{B}_1$ to $\mathcal{B}_2$, we win.

Categorical approach: We will freely use the categorical language here (i.e. categories, functors, adjoints, universal properties). Given A a ring, we denote by Alg_A the category of A-algebras. We have the obvious forgetful functor $\mathrm{Alg}_S \to \mathrm{Alg}_R$. Let us show that $- \otimes_R S$ defines a left adjoint.

Given $A \in \mathrm{Alg}_R, B \in \mathrm{Alg}_S$, we have to show that there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Alg}_S}(A \otimes_R S, B) \to \mathrm{Hom}_{\mathrm{Alg}_R}(A, B)$$

Given $f : A \to B$ a map of R-algebras, define

$$f' : A \otimes S \to R$$

by $f'(a \otimes s) = sf(a)$, and conversely given a map $f' : A \otimes_R S \to B$ of S-algebras, define $f : A \to B$ via $f(a) = f'(a \otimes 1)$. We leave the fact that this gives a well-defined bijection to the reader (note that we could replace the word "algebras" by "modules" and this would work exactly the same way).

Note that if A is any ring, and B is an A-algebra,

$$\mathrm{Hom}_{\mathrm{Alg}_A}(A[x_1, \dots, x_n], B) \cong \prod_{i=1}^{n} B$$

by definition of a polynomial ring (we can send the x_i's wherever we want, and this defines a ring map form the polynomial algebra).

From the above discussion, we obtain that if T is any S algebra, we have a natural bijection

$$\mathrm{Hom}_{\mathrm{Alg}_S}(R[x_1, \dots, x_n] \otimes_R S, T) \cong \mathrm{Hom}_{\mathrm{Alg}_R}(R[x_1, \dots, x_n], T) \cong \prod_{i=1}^{n} T \cong \mathrm{Hom}_{\mathrm{Alg}_S}(S[x_1, \dots, x_n], T)$$

so both $R[x_1, \dots, x_n] \otimes_R S$ and $S[x_1, \dots, x_n]$ share the same universal property in the category of S-algebras, so there is a natural isomorphism between these two objects. To find it explicitly, we simply have to see what

$$id \in \mathrm{Hom}_{\mathrm{Alg}_S}(S[x_1, \dots, x_n], S[x_1, \dots, x_n])$$

corresponds to in $\mathrm{Hom}_{\mathrm{Alg}_S}(R[x_1, \dots, x_n] \otimes_R S, S[x_1, \dots, x_n])$. Unraveling the definitions gives us that this morphism is exactly the one given with the previous strategy.

Now let us work out the general case (i.e. I is not necessarily 0). We have a short exact sequence

$$0 \to I \to R[x_1, \dots, x_n] \to R[x_1, \dots, x_n]/I \to 0$$

Tensoring by S gives the exact sequence

$$I \otimes_R S \to R[x_1, \dots, x_n] \otimes_R S \to R[x_1, \dots, x_n]/I \otimes_R S \to 0$$

Note that the composition

$$I \otimes_R S \to R[x_1, \dots, x_n] \otimes_R S \cong S[x_1, \dots, x_n]$$

simply sends $\sum_i p_i \otimes s_i$ to $\sum_i p_i s_i$, so by definition its image is (I), whence we deduce that

$$R[x_1, \dots, x_n]/I \otimes_R S \cong S[x_1, \dots, x_n]/(I)$$

It is striaghtforward to check that this map is not only an isomorphism of S-modules, but actually S-algebras.

(2) Since $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$, we see by the previous point that

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[x]/(x^2 + 1)$$

by the Chinese remainder theorem,

$$\mathbb{C}[x]/(x^2 + 1) = \mathbb{C}[x]/(x + i) \times \mathbb{C}[x]/(x - i) \cong \mathbb{C} \times \mathbb{C}$$

(3) Let us show the following result: let k be a field of characteristic $p > 0$ and $a \in k \setminus k^p$, and let $a^{1/p}$ be a p'th root living in some higher extension L of k. Then

$$k(a^{1/p}) \cong k[t]/(t^p - a)$$

Proof. The only thing to show is that $t^p - a$ is irreducible, so let us write by contradiction that $t^p - a = \alpha(t)\beta(t)$. Since in L, $t^p - a = (t - a^{1/p})^p$, we can write $\alpha(t) = (t - a^{1/p})^n$ and $\beta(t) = (t - a^{1/p})^m$ for some $m + n = p$. Therefore we get

$$k[t] \ni \alpha(t) = t^n - nt^{n-1}a^{1/p} + \text{lower order terms}$$

so since $a^{1/p} \notin k$, we must have $n = 0 \in k$, so since k has characteristic p either $n = 0 \in \mathbb{Z}$ or $n = p \in \mathbb{Z}$. In other words, either $\alpha(t)$ or $\beta(t)$ is a unit, hence we win. □

From the above, we deduce that

$$\mathbb{F}_p(x) \otimes_{\mathbb{F}_p(x^p)} \mathbb{F}_p(x) \cong \mathbb{F}_p(x^p)[t]/(t^p - x^p) \otimes_{\mathbb{F}_p(x^p)} \mathbb{F}_p(x) \cong \mathbb{F}_p(x)[t]/(t^p - x^p) = \mathbb{F}_p(x)[t]/(t-x)^p$$

□

Exercise 2. Let M be an A-module, and let $\mathfrak{a}$ be an ideal in A. Show that the following are equivalent:

(1) $M = 0$,
(2) $M_{\mathfrak{p}} = 0$, for every prime ideal $\mathfrak{p} \subseteq A$,
(3) $M_{\mathfrak{m}} = 0$, for every maximal ideal $\mathfrak{m} \subseteq A$.

