EPFL - Fall 2025	Zsolt Patakfalvi
Rings and modules	**Exercises**
Sheet 5	16 October 2025

Exercise 1. Let R be a commutative ring. The *projective dimension* of an R-module M is the smallest integer $n \geq 0$ such that there exists a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0$$

of M. We write $\operatorname{projdim}(M) = n$, and if no finite projective resolution exists, this number is by definition ∞.

In our case, we focus on the ring $R = k[x,y]/(x^2 - y^3)$ and $M = R/(x,y)$. The goal is to show that M does not have finite projective dimension. Proceed as follows:

(1) Compute the dimension as a k-vector space of $\operatorname{Ext}^1_R(M,M)$.
(2) Show that there is a short exact sequence

$$0 \to M \to R/y \to M \to 0.$$

(3) Use the two points above to show that $\operatorname{Ext}^i_R(M,M) \neq 0$ for all $i \geq 0$.
(4) Conclude that $\operatorname{projdim}(M) = \infty$.

Remark 0.1. A celebrated theorem of Serre states that a ring R is *regular* if and only if every module M over R has finite projective dimension. Without going into details, regular means that the associated algebraic variety looks "good" (e.g. would be a smooth manifold over the complex numbers). This gives a very important application of Ext-functors in commutative algebra, since they help detect the projective dimension of modules (and hence regularity of the ring).

In the case above, note that the associated variety (here $\{(x,y) \in \mathbb{R}^2 | x^2 = y^3\}$ if $k = \mathbb{R}$) doesn't look good at the origin (draw this curve!), it has a so-called cusp singularity, and hence it is not regular. This exercise is then about verifying Serre's theorem in a special example.

Exercise 2. For two short exact sequences

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

and

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

we say that there is a map between them if there exists morphisms $f_i : M_i \to N_i$, for $1 \leq i \leq 3$ and a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & 0. \end{array}$$

Show that whenever there is a map between two short exact sequences, then there is an induced map between long exact sequences of Ext-modules, making the suitable diagram commute.

Exercise 3. In this exercise we prove the the two *4-lemmas*. To this end, suppose that we have a commuting diagram with exact rows:

$$\begin{array}{ccccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \xrightarrow{f_3} & D \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\ A' & \xrightarrow{f_1'} & B' & \xrightarrow{f_2'} & C' & \xrightarrow{f_3'} & D' \end{array}$$

(1) Show that if a and c are surjective and d is injective, then b is an surjective.
(2) Show that if b and d are injective and a is surjective, then c is a injective.

Exercise 4. (1) Set $k = \mathbb{F}_p$ and $G = \mathbb{Z}/p\mathbb{Z}$. Find all the submodules (i.e. ideals) of $R = k[G]$. [1]
[*Hint:* To understand $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ in terms of more common rings, it might be a good idea to look for ring morphisms $\mathbb{F}_p[x] \to \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ and investigate both kernel and image.]
(2) For $p = 2$, let x denote a generator of G and set $M = (x+1) \subseteq k[G]$. Compute $\mathrm{Ext}^i_R(M, M)$ for all $i \geq 0$.

Exercise 5. In this exercise we define injective modules and prove *Baer's criterion.* Let R be a (not necessarily commutative) ring; any R-module and any R-morphism appearing in this exercise will be a left R-module resp. a morphism of left R-modules.
We say that an R-module Q is injective if it satisfies the following property:
Whenever we have an injective R-morphism $f : X \hookrightarrow Y$ and an R-morphism $g : X \to Q$, then there exists an R-morphism $h : Y \to Q$ making the following diagram commute:

$$\begin{array}{ccc} X & \overset{f}{\hookrightarrow} & Y \\ \downarrow{\scriptstyle g} & \swarrow{\scriptstyle h} & \\ Q & & \end{array}$$

We will prove the following:

Theorem (Baer's Criterion)**.** *Suppose that the left R-module Q has the property that if I is any left ideal of R and $f : I \to Q$ is an R-morphism, there exists an R-morphism $F : R \to Q$ extending f. Then Q is an injective R-module.*

We will prove *Baer's criterion* in several steps. Assume that the R-module Q satisfies Baer's criterion.

(1) Let X, Y be R-modules, and assume that Y is *cyclic* (generated by $b \in Y$). Let $f : X \hookrightarrow Y$ be an injective R-morphism. Show that for every R-morphism $g : X \to Q$,

[1] Recall that $k[G]$ is defined as the set of functions $f : G \to k$, where addition is defined pointwise, and multiplication is defined by convolution:

$$(f \cdot g)(z) := \sum_{\substack{x,y \in G \\ xy = z}} f(x)g(y) \quad \text{for all } f, g \in k[G],\ z \in G.$$

Now for $g \in G$, let $\delta_g \in k[G]$ be the function taking the value 1 at g and 0 everywhere else. Then $\{\delta_g\}_{g \in G}$ is a k-basis of $k[G]$. Furthermore, one can verify that the map $g \mapsto \delta_g$ is an injective group morphism $G \hookrightarrow k[G]^\times$ (in particular, $1_{k[G]} = \delta_{e_G}$). Usually people just write g instead of δ_g by abuse of notation, but as the underlying sets of $\mathbb{F}_p$ and $\mathbb{Z}/p\mathbb{Z}$ coincide, it is probably better to distinguish them in the notation.

there exists an R-morphism $h : Y \to Q$ making the appropriate diagram commute.
[*Hint:* Identify X with a submodule of Y and consider the subset I of R defined by $I = \{r \in R : rb \in X\}$.]

(2) Let X, Y be left R-modules with an injective R-morphism $f : X \hookrightarrow Y$ (we identify X with its image under f). Let $b \in Y$ be arbitrary. With a similar approach as in the previous point, prove that any R-morphism $g : X \to Q$ can be extended to an R-morphism $h : X + Rb \to Q$ making the appropriate diagram commute.

(3) Use *Zorn's Lemma* to conclude the proof.

Axiom 1 (Zorn's Lemma / Axiom of Choice)**.** If $(\mathcal{P}, \leq)$ is a partially ordered set with the property that every totally ordered subset (often called a chain) has an upper bound, then there exists a maximal $M \in \mathcal{P}$. (that is, for $N \in \mathcal{P}$, we have $M \nleq N$)

[*Hint:* Try to think of what it means for one partial extension of $g : X \to Q$ to be smaller than another.]

Exercise 6. Use Baer's Criterion to show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.