

Exercise 1. Show that the following holds for a R -module M of finite length $l(M)$ (i.e., an R -module M that admits a composition series of finite length).

- (1) If there is a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R -modules, then $l(M) = l(M') + l(M'')$.

- (2) If $N <_R M$ is a proper submodule then $l(N) < l(M)$.
 (3) Use (2) to show that any strict chain of submodules in M (not necessary a maximal chain, i.e. not necessarily a composition series) has length smaller than or equal to $l(M)$. Conclude that a module M is of finite length if and only if M is both Noetherian and Artinian.

Exercise 2. Let R be a commutative ring and let M be a finitely generated module over R . Let $f : M \rightarrow M$ be an R -module homomorphism.

- (1) Suppose that R is a Noetherian ring.
 (i) Does injectivity of f implies surjectivity?
 (ii) Does surjectivity of f implies injectivity?
 (2) Suppose that M is a module of finite length, show that f is injective if and only if f is surjective.

Exercise 3. (1) Let R be a PID, and let $f \in R$ be a product of $n \geq 0$ prime elements. Prove that the length of $R/(f)$ as an R -module is equal to n .

- (2) Let $f \in \mathbb{R}[x]$ be a nonzero polynomial with exactly $n \geq 0$ non-real roots (counted with multiplicity). Prove that

$$\dim_{\mathbb{R}}(\mathbb{R}[x]/(f)) - \text{length}_{\mathbb{R}[x]}(\mathbb{R}[x]/(f)) = n/2$$

- (3) Let M be a \mathbb{Z} -module. Prove that M has finite length if and only if it is finite (as a set).
 (4) Give an example of a ring and a module over this ring which has finite length but infinitely many submodules.

Exercise 4. (1) Let $n, m > 0$ be integers, let k be a field and let $R := k[x, y]$. Show that the R -module

$$M := k[x, y]/(x^n, y^m)$$

has length nm .

Hint: Exercise 1 can be useful to decompose this computation into easier ones, allowing some induction argument. The same applies for the next point.

- (2) Let $p > 0$ be a prime number. Compute the length of

$$\mathbb{Z}[x]/(p^2, x^2 - p),$$

as a module over the ring $\mathbb{Z}[x]$.

Exercise 5. Let R be a commutative Noetherian ring. Are the following rings Noetherian? Are they Artinian?

- (1) $R[x, \frac{1}{x}] := \{\sum_{i=-m}^n a_i x^i : a_i \in R, m, n \in \mathbb{N}\}$.
- (2) $R[x_1, x_2, x_3, \dots]$.
- (3) $R[[x]]$, the ring of formal power series¹ with coefficients in R .

Hint: For an ideal I and each $n \in \mathbb{N}$, let $I_n := \{a_n : \exists \sum_{i=n}^{\infty} a_i x^i \in I\}$. Then adapt the proof of the Hilbert basis theorem.

- (4) $C^0(\mathbb{R})$, the ring of continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ with pointwise operations.
- (5) $\mathbb{R}[x] / ((x-1)^2 x)$.

Exercise 6. Computing a presentation of an R -module M means explicitly determining an exact sequence of the form $R^{\oplus t} \xrightarrow{\eta} R^{\oplus s} \xrightarrow{\varepsilon} M \rightarrow 0$. Do the following computations.

- (1) Compute a presentation of the \mathbb{Z} -module

$$M := \mathbb{Z}(2, 9) + \mathbb{Z}(4, 3) + \mathbb{Z}(6, 8) \subseteq \mathbb{Z} \oplus \mathbb{Z}.$$

- (2) Let $R = \text{Mat}_{2 \times 2}(\mathbb{Z})$ be the ring of 2×2 -matrices over \mathbb{Z} . Compute a presentation of the left R -module

$$M := R \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + R \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \subseteq R.$$

¹ $R[[x]] = \{\sum_{i=0}^{\infty} a_i x^i : a_i \in R\}$, where multiplication and addition are defined formally, as what you think they should be. These are purely formal objects: there is no requirement for any kind of convergence.