Moreover, suppose that M is a finitely generated A-module, under this assumption prove that $M = \mathfrak{a}M$ if and only if $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ satisfying $\mathfrak{a} \subseteq \mathfrak{m}$.

Proof. The implications $(1) \implies (2) \implies (3)$ are obvious. Note also that by Exercise 7, the implication $(3) \implies (2)$ is also straightforward: if (3) holds and $\mathfrak{p}$ is any prime ideal, then let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. Set $T = R \setminus \mathfrak{m}$ and $S = R \setminus \mathfrak{p}$ so that $T \subseteq S$, and define $\tilde{S} \subseteq T^{-1}R$ as in Exercise 7. Then we have

$$M_{\mathfrak{p}} = S^{-1}M \cong \tilde{S}^{-1}(T^{-1}M) = \tilde{S}^{-1}M_{\mathfrak{m}} = 0,$$

as any localization of the zero module is the zero module. Thus (2) holds as well.
Now to prove $(3) \Rightarrow (1)$, assume by contradiction that $M \neq 0$ but that $M_{\mathfrak{m}} = 0$, for every maximal ideal $\mathfrak{m}$. Then there exists $x \in M \setminus \{0\}$, and in particular $\operatorname{Ann}(x) \neq A$. Consider the inclusion $Ax \hookrightarrow M$ and let $\mathfrak{m}$ be a maximal ideal of A containing $\operatorname{Ann}(x)$. As localisation is exact, localisation at $\mathfrak{m}$ preserves injectivity, so $(Ax)_{\mathfrak{m}} \hookrightarrow M_{\mathfrak{m}} = 0$ is still injective. Therefore $(Ax)_{\mathfrak{m}} = 0$, which implies in particular that $x/1$ is equal to 0 inside $(Ax)_{\mathfrak{m}}$. By definition, this means that there exists $t \in A \setminus m$ such that $tx = 0$, which contradicts $\operatorname{Ann}(x) \subseteq \mathfrak{m}$. Hence we must have $M = 0$, and thus we proved the equivalence of the three statements.

Now to the second part. We have $M = \mathfrak{a}M$ if and only if ${}^{M}/_{\mathfrak{a}M} = 0$, which by the above is equivalent to $({}^{M}/_{\mathfrak{a}M})_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ of A. By exactness of taking localization (see Exercise 6.3 of sheet 11), we have $({}^{M}/_{\mathfrak{a}M})_m \cong {}^{M_{\mathfrak{m}}}/_{(\mathfrak{a}M)_{\mathfrak{m}}}$, and notice that $(\mathfrak{a}M)_{\mathfrak{m}}$ can be naturally identified with the submodule $(\mathfrak{a}A_{\mathfrak{m}})M_{\mathfrak{m}}$ of $M_{\mathfrak{m}}$ (as the localization of the inclusion $\mathfrak{a}M \hookrightarrow M$ at $\mathfrak{m}$ has image $(\mathfrak{a}A_{\mathfrak{m}})M_{\mathfrak{m}}$).

Thus $(M/\mathfrak{a}M)_m$ is zero iff $M_\mathfrak{m} = (\mathfrak{a}A_\mathfrak{m})M_\mathfrak{m}$. If $\mathfrak{a}$ is not contained in the maximal ideal $\mathfrak{m}$ then $\mathfrak{a}$ contains a unit of $A_\mathfrak{m}$ and thus $M_\mathfrak{m} = (\mathfrak{a}A_\mathfrak{m})M_\mathfrak{m}$. Therefore, $M = \mathfrak{a}M$ if and only if $M_\mathfrak{m} = (\mathfrak{a}A_\mathfrak{m})M_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ satisfying $\mathfrak{a} \subseteq \mathfrak{m}$. Finally, observe that if $M_\mathfrak{m}$ then trivially $M_\mathfrak{m} = (\mathfrak{a}A_\mathfrak{m})M_\mathfrak{m}$. On the other hand, if $M_\mathfrak{m} = (\mathfrak{a}A_\mathfrak{m})M_\mathfrak{m}$, then as $\mathfrak{a} \subseteq \mathfrak{m}$ we also have $M_\mathfrak{m} = (\mathfrak{m}A_\mathfrak{m})M_\mathfrak{m}$. By applying Nakayama's Lemma (Exercise 4.2 on sheet 9) to the finitely generated $A_\mathfrak{m}$-module $M_\mathfrak{m}$ and the local ring $(A_\mathfrak{m}, \mathfrak{m}A_\mathfrak{m})$, this implies $M_\mathfrak{m} = 0$. So $M_\mathfrak{m} = (\mathfrak{a}A_\mathfrak{m})M_\mathfrak{m}$ for $\mathfrak{a} \subseteq \mathfrak{m}$ if and only if $M_\mathfrak{m} = 0$. By combining all of the above, we hence obtain that $M = \mathfrak{a}M$ if and only if $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$ with $\mathfrak{a} \subseteq \mathfrak{m}$. □

Exercise 3. Let $R = F[x]$, where F is a field.

(1) If F is algebraically closed, then show that for every prime ideal $\mathfrak{p}$ of R, either $R_\mathfrak{p} \cong F(x)$ or $R_\mathfrak{p} \cong F[x]_{(x)}$, where these isomorphisms are isomorphisms of F-algebras. Show that the above two cases are not isomorphic.

(2) If $F = \mathbb{R}$, then show that up to ring isomorphism there are three possibilities for $R_\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of $F[x]$.
[*Hint:* To tell the three cases apart, consider the residue field, to show that there are only three cases, apply linear transformations to x.]

