EPFL - Fall 2025	Zsolt Patakfalvi
Rings and modules	**Exercises**
Sheet 13	18 December 2025

Exercise 1. Let R be a Noetherian ring. Show that R has only finitely many minimal prime ideals.

Hint: Reformulate this statement into a topological one by using spectra

Exercise 2. Let F be a field and let R be a ring, let $I = (f) \subseteq F[x]$ be a principal ideal, and let $\phi : F[x] \to R$ be a ring morphism. If we speak of extensions and contractions of ideals in this exercise, they are always understood to be with respect to ϕ. Let g be a generator of the ideal $I^{ec} \subseteq F[x]$, and note that g is uniquely defined up to multiplication by a unit. Give a formula for g in terms of the prime factors of f when ϕ is

(1) the localization $F[x] \to F[x]_x$.
(2) the localization $F[x] \to F[x]_{(x)}$ (i.e. localization at the prime ideal $(x) \subseteq F[x]$).

Additionally, characterize in both cases when $I^{ec} = I$, in terms of the prime factors of f.

Exercise 3. If $S \subseteq R$ is a ring extension and $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of S resp. R, respectively, we say that $\mathfrak{q}$ lies above $\mathfrak{p}$ if and only if $\mathfrak{q}^c = \mathfrak{p}$. Show the following:

(1) Let R be a UFD. Then an ideal $\mathfrak{p} \subseteq R$ is a prime ideal of height 1 if and only there exists an irreducible element $f \in R$ such that $\mathfrak{p} = (f)$.
(2) If $S \subseteq R$ is an integral extension and $\mathfrak{p} \subseteq S$ is a prime ideal, then all prime ideals lying over $\mathfrak{p}$ have height at most that of $\mathfrak{p}$, with equality for at least one of them. [*Hint:* Localize at $\mathfrak{p}$.]
(3) If $S \subseteq R$ is an integral extension of domains, then all primes of R lying over height 1 primes of S are of height 1.
(4) The ideal $\mathfrak{p} = (x^2 + y^2 + 1) \subseteq \mathbb{C}[x^2, y^2]$ is a height 1 prime, and there is a single prime in $\mathbb{C}[x, y]$ lying over it.

Exercise 4. Let R be a ring which is the quotient of a polynomial ring over an algebraically closed field F by a radical ideal. This naturally determines an algebraic set X whose coordinate ring is R. Noether normalisation says there is a subring $S \subseteq R$ such that $S \cong F[t_1, ..., t_r]$ and R is an integral extension of S. Give a geometric interpretation of Noether normalisation. That is, the inclusion $S \to R$ corresponds to a morphism f of algebraic sets. Prove that the fibres of f are finite, i.e. the preimage of any point in F^r under f consists of a finite set of points in X.

Exercise 5. Let F be an algebraically closed field. Calculate a primary decomposition for the ideals

(1) $(x^4 - 2x^3 - 4x^2 + 2x + 3) \subseteq F[x]$,
(2) $(x^2, xy^2) \subseteq F[x, y]$,
(3) $(x^2, xy, xz, yz) \subseteq F[x, y, z]$.

Exercise 6. Let $T \subseteq R$ be a multiplicative subset of a ring R and let $\{I_i\}_{1 \le i \le n}$ be finitely many ideals in R. By extension and contraction of ideals we shall mean extension and contraction via the natural morphism $R \to T^{-1}R$. Prove the following:

(1) $(\bigcap_i I_i)^{ec} = \bigcap_i I_i^{ec}$

(2) $\left(\bigcap_i I_i\right)^e = \bigcap_i I_i^e$

(3) Show that $T^{-1}\left(R/I\right) \cong T^{-1}R\Big/I^e$ as R-modules. Use this to endow $T^{-1}\left(R/I\right)$ with a ring structure, so that it becomes in fact an isomorphism of rings.

(4) If I is primary, and $u \notin \sqrt{I}$, then $(I : u) = I$

(5) For an ideal I of a ring R admitting a finite primary decomposition, let $I = \bigcap_i I_i$ be such a primary decomposition, and show the following

  (i) $I^e = \bigcap_{T\cap I_i=\varnothing} I_i^e$,

  (ii) $I^{ec} = \bigcap_{T\cap I_i=\varnothing} I_i$

(6) From now on, let $R = F[x,y]$ for a field F, $I_1 = (x)$, $I_2 = \mathfrak{m}^s$ where $\mathfrak{m} = (x,y)$ and $s > 1$ is some integer, $I_3 = (x, y-1)^2$, and $\mathfrak{p} \subseteq R$ a prime ideal for which we set $T = R \setminus \mathfrak{p}$. Show that

  (i) if $\mathfrak{p} = (x)$, then $T^{-1}\left(R/I_1 \cap I_2 \cap I_3\right) \cong F(y)$.

  (ii) if $\mathfrak{p} = (x,y)$, then $T^{-1}\left(R/I_1 \cap I_2 \cap I_3\right) \cong T^{-1}R\Big/I_1^e \cap I_2^e$

  (iii) if $\mathfrak{p} = (x,y)$, compute the smallest integer n such that $\left(\frac{x}{1}\right)^n \in T^{-1}\left(R/I_1 \cap I_2 \cap I_3\right)$ is zero.