

As always in the course, we fix the following notation:

- R is a ring,
- k is a field,
- if not specified, “module” means “left R -module”, and
- M and N are modules.

Exercise 1. (1) A simple module is a module that has only trivial submodules. Show that any simple module is cyclic.

(2) Let $m \in M$ be an element. We define the annihilator of m by

$$\text{Ann}_R(m) = \{ r \in R \mid rm = 0 \}$$

We only write $\text{Ann}(m)$ if the base ring is clear from the context.

Show that $\text{Ann}(m)$ is a left ideal of R and that the cyclic module Rm is isomorphic to the module $R/\text{Ann}(m)$.

(3) Let M be a simple $k[x]$ -module. Prove that $M \cong k[x]/(f)$ where f is an irreducible polynomial in $k[x]$ and (f) denotes the ideal generated by f .

(4) Which of the following \mathbb{Z} -modules are simple?

- (a) \mathbb{Z}
- (b) $\mathbb{Z}/6\mathbb{Z}$
- (c) $\mathbb{Z}/7\mathbb{Z}$

Exercise 2. Let R be a ring, M a left R -module and $m \in M$.

(1) In the previous exercise you proved that $\text{Ann}(m)$ is a left ideal of R . Give an example to show that $\text{Ann}(m)$ might *not* be a two sided ideal of R .

(2) Define the *annihilator* of M to be

$$\text{Ann}_R(M) = \{ r \in R \mid rM = 0 \} = \{ r \in R \mid \forall m \in M: rm = 0 \}$$

Prove that $\text{Ann}(M)$ is a two sided ideal of R .

(3) Let $\phi: S \rightarrow R$ be a surjective homomorphism of rings and M a module over S . Show that we can endow an R -module structure given by $r \cdot m = s \cdot m$ for any $s \in \phi^{-1}(r)$ and $m \in M$ if and only if $\ker \phi \subseteq \text{Ann}(M)$.

(4) For example, let $S = k[x]$ and $M = k[x]$ (with the standard action). Then M/f^2M is a $k[x]/(f^2)$ -module for any $0 \neq f \in k[x]$. In addition, if f is not invertible, then M/f^2M is not a $k[x]/(f)$ -module.

Exercise 3. Answer the following questions. Provide an explanation by a proof or a counterexample.

(1) Suppose that R is a Noetherian ring. Let $S \subset R$ be a subring. Is it true that S is Noetherian?

(2) Let R be a commutative Artinian ring. Is every prime ideal of R maximal?

Exercise 4. Let $I \subseteq R$ be an ideal.

(1) Show that

$$IM = \left\{ \sum_{i=1}^d r_i m_i \mid 1 \leq d \in \mathbb{Z}, r_i \in I, m_i \in M \right\}$$

is an R -submodule of M .

(2) Show that M/IM is an R/I -module with scalar multiplication given by

$$(x + I)(y + IM) = xy + IM.$$

From now on, fix $R := k[x, y]$, M the R -submodule generated by the element $(x, y) \in R \oplus R =: N$, and let I be the maximal ideal $I = Rx + Ry$ of R . Note that $R/I \cong k$ via the homomorphism $R \rightarrow k$ that evaluates x and y to 0.

(3) Show that $M \subseteq IN$ and hence $I(N/M) = IN/M$ as R -submodules of N/M .

(4) Show that L/IL is a two dimensional vector-space over k , where $L = N/M$.

Now, we change a little bit our setup, and we redefine M :

(5) Let M be the submodule generated by the two elements $(x, 0)$ and $(0, y)$ of $R \oplus R =: N$.
Is $N/M \cong R$?

[Hint: look at $\text{Ann}(N/M)$.]

Exercise 5. Let

$$0 \rightarrow M \rightarrow N \rightarrow N/M \rightarrow 0$$

be a short exact sequence of R -modules. For each of the following assertions either prove that the assertion holds or provide a counterexample.

- (1) If M and N/M are finitely generated, then N is too.
- (2) Conversely, if N is finitely generated, then N/M is finitely generated too.
- (3) If N is finitely generated, then M is finitely generated too.