(3) Show that if F is algebraically closed, then $F[x, y]$ has infinitely many prime ideals $\mathfrak{p}$ for which $F[x, y]_\mathfrak{p}$ are pairwise non-isomorphic F-algebras. For this, you can use the following theorem of algebraic geometry:

Theorem. *There exists a sequence of irreducible polynomials $(f_d)_{d\in\mathbb{N}\setminus\{0,2\}}$ in $F[x, y]$ such that f_d is of degree d and such that the fields* $\mathrm{Frac}\left(F[x, y]/(f_d)\right)$ *are pairwise non-isomorphic as F-algebras.*

Proof. Let us first prove a useful result which we will use throughout this solution.

Lemma 0.1. *Let R, S be two local rings with respective maximal ideals $\mathfrak{m}_R$ and $\mathfrak{m}_S$. If $R \cong S$, then we also have an isomorphism of residue fields $R/\mathfrak{m}_R \cong S/\mathfrak{m}_S$.*

Proof. Recall that given a local ring T, its maximal ideal is exactly the set of non-invertible elements of T, which is certainly a notion preserved by isomorphisms.

Thus, in our case, an isomorphism $\theta : R \to S$ must satisfy $\theta^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$, so it induces an isomorphism of residue fields. □

(1) Every non-zero prime ideal of $F[x]$ is principal of the form $(x - a)$ since F is algebraically closed. We have $F[x]_{(0)} = F(x)$, hence it is sufficient to prove that there is a F-algebra isomorphism $F[x]_{(x-a)} \cong F[x]_{(x-b)}$ for all $a, b \in F$. First, consider the F-algebra endomorphism $\phi_{a,b} : F[x] \to F[x]$ obtained by mapping x to $x+a-b$. Then the composition $F[x] \overset{\phi_{a,b}}{\to} F[x] \to F[x]_{(x-b)}$ maps every element not divisible by $x - a$ to a unit in $F[x]_{(x-b)}$, and thus induces an F-algebra map $\overline{\phi_{a,b}} : F[x]_{(x-a)} \to F[x]_{(x-b)}$, which sends $f(x)/g(x)$ to $f(x + a - b)/g(x + a - b)$. It is thus clear that $\overline{\phi_{a,b}}$ and $\overline{\phi_{b,a}}$ are mutually inverse, and hence $F[x]_{(x-a)} \cong F[x]_{(x-b)}$ for all $a, b \in F$. Finally, there is an inclusion $F[x]_{(x)} \to F(x)$, but the two rings aren't isomorphic as $x \in F[x]_{(x)}$ is a non-zero non-unit, but $F(x)$ is a field.

(2) There are three options for prime ideals in $\mathbb{R}[x]$ we have that $p = 0$ or p is principal generated by $(x - a)$ for $a \in \mathbb{R}$ or p is principal generated by a degree two polynomial with no real roots. With the same proof as in the previous point one has

$\mathbb{R}[x]_{(x-a)} \cong \mathbb{R}[x]_{(x-b)}$ for all $a, b \in \mathbb{R}[x]$. Now let $x^2 + bx + c$ be a monic quadratic polynomial without real roots (we can assume monicity without loss of generality). That is, we have $d^2 := c - b^2/4 > 0$. Then the linear change of coordinates where x is replaced by $dx + e$ where $e : -b/2$ transforms $x^2 + bx + c$ into $d^2(x^2 + 1)$. Another way of putting this, is that under the $\mathbb{R}$-agebra map $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ which sends x to $dx + e$, the polynomial $x^2 + bx + c$ is mapped to $d^2(x^2 + 1)$. Therefore, the composition $\mathbb{R}[x] \xrightarrow{\phi} \mathbb{R}[x] \to \mathbb{R}[x]_{d^2(x^2+1)}$ maps elements outside of (x^2+bx+c) to units, and thus we obtain an induced map of $\mathbb{R}$-algebras $\overline{\phi} : \mathbb{R}[x]_{(x^2+bx+c)} \to \mathbb{R}[x]_{(d^2(x^2+1))} = \mathbb{R}[x]_{(x^2+1)}$. By performing the inverse linear substitution (i.e. mapping x to $(x - e)/d$) one can construct an inverse to $\overline{\phi}$ with the same argument, and thus we obtain that $\mathbb{R}[x]_{(x^2+bx+c)} \cong \mathbb{R}[x]_{(x^2+1)}$ for all quadratic irreducible polynomials $x^2 + bx + c \in \mathbb{R}[x]$. So to conclude, we need to show that $\mathbb{R}(x)$, $\mathbb{R}[x]_{(x)}$ and $\mathbb{R}[x]_{(x^2+1)}$ are pairwise non-isomorphic. Notice that $x \in \mathbb{R}[x]_{(x)}$ and $x^2 + 1 \in \mathbb{R}[x]_{(x^2+1)}$ are non-zero non-units, and thus $\mathbb{R}(x)$ is not isomorphic to $\mathbb{R}[x]_{(x)}$ nor to $\mathbb{R}[x]_{(x^2+1)}$. Now the residue field of $\mathbb{R}[x]_{(x)}$, i.e. $\mathbb{R}[x]_{(x)} / x \cdot \mathbb{R}[x]_{(x)}$ is, by Exercise 6.4 on sheet 11, isomorphic to $\mathrm{Frac}\left(\mathbb{R}[x]/(x)\right) \cong \mathbb{R}$. By the same argument, the residue field of $\mathbb{R}[x]_{(x^2+1)}$ is isomorphic to $\mathrm{Frac}\left(\mathbb{R}[x]/(x^2+1)\right) \cong \mathrm{Frac}(\mathbb{C}) = \mathbb{C}$. As $\mathbb{R} \ncong \mathbb{C}$ we conclude that $\mathbb{R}[x]_{(x)}$ and $\mathbb{R}[x]_{(x^2+1)}$ are non-isomorphic.

