EPFL - Fall 2025	Zsolt Patakfalvi
Rings and modules	**Exercises**
Sheet 3	2 October 2025

Exercise 1. Do the following:

(1) Calculate the Smith normal form of the following matrix over $\mathbb{Z}$.

$$\begin{pmatrix} 1 & 9 & 1 \\ -2 & -6 & 0 \\ 2 & -8 & 2 \\ -1 & 1 & 5 \end{pmatrix}$$

(2) (i) Find a direct sum of cyclic $\mathbb{Z}$-modules isomorphic to the $\mathbb{Z}$-module M with generators e_1, e_2, e_3, e_4 and relations

$$e_1 - 2e_2 + 2e_3 - e_4 = 0$$
$$9e_1 - 6e_2 - 8e_3 + e_4 = 0$$
$$e_1 + 2e_3 + 5e_4 = 0$$

[*Hint/Remark:* By definition, M is the quotient of the free $\mathbb{Z}$-module on 4 generators $\bigoplus_{i=1}^{4} \mathbb{Z}e_i$ by the submodule generated by $e_1 - 2e_2 + 2e_3 - e_4$, $9e_1 - 6e_2 - 8e_3 + e_4$ and $e_1 + 2e_3 + 5e_4$. Notice that in the quotient, $e_1, \dots, e_4$ then satisfy exactly these relations.]

(ii) Explicitly give 'nice' generators of M, in terms of the original generators e_1, e_2, e_3, e_4. Here, $f_1, \dots, f_s$ are 'nice' generators if the relations they satisfy are generated by relations of the form $m_i f_i = 0$, where $m_1, \dots, m_s \in \mathbb{Z}$ are integers.

Exercise 2. Let $R = \mathbb{Q}[x]$. Find a direct sum of cyclic R-modules isomorphic to the R-module with generators e_1, e_2 and relations

$$x^2 e_1 + (x+1)e_2 = 0$$
$$(x^3 + 2x + 1)e_1 + (x^2 - 1)e_2 = 0$$

.

Exercise 3. Give an example of an infinitely generated $\mathbb{Z}$-module which is *not* an (infinite) direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ for various choices of n.

Remark 0.1. This shows that the fundamental theorem of finitely generated modules over PID's does not hold for infinitely generated modules.

Exercise 4. Let $R = \mathbb{Z}[x]$ and consider the matrix $A = \begin{pmatrix} 2 & x \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(R)$.

(1) Show that A is not equivalent to a diagonal matrix. The equivalence that we consider here is the one introduced in the lectures, that is, up to left or right multiplication by an invertible matrix.

(2) Show that the cokernel of the map $A : R^{\oplus 2} \to R^{\oplus 2}$ is isomorphic to a direct sum of cyclic R-modules, but is not isomorphic to an R-module of the form $R^{\oplus m} \oplus \bigoplus_{i=1}^{n} R/(a_i)$ where $a_1, \dots, a_n \in R \setminus \{0\}$.

(3) Show that $(2, x)$ is not isomorphic to a direct sum of cyclic R-modules.

Exercise 5. Show that an exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

of R-modules induces an exact sequence

$$0 \longrightarrow \mathrm{Tors}(M) \longrightarrow \mathrm{Tors}(N) \longrightarrow \mathrm{Tors}(L) \ ,$$

but not necessarily an exact sequence

$$0 \longrightarrow \mathrm{Tors}(M) \longrightarrow \mathrm{Tors}(N) \longrightarrow \mathrm{Tors}(L) \longrightarrow 0 \ .$$

Exercise 6. Let $M \in \mathrm{Mat}(n \times n, k)$ for a field k. Show that there is a basis with respect to which M is block diagonal with blocks of the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & \ddots & 0 & a_1 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & a_{d-2} \\ 0 & 0 & \dots & 1 & a_{d-1} \end{pmatrix}$$

Hint: M acts naturally on some n-dimensional k-vector space V. Consider V as a $k[x]$-module via $f \cdot v = f(M)(v)$ and use the classification of finitely generated modules over a PID.