EPFL - Fall 2025	Zsolt Patakfalvi
Rings and modules	**Exercises**
Sheet 8	13 November 2025

Exercise 1 (Nullstellensatz for $\operatorname{Spec} R$)**.** Let R be a commutative ring. Given a closed subset $Z \subseteq \operatorname{Spec} R$, define $I(Z) := \{f \in R,\ Z \subseteq V(f)\}$. Show that $I(Z)$ is an ideal, and that for all ideals $I \subseteq \operatorname{Spec} R$,

$$I(V(I)) = \sqrt{I}$$

In particular, show that for all ideals I, J of R,

$$V(I) = V(J) \iff \sqrt{I} = \sqrt{J}$$

Exercise 2. Let R be a commutative ring and $I \subseteq R$ be a radical ideal. Show that I is prime if and only if $V(I)$ is an irreducible topological space.

Exercise 3. Let $R = \mathbb{C}[x, y, z]$ and $I = (xy - z^2, x^2 - y^2) \subseteq R$. Identify $V(I) \subset \mathbb{C}^3$. Notice that this naturally breaks into smaller algebraic sets. What are the ideals of each piece?

Exercise 4. Let F be an algebraically closed field. Let X and Y be algebraic sets in F^n.

(1) Prove that $I(X \cup Y) = I(X) \cap I(Y)$
(2) By considering $X = V(x^2 - y)$ and $Y = V(y)$ for the ideals $(x^2 - y)$ and (y) in $F[x, y]$, show that it need not be true that $I(X \cap Y) = I(X) + I(Y)$.
(3) Prove that in general $\sqrt{I(X) + I(Y)} = I(X \cap Y)$.

Review exercises for material from "Anneaux et corps"

Exercise 5. Show that $x^3 + y^7 \in k[x, y]$ is irreducible.
[*Hint:* Use the consequence of Gauss's theorem saying that for a unique factorisation domain R and a primitive polynomial $f \in R[t]$, we have that f is irreducible in $\operatorname{Frac}(R)[t]$ if and only if it is irreducible in $R[t]$.]

Exercise 6. Let $R = k[x, y, z]$. Show that $(xz^3 + yz^3 - y^2z^2 + xyz - xy)$ is a prime ideal of R.
[*Hint*: Use Eisenstein's Criterion.]

Exercise 7. Solve the following exercises:

(1) Consider the polynomial $f = X^3Y + X^2Y^2 + Y^3 - Y^2 - X - Y + 1$ in $\mathbb{C}[X, Y]$. Write it as an element of $(\mathbb{C}[X])[Y]$, that is collect together terms according to powers of Y, and then use Eisenstein's criterion to show that f is prime in $\mathbb{C}[X, Y]$.
(2) Let F be any field. Show that the polynomial $f = X^2 + Y^2 - 1$ is irreducible in $F[X, Y]$, unless F has characteristic 2. What happens in that case?

Exercise 8. Show the following:

(1) Let $F \subseteq L$ be a field extension, and suppose $a_1, \ldots, a_n$ are elements of L which are algebraically independent over F. Prove that $F(a_1, \ldots, a_n)$ is isomorphic to the fraction field of the polynomial ring $F[x_1, \ldots, x_n]$.

(2) Let $F \subseteq L$ be a field extension. Show that a subset of L is a transcendence basis for L over F if and only if it is a maximal algebraically independent set. As a consequence, use Zorn lemma to show that a transcendence basis exists for any field extension $F \subseteq L$.

Exercise 9. Prove that if $F \subseteq K \subseteq L$ are field extensions such that $\operatorname{trdeg}_F L < \infty$, then $\operatorname{trdeg}_F L = \operatorname{trdeg}_F K + \operatorname{trdeg}_K L$