(3) Let $(f_d)_d$ be as in the theorem; we will show that $(F[x,y]_{(f_d)})_d$ are pairwise non-isomorphic for $d \in \mathbb{N} \setminus \{0, 2\}$. Suppose that there is an isomorphism $\phi : F[x,y]_{(f_d)} \to F[x,y]_{(f_{d'})}$ for some d, d'. Then the residue fields must be isomorphic too. However recall that in general, given a ring R and a prime ideal $\mathfrak{p}$, the maximal ideal of $R_\mathfrak{p}$ is $\mathfrak{p}R_\mathfrak{p}$ and the residue field is isomorphic to $\mathrm{Frac}(R/\mathfrak{p})$.

Using this fact in our case contradicts the choices of f_d and $f_{d'}$.

□

Exercise 4. Let F be an algebraically closed field.

(1) List the prime ideals of $R = F[x,y]/(xy)$.

[*Hint:* Consider the implications of a containment $xy \in \mathfrak{p}$, for a prime ideal $\mathfrak{p}$. Consider the projections $R \to R/(x)$ and $R \to R/(y)$ and use that you know the prime ideals of $F[y]$ and $F[x]$.]

(2) Show that for all prime ideals $\mathfrak{p}$ of R, $R_\mathfrak{p}$ falls into three cases up to F-algebra isomorphism, one which is a field, one which is a domain but not a field and one which is not a domain.

Proof. (1) The prime ideals of $R = F[x,y]/(xy)$ corresponds to prime ideals inside $F[x,y]$ containing xy. If $xy \in \mathfrak{p}$ for $\mathfrak{p}$ prime, then either $(x) \subseteq \mathfrak{p}$ or $(y) \subseteq \mathfrak{p}$. Suppose $(x) \subseteq \mathfrak{p}$, then the image $\mathfrak{q}$ of $\mathfrak{p}$ under the projection $F[x,y] \twoheadrightarrow F[x,y]/(x) \cong F[y]$ is prime (where the last isomorphism is given by setting x to 0). As F is algebraically closed, $\mathfrak{q}$ must be either (0), or of the form $\mathfrak{q} = (y - b)$ for some $b \in F$. As $\mathfrak{p}$ is the preimage of $\mathfrak{q}$, we obtain that $\mathfrak{p}$ is either equal to (x), or equal to $(x, y - b)$, and it is straightforward to see that any such ideal is prime. By doing the same argument where the roles of x and y are swapped, we hence conclude that prime ideals of $F[x,y]$ containing xy are precisely (x), (y), $(x-a,y)$ for $a \in F$ and $(x,y-b)$ for $b \in F$. Hence the prime ideals of R are precisely $(\overline{x})$, $(\overline{y})$, $(\overline{x-a},\overline{y})$ for $a \in F$ and $(\overline{x},\overline{y-b})$ for $b \in F$, where we use $\overline{\bullet}$ to denote the class of an element.

(2) For this exercise, it is useful to know (and prove) the following lemma.

Lemma 1. *Let R be a ring with multiplicative subset T and ideal I. Let $S = R/I$ and let $\overline{T}$ be the image of T under $R \twoheadrightarrow S$. Then there is a natural ring isomorphism $\overline{T}^{-1}S \cong T^{-1}R/I\cdot T^{-1}R$.*

Proof. Consider the composition $R \to T^{-1}R \to T^{-1}R/I\cdot T^{-1}R$. As every element of I is mapped to 0, this induces a map $S \to T^{-1}R/I\cdot T^{-1}R$ which sends $r+I$ to $\frac{r}{1} + I\cdot T^{-1}R$. In particular, let $\bar{t} \in \overline{T}$ be arbitrary, and write $\bar{t} = t + I$ for a $t \in T$. Then $\bar{t}$ is mapped to $\frac{t}{1} + I\cdot T^{-1}R$, which has inverse $\frac{1}{t} + I\cdot T^{-1}R$. Hence every element of $\overline{T}$ is mapped to a unit, and thus we obtain a ring map $\overline{T}^{-1}S \to T^{-1}R/I\cdot T^{-1}R$, given by sending $\frac{r+I}{t+I}$ (with $t \in T$) to $\frac{r}{t} + I\cdot T^{-1}R$.

On the other hand, consider the composition $R \to S \to \overline{T}^{-1}S$. Then an element $t \in T$ is mapped to $(t+I)/1$, which is a unit since $t+I \in \overline{T}$. Hence we obtain an induced map $T^{-1}R \to \overline{T}^{-1}S$ sending $\frac{r}{t}$ to $\frac{r+I}{t+I}$. Notice that every element of the form $r/1$ with $r \in I$ is mapped to 0 by this mapped, and thus the ideal generated by elements of this form, i.e. $I\cdot T^{-1}R$, is in the kernel. Hence we obtain a map $T^{-1}R/I\cdot T^{-1}R \to \overline{T}^{-1}S$ which maps $\frac{r}{t} + I\cdot T^{-1}R$ to $\frac{r+I}{t+I}$. It is then easy to see that this is inverse to the morphism constructed in the previous paragraph. □

Now to the exercise. By the above Lemma, we have

$$\left(F[x,y]/(xy)\right)_{(\overline{x})} \cong (F[x,y]\setminus(x))^{-1}F[x,y]/(xy)\cdot(F[x,y]\setminus(x))^{-1}F[x,y] =$$

$$= F[x,y]_{(x)}/x\cdot F[x,y]_{(x)} \cong \operatorname{Frac}\left(F[x,y]/(x)\right) \cong F(y)$$

where in the second to last isomorphism we Exercise 6.4 on sheet 11. By swapping the roles of x and y, one obtains $R_{(\overline{y})} \cong F(x)$.

