

**Exercise 1.** Show the following:

- (1) Prove that the only prime ideal of height zero in a domain is the ideal  $(0)$ .
- (2) Prove that a prime ideal of height 1 in a UFD is principal.
- (3) Compute the prime ideals of height zero in  $\mathbb{R}[x, y]/(xy)$ .  
[*Hint:* Recall that there is a one-to-one correspondence between the prime ideals  $R$  containing  $I$  and the prime ideals of  $R/I$ .]

**Exercise 2.** Show the following:

- (1) If  $R$  is a domain with  $\dim R = 0$ , then  $R$  is a field.
- (2) We say that a ring  $R$  is reduced if there are no nilpotent elements in  $R$ . That is, if  $r \in R$  is such that  $r^n = 0$  for some  $n$ , then  $r = 0$ . Give an example of a reduced ring  $R$  of dimension zero which is not a field.

**Exercise 3.** Solve the following problems:

- (1) Prove that every Artinian ring has dimension 0.
- (2) Compute the dimension of the ring  $\mathbb{Z}[x]/(4, x^2)$ .

**Exercise 4** (Nakayama's Lemma). Let  $R$  be a commutative ring and let  $M$  be a finitely generated  $R$ -module. Show the following:

- (1) Let  $I$  be an ideal of  $R$  such that  $IM = M$ . Then there exists  $x \in 1 + I$  such that  $xM = 0$ .  
[*Hint:* The proof is similar to the direction  $(3) \Rightarrow (1)$  in Proposition 8.1.4 of the lecture notes.]
- (2) Suppose now that the ring  $R$  is local, i.e., that there is a unique maximal ideal  $\mathfrak{m}$  of  $R$ . Show that if  $\mathfrak{m}M = M$ , then  $M = 0$ .
- (3) For a ring  $R$  denote by  $\text{Jac}(R)$  the intersection of all maximal ideals of  $R$ ; this is called the *Jacobson radical* of  $R$  (note also that  $\text{nil}(R) \subseteq \text{Jac}(R)$ ). Show that if there is an ideal  $I \subset \text{Jac}(R)$  such that  $IM = M$ , then this implies that  $M = 0$ . This generalizes the previous point to any ring.

[*Hint:* Prove that in (2), (3) the element  $x$ , whose existence is assured by (1), is in fact invertible.]

*Remark 0.1.* Nakayama's lemma is a very powerful tool in commutative algebra and algebraic geometry, so keep it in mind this exists. You should really remember it!

To give a hint of its tremendous power, recall you had an exercise about showing that if  $R$  is a commutative ring,  $M$  a finitely generated module and  $f: M \rightarrow M$  a surjective endomorphism, then  $f$  is an isomorphism. Actually, the statement follows immediately by considering  $M$  as an  $R[x]$ -module via  $x \cdot m = f(m)$ , and taking  $I = (x)$  in (1).

Recall that when you proved it in an early exercise sheet, you had a Noetherian assumption on  $R$  (and it was fundamental to the proof, have fun trying to prove it directly without this assumption!). With this argument, you don't need it!

**Exercise 5.** Let  $R$  be a commutative ring which is an integral domain but not a field, and let  $F$  be the fraction field of  $R$ . Show that  $F$  is not finitely generated as an  $R$ -module.

**Exercise 6.** Let  $R = \mathbb{F}_q[[t]]$  be the ring of power-series in the variable  $t$  over the finite field with  $q$  elements  $\mathbb{F}_q$ .

Recall that as a set,  $R$  is the set of formal power-series  $f = \sum_{n \geq 0} a_n t^n$  with coefficients  $a_n \in \mathbb{F}_q$ . For two such power series,  $\sum_{n \geq 0} a_n t^n$  and  $\sum_{n \geq 0} b_n t^n$ , one defines the addition to be the power-series  $\sum_{n \geq 0} (a_n + b_n) t^n$  and multiplication to be the power-series  $\sum_{n \geq 0} (\sum_{k=0}^n a_k b_{n-k}) t^n$ . Recall (or do) the two following exercises from "Anneaux et corps":

- (1) If  $f \in R \setminus (t)$ , then  $f$  is invertible (and hence  $R$  is a local ring with maximal ideal  $(t)$ ).
- (2) A formal Laurent series over the field  $\mathbb{F}_q$  is defined in a similar way to a formal power series, except that we also allow finitely many terms of negative degree. That is, series of the form  $f = \sum_{n \geq N} a_n t^n$  where for some  $N \in \mathbb{Z}$ . Define a natural ring structure on this set and show that with this ring structure the ring of formal Laurent series over  $\mathbb{F}_q$ , usually denoted  $\mathbb{F}_q((t))$ , is equal to the fraction field of  $R$ .

Now let us go to the actual exercise:

- (3) Show that  $\text{trdeg}_{\mathbb{F}_q}(\text{Frac}(R))$  is infinite.  
[Hint: show that  $\mathbb{F}_q(t_1, \dots, t_r)$  is countable, and  $R$  is not.]
- (4) Show that  $\dim R = 1$  and hence show that Theorem 6.1.12 in the course notes does not work with non-finitely-generated algebras.

**Exercise 7.** Let  $R$  be a Noetherian local ring (i.e. it has a unique maximal ideal) with maximal ideal  $\mathfrak{m}$ , and set  $k = R/\mathfrak{m}$ . Furthermore, fix a finitely generated module  $M$  over  $R$ .

- (1) Show that if  $f: M \rightarrow N$  is a morphism of finitely generated modules, such that the induced map  $M/\mathfrak{m}M \rightarrow N/\mathfrak{m}N$  is surjective. Show that  $f$  is then surjective.
- (2) A *minimal free resolution* of  $M$  is a resolution

$$\cdots \rightarrow F_n \xrightarrow{d_n} F_{n-1} \cdots \rightarrow F_0$$

of  $M$  such that for all  $n$ ,  $F_n$  is free of finite rank and  $\text{im}(d_n) \subseteq \mathfrak{m}F_{n-1}$ . Show that  $M$  admits a minimal free resolution.

- (3) Fix a minimal free resolution  $F_\bullet$  of  $M$ . Then show that for all  $n \geq 0$ ,  $\text{Ext}^n(M, k) \neq 0$  if and only if  $F_n \neq 0$ .
- (4) Deduce the surprising fact that if  $\text{Ext}^{n+1}(M, k) \neq 0$ , then  $\text{Ext}^n(M, k) \neq 0$ .
- (5) Show that a finitely generated projective module over  $R$  is free.

*Hint for this exercise (and for your potential future life as a commutative algebraist/algebraic geometer): Nakayama's lemma is your friend.*