Exercise 6. (1) Let

$$0 \rightarrow M \rightarrow N \rightarrow N/M \rightarrow 0$$

be a short exact sequence of R -modules. For each of the following assertions either prove that the assertion holds or provide a counterexample.

- If N is free, then N/M is free.
 - If N is free, then M is free.
 - If M and N/M are free, then N is free.
- (2) Let $R = \mathbb{Z}$. Is $\mathbb{Z}[x]/(x^2 + 1)\mathbb{Z}[x]$ a free R -module? How about $\mathbb{Z}[x]/(2x^2)\mathbb{Z}[x]$?
Is \mathbb{Q} a free R -module? Is it finitely generated?

Optional exercise. Not on the exam. Suggested if you are seriously interested in algebra.

Exercise 7. Let k be a field. In this exercise, we want to understand *differential operators* on $k[x]$. To this end, define the operator $\frac{\partial}{\partial x} \text{End}_k(k[x])$ by the usual rule

$$\frac{\partial}{\partial x}(x^n) := nx^{n-1}.$$

Define also $x \in \text{End}_k(k[x])$ defined by multiplication by x . Finally, define the subring $\mathcal{D} \subseteq \text{End}_k(k[x])$ to be the sub- k -algebra generated by x and $\frac{\partial}{\partial x}$.

We will show that this non-commutative ring behaves very differently, whether we work in characteristic zero or in positive characteristic.

- (1) Show that a basis of \mathcal{D} as a k -vector space is given by the elements $x^i \left(\frac{\partial}{\partial x}\right)^j$, where $(i, j) \in \mathbb{N}^2$ if $\text{char } k = 0$, and $i \in \mathbb{N}$ and $j \in \{0, 1, \dots, p-1\}$ if $\text{char } k = p > 0$.
- (2) Now we change the perspective and consider a quotient of the free k -algebra on two generators $\mathcal{D}^{form} = k\langle u, v \rangle / (uv - vu - 1)$. Prove that in \mathcal{D}^{form} we have the identity

$$uP(v) = \frac{\partial}{\partial v}P(v) + P(v)u$$

for all polynomials $P(v) \in k[v]$. Use this to prove that \mathcal{D}^{form} is generated as a k -vector space by $\{v^j u^i \mid (i, j) \in \mathbb{N}^2\}$.

- (3) Show that there are well defined ring homomorphisms ϕ and ψ from \mathcal{D}^{form} to $\text{End}_k(k[x])$, such that $\phi(u) = \frac{\partial}{\partial x}$ and $\phi(v) = x$, as well as $\psi(u) = x$ and $\psi(v) = -\frac{\partial}{\partial x}$. Show that ϕ and ψ are surjective onto \mathcal{D} , and define an isomorphism between \mathcal{D} and \mathcal{D}^{form} if and only if $\text{char}(k) = 0$.
- (4) Show that $k[x]$ is simple as a left \mathcal{D} -module (with left \mathcal{D} -module structure given by the inclusion $\mathcal{D} \subset \text{End}_k(k[x])$) in the case when $\text{char } k = 0$.
- (5) Determine the left submodules of $k[x]$ as a \mathcal{D} -module when $\text{char } k = 2$.

Remark 0.1. For any field k , regardless of the characteristic, one can define the notion of a differential operator. In characteristic zero, this definition agrees with our \mathcal{D} above, but this is not the case in characteristic $p > 0$. In other words, there are more exotic differential operators in positive characteristic, not spanned by $\frac{\partial}{\partial x}$ and $x!$ An example is what we usually write

$$\delta := \frac{1}{p} \left(\frac{\partial}{\partial x} \right)^p,$$

defined as

$$\delta(x^n) = \frac{n(n-1)\dots(n-p+1)}{p} x^{n-p},$$

where we note that, even if we work in characteristic p , the element

$$\frac{n(n-1)\dots(n-p+1)}{p}$$

makes sense in k , since it lives in \mathbb{Z} and there is a natural map $\mathbb{Z} \rightarrow k$

With this more general notion of differential operators, $k[x]$ becomes again simple (in any characteristic!).