Now let $b \in F\setminus\{0\}$, then

$$\left(F[x,y]/(xy)\right)_{(\overline{x},\overline{y-b})} \cong (F[x,y]\setminus(x,y-b))^{-1}F[x,y]/(xy)\cdot(F[x,y]\setminus(x,y-b))^{-1}F[x,y] =$$

$$= F[x,y]_{(x,y-b)}/x\cdot F[x,y]_{(x,y-b)} \cong F[y]_{(y-b)}$$

where the last isomorphism is induced by sending x to 0 (or identifying $F[y] = F[x,y]/(x)$ and using the Lemma). Again by swapping the roles of x and y we obtain $\left(F[x,y]/(xy)\right)_{(\overline{x-a},\overline{y})} \cong F[x]_{(x-a)}$ for all $a \in F\setminus\{0\}$. These are all isomorphic by the proof of point Exercise 3.1, and are a domain which isn't a field.

Finally, $(F[x,y]/(xy))_{(\overline{x},\overline{y})}$ is not a domain, since neither $\overline{x}/1$ nor $\overline{y}/1$ are zero, but their product is 0.

To sum up, up to a linear coordinate change we have $R_{\mathfrak{p}} \cong F(y)$ a field, $R_{\mathfrak{p}} \cong F[y]_{(y)}$ which is a domain but not a field or $R_{\mathfrak{p}} = (F[x,y]/(xy))_{(\overline{x},\overline{y})}$ which is not a domain.

□

Exercise 5. Let R be a ring.

(1) Let $T \subseteq R$ a multiplicatively closed subset of R. Let $\mathfrak{q}$ be a prime ideal of $T^{-1}R$. Let $\mathfrak{q}^c$ be the contraction of q under $R \to T^{-1}R$. Prove that $\operatorname{ht}(\mathfrak{q}) = \operatorname{ht}(\mathfrak{q}^c)$.
(2) Let $\mathfrak{p}$ be a prime ideal of R. Prove that $\operatorname{ht}(\mathfrak{p}) = \dim R_{\mathfrak{p}}$.

Proof. The proof consists of the following steps based on the observation that both heights and dimensions are defined in terms of chains of ideals.

(1) Prime ideals of $T^{-1}R$ are in one-to-one correspondence with prime ideals of R that do not intersect T. A strictly increasing chain of prime ideals ending in $\mathfrak{q}$ induces a strictly increasing chain of prime ideals ending in $\mathfrak{q}^c$ by contraction. Conversely, if $\mathfrak{p} \subseteq \mathfrak{q}^c$ is prime, then in particular it must avoid T (as otherwise $\mathfrak{q}$ would contain a unit), and thus in a strictly increasing chain of prime ideals ending in $\mathfrak{q}^c$ induces a strictly increasing chain of prime ideals ending in $\mathfrak{p}$ by extension.
(2) Prime ideals of $R_{\mathfrak{p}}$ are in an inclusion preserving one-to-one correspondence with prime ideals of R avoiding $R \setminus \mathfrak{p}$, i.e. contained in $\mathfrak{p}$.

□

Exercise 6. Let $S \to R$ be a morphism of rings. Show that a prime ideal $\mathfrak{p}$ of S is the contraction of a prime ideal of R if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.
[*Hint:* For one direction use ideas from the proof of Going-Up Theorem (Proposition 9.4.2 of the lecture notes).]

Proof. Recall that if $\mathfrak{p}$ is an ideal of S and $\mathfrak{q}$ is an ideal of R then there are always containments $\mathfrak{q}^{ce} \subseteq \mathfrak{q}$ and $\mathfrak{p}^{ec} \supseteq \mathfrak{p}$. If there exists a prime ideal $\mathfrak{q}$ of R such that $\mathfrak{p} = \mathfrak{q}^c$, then $\mathfrak{p}^e = \mathfrak{q}^{ce} \subseteq \mathfrak{q}$ and therefore $\mathfrak{p}^{ec} \subseteq \mathfrak{q}^c = \mathfrak{p}$. Since the inclusion $\mathfrak{p} \subseteq \mathfrak{p}^{ec}$ holds always this shows that $\mathfrak{p}^{ec} = \mathfrak{p}$.
Conversely, denote $R_{\mathfrak{p}} := (\phi(S \setminus \mathfrak{p}))^{-1}R$ (this is a common notation so remember it) where $\phi : S \to R$ is the ring morphism from the statement. If $\mathfrak{p}^{ec} = \mathfrak{p}$ holds, then the ideal $\mathfrak{p}^e$ doesn't meet the image of $S \setminus \mathfrak{p}$ in R. Thus $\mathfrak{p}^e R_{\mathfrak{p}}$ is a proper ideal of $R_{\mathfrak{p}}$. Let $\mathfrak{m}$ be a maximal ideal of $R_{\mathfrak{p}}$ that contains $\mathfrak{p}^e R_{\mathfrak{p}}$. Let $\mathfrak{q} \subseteq R$ be the contraction of $\mathfrak{m}$ along $R \to R_{\mathfrak{p}}$. Then $\mathfrak{q}$ is a prime ideal of R that doesn't intersect the image of $S \setminus \mathfrak{p}$ in R, and $\mathfrak{p}^e \subseteq \mathfrak{q}$. Hence, $\mathfrak{p} = \mathfrak{p}^{ec} \subseteq \mathfrak{q}^c$, and $\mathfrak{q}^c \subseteq \mathfrak{p}$ as $\mathfrak{q}^c \cap (S \setminus \mathfrak{p}) = \varnothing$. □

Exercise 7. Let R be a ring, let M be an R-module and let $T, S \subseteq R$ be two multiplicatively closed subsets of R. Define $ST := \{st \mid s \in S, t \in T\}$ and $\tilde{S} := \{s/1 \mid s \in S\} \subseteq T^{-1}R$.

