EPFL - Fall 2025	Zsolt Patakfalvi
Rings and modules	**Exercises**
Sheet 7	6 November 2025

Exercise 1. Let F be an algebraically closed field, and let I, J be ideals of $R = F[x_1, ..., x_n]$. Prove that $\sqrt{I} \subseteq \sqrt{J}$ if and only if $V(J) \subseteq V(I)$.

Exercise 2. Let F be an algebraically closed field, and let I, J be ideals of $R = F[x_1, ..., x_n]$. Show that

(1) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$
(2) $V(I) \cap V(J) = V(I + J)$

Exercise 3. Let R be a commutative ring, and let I, J be ideals of R. In both $\operatorname{Spec}(R)$ and $m - \operatorname{Spec}(R)$, show that

(1) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$
(2) $V(I) \cap V(J) = V(I + J)$

Exercise 4.
- Let R, S be commutative rings, and let $f : R \to S$ be a ring morphism. Show that there is an induced continuous map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$.
- Let R be a ring and I an ideal. Show that the morphism $\operatorname{Spec}(R/I) \to \operatorname{Spec}(R)$ induced by the quotient map corresponds to the inclusion of the closed subset $V(I) \subseteq \operatorname{Spec}(R)$.

Exercise 5. Prove that $Z = \{(u^3, u^2 v, uv^2, v^3) : u, v \in \mathbb{C}\} \subset \mathbb{C}^4$ is an algebraic set (i.e. there exists an ideal I of $\mathbb{C}[x_1, x_2, x_3, x_4]$ such that $Z = V(I)$). Find $I(Z)$.
[*Hint:* Make sure you have everything!]

Exercise 6. Let F be an algebraically closed field, and $X \subseteq F^m$ an algebraic set with ideal $I = I(X)$. Define the coordinate ring $A(X)$ of X to be $A(X) := F[x_1, \ldots, x_m]\big/I$. Notice that every element of $A(X)$ naturally defines a set-map from X to F, and thus one may think of $A(X)$ as the set of global algebraic functions on X.

(1) If $X = V(I) \subseteq F^m$, and $Y = V(J) \subseteq F^n$ are algebraic sets with ideals $I = I(X)$ and $J = I(Y)$, then a morphism $f : X \to Y$ is defined to be a set-map from the points of X to the points of Y, for which the following holds: there exists a vector $(h_1, \ldots, h_n)$ of polynomials $h_i \in F[x_1, \ldots, x_m]$, such that for every $\underline{a} \in X$ we have $f(\underline{a}) = (h_1(\underline{a}), h_2(\underline{a}), \ldots, h_n(\underline{a})) \in Y$.

Show that whenever there is a morphism $f : X \to Y$ of algebraic sets as defined above, there is a unique homomorphism of F-algebras $\lambda_f : A(Y) \to A(X)$, such that the following diagram commutes.

$$\begin{array}{ccc} F[y_1, \ldots, y_n] & \xrightarrow{y_i \mapsto h_i} & F[x_1, \ldots, x_m] \\ \downarrow & & \downarrow \\ A(Y) & \xrightarrow{\quad\lambda_f\quad} & A(X) \end{array}$$

Here the vertical arrows are the quotient maps stemming from the definition of $A(X)$ and $A(Y)$, and the top horizontal map is given by sending y_i to $h_i(x_1, ..., x_m)$.

(2) With setup as above, show that if there is a homomorphism of F-algebras $\lambda : A(Y) \to A(X)$, then there is a morphism $f : X \to Y$ such that $\lambda = \lambda_f$. Furthermore, all choices of f are the same (as set-maps from the points of X to the points of Y).

Exercise 7. Let F be an algebraically closed field. Let X be an algebraic set in F^n with ideal $I(X) = I$. Prove that points of F^n contained in X are naturally in bijection with maximal ideals of the coordinate ring $A(X) = F[x_1, ..., x_n] \big/ I$.