(1) Show that ST and $\tilde{S}$ are multiplicatively closed subsets of R resp. $T^{-1}R$.
(2) Show that there exists a ring morphism $\tilde{S}^{-1}(T^{-1}R) \to (ST)^{-1}R$ sending $(r/t)/(s/1) \in \tilde{S}^{-1}(T^{-1}R)$ to $r/(st) \in (ST)^{-1}R$. Show further that this is an isomorphism.
(3) Show that $\tilde{S}^{-1}(T^{-1}M)$ and $(ST)^{-1}M$ are isomorphic as $(ST)^{-1}R$-modules, where the $(ST)^{-1}R$-module structure of $\tilde{S}^{-1}(T^{-1}M)$ is provided via the isomorphism of the previous point.
(4) Show that if $T \subseteq S$ then $ST = S$, and formulate the results of points (2) and (3) in this case.

Proof. (1) Note that $1 \in S \cap T$ and thus $1 = 1 \cdot 1 \in ST$. Furthermore, if $s, s' \in S$ and $t, t' \in T$ then $(st)(s't') = (ss')(tt') \in ST$ as $ss' \in S$ and $tt' \in T$. Hence ST is multiplicatively closed. As for $\tilde{S}$, note that if $\phi : R \to R'$ is any ring morphism, then

$\phi(S) \subseteq R'$ is multiplicatively closed as $\phi(1) = 1$ and ϕ preserves multiplication. So as $\tilde{S}$ is the image of S under the localisation morphism $R \to T^{-1}R$, we conclude that it is a multiplicatively closed subset of $T^{-1}R$.

(2) Denote by $\iota_T : R \to T^{-1}R$, $\iota_{ST} : R \to (ST)^{-1}R$ and $\iota_{\tilde{S}} : T^{-1}R \to \tilde{S}^{-1}(T^{-1}R)$ the localization morphisms. As $T \subseteq ST$, the morphism ι_{ST} sends every element of T to a unit. Hence by the universal property of localization, there exists a ring morphism $\iota_{T,ST} : T^{-1}R \to (ST)^{-1}R$ such that $\iota_{T,ST} \circ \iota_T = \iota_{ST}$. This implies that any $\frac{r}{t} \in T^{-1}R$ is mapped to $\frac{r}{t} \in (ST)^{-1}R$. Now let $s/1 \in \tilde{S}$ be arbitrary. Then $\iota_{T,ST}$ sends $s/1$ to $s/1 \in (ST)^{-1}R$, which is a unit (with inverse $1/s$). Hence by the universal property of localization, there exists a ring morphism $\phi : \tilde{S}^{-1}(T^{-1}R) \to (ST)^{-1}R$ such that $\phi \circ \iota_{\tilde{S}} = \iota_{T,ST}$. This implies that ϕ sends any $(r/t)/(s/1) \in \tilde{S}^{-1}(T^{-1}R)$ to $\iota_{S,ST}(r/t)(\iota_{S,ST}(s/1))^{-1} = r/(ts) \in (ST)^{-1}R$, so this is the morphism we sought to construct.

To prove that ϕ is an isomorphism, we construct an inverse. Note that $\iota_{\tilde{S}} \circ \iota_T : R \to \tilde{S}^{-1}(T^{-1}R)$ sends any $st \in ST$ to $(st/1)/(1/1)$, which has inverse $(1/t)/(s/1) \in \tilde{S}^{-1}(T^{-1}R)$. Indeed, we have

$$\left(\left(\frac{st}{1}\right)/\left(\frac{1}{1}\right)\right) \cdot \left(\left(\frac{1}{t}\right)/\left(\frac{s}{1}\right)\right) = \left(\left(\frac{s}{1}\right)/\left(\frac{s}{1}\right)\right) = 1_{\tilde{S}^{-1}(T^{-1}R)}.$$

Hence by the universal property of localization, there exists a ring morphism $\psi : (ST)^{-1}R \to \tilde{S}^{-1}(T^{-1}R)$ such that $\psi \circ \iota_{ST} = \iota_{\tilde{S}} \circ \iota_T$. This implies that any $r/(st) \in \tilde{S}^{-1}R$ is mapped to

$$\psi(r/(st)) = (\iota_{\tilde{S}} \circ \iota_T(r)) \cdot (\iota_{\tilde{S}} \circ \iota_T(st))^{-1} = \left(\left(\frac{r}{1}\right)/\left(\frac{1}{1}\right)\right) \cdot \left(\left(\frac{1}{t}\right)/\left(\frac{s}{1}\right)\right) = \left(\left(\frac{r}{t}\right)/\left(\frac{s}{1}\right)\right).$$

Hence ϕ and ψ are mutually inverse, and thus isomorphisms.

(3) The structure of $\tilde{S}^{-1}(T^{-1}M)$ as an $(ST)^{-1}R$-module is given by the formula

$$\frac{r}{st} \cdot \left(\left(\frac{m}{t'}\right)/\left(\frac{s'}{1}\right)\right) := \psi\left(\frac{r}{st}\right)\left(\left(\frac{m}{t'}\right)/\left(\frac{s'}{1}\right)\right) = \left(\left(\frac{rm}{tt'}\right)/\left(\frac{ss'}{1}\right)\right).$$

Tensor approach: Note that by Exercise 5 of Sheet 11, we have

$$(ST)^{-1}M \cong (ST)^{-1}R \otimes_R M$$

and

$$\tilde{S}^{-1}(T^{-1}M) \cong \tilde{S}^{-1}(T^{-1}R) \otimes_{T^{-1}R} (T^{-1}R \otimes_R M).$$

Note that we have

$$\begin{aligned}\tilde{S}^{-1}(T^{-1}R) \otimes_{T^{-1}R} (T^{-1}R \otimes_R M) &\cong (\tilde{S}^{-1}(T^{-1}R) \otimes_{T^{-1}R} T^{-1}R) \otimes_R M \cong \\ &\cong \tilde{S}^{-1}(T^{-1}R) \otimes_R M \cong (ST)^{-1}R \otimes_R M,\end{aligned}$$

at the very least as R-modules. By following the chain of isomorphisms, the above isomorphism is given on simple tensors by mapping $(r/t)/(s/1) \otimes (r'/t' \otimes m)$ to $((rr')/(tt's)) \otimes m$. It is then straightforward to check that this map is in fact $(ST)^{-1}R$-linear, and thus an isomorphism of $(ST)^{-1}R$-modules.

Pure localization approach: Denote by $\iota_T^M : M \to T^{-1}M$, $\iota_{ST}^M : M \to (ST)^{-1}M$ and $\iota_{\tilde{S}}^M : T^{-1}M \to \tilde{S}^{-1}(T^{-1}M)$ the localization morphisms. Recall that $\tilde{S}^{-1}(T^{-1}M)$ is naturally an R-module, via the localization morphisms (i.e. multiplication by r is multiplication by $(r/1)/(1/1)$). Notice that multiplication by any $st \in ST$ on $\tilde{S}^{-1}(T^{-1}M)$ is invertible, with inverse being multiplication by $(1/t)/(s/1)$. Hence by the universal property of localization of a module (see the solution of Exercise 1 on Sheet 10), $\tilde{S}^{-1}(T^{-1}M)$ naturally has the structure of an $(ST)^{-1}R$-module via the formula

$$\frac{r}{st} \cdot \left(\left(\frac{m}{t'}\right) / \left(\frac{s'}{1}\right)\right) := ((r/1)/(1/1)) \cdot (((st)/1)/(1/1))^{-1} \cdot \left(\left(\frac{m}{t'}\right) / \left(\frac{s'}{1}\right)\right) = \left(\left(\frac{rm}{tt'}\right) / \left(\frac{ss'}{1}\right)\right),$$

and there exists an $(ST)^{-1}M$-module morphism $\psi^M : (ST)^{-1}M \to \tilde{S}^{-1}(T^{-1}M)$ such that $\psi^M \circ \iota_{ST}^M = \iota_{\tilde{S}}^M \circ \iota_T^M$. Notice that the $(ST)^{-1}M$-module structure on $\tilde{S}^{-1}(T^{-1}M)$ is the same as the one defined via the isomorphism of the previous point, and that ψ^M maps an element $m/(st)$ to $(m/t)/(s/1)$.
Now either one constructs an inverse to ψ^M with a similar procedure, or one proves directly that ψ^M is an isomorphism. We will do the latter for once: if $y := (m/t)/(s/1) \in \tilde{S}^{-1}(T^{-1}M)$ is arbitrary, then ψ^M maps $x := m/(ts)$ to $(m/t)/(s/1)$, so ψ^M is surjective. Finally, suppose that ψ^M maps some $m/(st) \in (ST)^{-1}M$ to 0. Then there exists $s'/1 \in \tilde{S}$ such that $(s'/1)(m/t) = 0$ inside $T^{-1}M$. Therefore, there exists $t' \in T$ such that $t's'm = 0$ inside M. But then as $t's' \in S$, this means $m/(st) = 0$ inside $(ST)^{-1}M$. Thus ψ^M is also injective, and hence an isomorphism.

(4) As $1 \in T$ we have $S \subseteq ST$. On the other hand, we have $ST \subseteq SS \subseteq S$ as S is multiplicatively closed, so $ST = S$. Hence point (2) gives $\tilde{S}^{-1}(T^{-1}R) \cong S^{-1}R$ as rings, and point (3) gives $\tilde{S}^{-1}(T^{-1}M) \cong S^{-1}M$ as $S^{-1}R$-modules.

□

Exercise 8. In Exercise 6 of sheet 10, we saw how to construct the tensor product of two R-algebras. The goal is to show the following result:

Proposition 0.2. *Let k be an algebraically closed field, and let R, S two finitely generated k-algebras which are domains. Then $R \otimes_k S$ is again a domain.*

During this exercise, you can freely use the following results (which you will see shortly) :

- Nullstellensatz (Theorem 6.5.4 from the notes)
- For any finitely generated k-algebra T and any maximal ideal $\mathfrak{m}$, the composition $k \to T \to T/\mathfrak{m}$ is an isomorphism (see the proof of the weak Nullstelensatz, which is Theorem 6.2.2 in the notes).

Proceed as follows:

(1) Let T be a finitely generated k-algebra which is a domain, and let $a_1, \dots a_s \in T$ be non-zero. Show that there is a maximal ideal $\mathfrak{m}$ of T such that $a_i \notin \mathfrak{m}$ for all i.
[*Hint*: write T as a quotient of a polynomial ring, and use Nullstellensatz.]

(2) Show that any element in $R \otimes_k S$ can be written as

$$\sum_i a_i \otimes b_i$$

with the b_i's linearly independent over k.

(3) Assume that

$$\left(\sum_i a_i \otimes b_i\right) \cdot \left(\sum_j a'_j \otimes b'_j\right) = 0$$

where both families $(b_i)_i$ and $(b'_j)_j$ are linearly independent. Let $\mathfrak{m}$ be a maximal ideal not containing any of the a_i, a'_j.
Show by applying the ring map

$$R \otimes_k S \to R/\mathfrak{m} \otimes_k S \cong S$$

that one of the factors must be zero, and hence conclude that $R \otimes_k S$ is a domain.

Proof. (1) We give two proofs of this part: one uses the intended way (which is more "geometric"), while the other one works over arbitrary fields (and is more "algebraic").

Note that in both cases, we may assume $s = 1$ (we will write $a = a_1$). Indeed, since T is a domain, $\prod_i a_i \neq 0$, so we reduce to the case $s = 1$ since maximal ideals are prime.

Intended way: Let us write $T = k[x_1, \ldots, x_n]/I$ (this is possible by definition of a finitely generated k-algebra). We need to find a maximal ideal in T which does not contain a. Let $b \in k[x_1, \ldots, x_n]$ be a lift of a. By the correspondence theorem, we need to find a maximal ideal in $k[x_1, \ldots, x_n]$ which contains I but not b.

By Nullstellensatz, this is equivalent to finding some $x \in k^n$ such that $x \in V(I)$ but $x \notin V(b)$. Indeed, if we had such an element, the maximal ideal $\mathfrak{m} := I(\{x\})$ would do the job by Nullstellensatz.

If such an x did not exist, then we would have $V(I) \subseteq V(b)$. Applying Nullstensatz would then give

$$b \in \sqrt{(b)} = I(V(b)) \subseteq I(V(I)) = \sqrt{I} = I$$

where the last equality holds since I is prime ($T = k[x_1, \ldots, x_n]/I$ is a domain). However, $b \in I$ implies that $a = 0$ (recall b is a lift of $a \in k[x_1, \ldots, x_n]/I$) which contradicts the hypothesis.

More general way: Let us show the following result:

Lemma 0.3. *Let k be an arbitrary field, and let $f : T \to S$ be a morphism of finitely generated k-algebras. Then for all maximal ideal $\mathfrak{m} \subseteq S$, $f^{-1}(\mathfrak{m})$ is maximal.*

Proof. The map f induces an injection

$$T/f^{-1}(\mathfrak{m}) \to S/\mathfrak{m}$$

Since $S/\mathfrak{m}$ is a field, we have

$$\operatorname{trdeg}_k(S/\mathfrak{m}) = \dim(S/\mathfrak{m}) = 0$$

Since $T/f^{-1}(\mathfrak{m}) \subseteq S/\mathfrak{m}$, we also have $\operatorname{trdeg}_k(T/f^{-1}(\mathfrak{m})) = 0$, and hence $\dim(T/f^{-1}(\mathfrak{m})) = 0$. This means by definition that any prime ideal of $\dim(T/f^{-1}(\mathfrak{m}))$ is maximal. Since $T/f^{-1}(\mathfrak{m})$ is a domain (the preimage of a prime ideal is always a prime ideal!), we deduce that (0) is maximal, so $T/f^{-1}(\mathfrak{m})$ is a field (i.e. $f^{-1}(\mathfrak{m})$ is maximal). □

Remark 0.4. This lemma above is completely wrong for non-finitely generated k-algebras! For example $k[x] \subseteq k(x)$ gives a counterexample ((0) is maximal in $k(x)$, but not in $k[x]$).

Now, the point is that T_a is again a finitely generated k-algebra! (indeed, we have $T_a \cong T[x]/(xa-1)$). Thus, given any maximal ideal $\mathfrak{m} \subseteq T_a$, its preimage $\mathfrak{m}^c$ will be maximal in T by the lemma above. Since it cannot contain a, we win.

(2) Let $\sum_{i\in I} r_i \otimes s_i \in R \otimes_k S$. If the elements s_i are linearly independent, we are fine. If not, we can write $s_j = \sum_{i\neq j} \alpha_i s_i$, we

$$\sum_{i\in I} r_i \otimes s_i = \sum_{i\neq j}(r_i \otimes s_i) + r_j \otimes \sum_{i\neq j} \alpha_i s_i = \sum_{i\neq j}(r_i \otimes s_i) + \sum_{i\neq j}(\alpha_i r_j) \otimes s_i = \sum_{i\neq j}(r_i + \alpha_i r_j) \otimes s_i$$

Note that in the right-hand side, s_j never appears. Since the index set I is finite, this process has to finish at some point.

(3) Let us show that $R \otimes_k S$ is a domain. Assume that

$$\left(\sum_i a_i \otimes b_i\right) \cdot \left(\sum_j a_j' \otimes b_j'\right) = 0$$

and assume that both families $(b_i)_i$ and $(b_j')_j$ are linearly independent (see the previous point). By contradiction, further assume that both elements above are non-zero. Therefore, $a_{i_1} \neq 0$ and $a_{j_1}' \neq 0$ for some i_1, j_1. By the first point, there exists a maximal ideal $\mathfrak{m}$ be a maximal ideal not containing a_{i_1} and a_{j_1}.

Since k is algebraically closed, $R/\mathfrak{m} \cong k$ by the weak Nullstellensatz. Let $\theta : R/\mathfrak{m} \to k$ denote an isomorphism. Thus there is a ring map $R \otimes_k S \to S$ is given by $\sum_i r_i \otimes s_i \mapsto \sum_i \theta(\overline{r_i}) s_i$. Applying our ring map above gives the element

$$\left(\sum_i \theta(\overline{a_i}) b_i\right) \cdot \left(\sum_j \theta(\overline{a_j'}) b_j\right) = 0$$

Since S is a domain, one of the two terms above is 0 (without loss of generality we may assume $\sum_i \theta(\overline{a_i}) b_i = 0$).

Since the b_i's are linearly independent, we have $\theta(\overline{a_i}) = 0$ for all i. However, θ is an isomorphism, so $\overline{a_{i_1}} = 0$. This is impossible since $a_{i_1} \notin \mathfrak{m}$ by assumption.

□