

The goal of this exercise is to see that the statement of Exercise 8 is wrong without the algebraically closed assumption.

**Exercise 1.** (1) Let  $R \rightarrow S$  be a morphism of commutative rings (thus making  $S$  an  $R$ -algebra), and let  $I$  be an ideal of  $R[x_1, \dots, x_n]$ . Then we have an isomorphism of  $S$ -algebras

$$R[x_1, \dots, x_n]/I \otimes_R S \cong S[x_1, \dots, x_n]/(I)$$

[*Hint: First show it for  $I = 0$ , and then deduce the general case using right exactness of the tensor product. The case  $I = 0$  can be handled by a direct computation, or by showing that both sides satisfy the same universal property.*]

(2) Show that

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$$

and hence it is not a domain (but it is nevertheless reduced!)

(3) Show that

$$\mathbb{F}_p(x) \otimes_{\mathbb{F}_p(x^p)} \mathbb{F}_p(x) \cong \mathbb{F}_p(x)[t]/(t-x)^p$$

which is not even reduced.

**Exercise 2.** Let  $M$  be an  $A$ -module, and let  $\mathfrak{a}$  be an ideal in  $A$ . Show that the following are equivalent:

- (1)  $M = 0$ ,
- (2)  $M_{\mathfrak{p}} = 0$ , for every prime ideal  $\mathfrak{p} \subseteq A$ ,
- (3)  $M_{\mathfrak{m}} = 0$ , for every maximal ideal  $\mathfrak{m} \subseteq A$ .

Moreover, suppose that  $M$  is a finitely generated  $A$ -module, under this assumption prove that  $M = \mathfrak{a}M$  if and only if  $M_{\mathfrak{m}} = 0$  for all maximal ideals  $\mathfrak{m}$  satisfying  $\mathfrak{a} \subseteq \mathfrak{m}$ .

[*Hint/Remark: Although the exercise can be solved without directly proving the implication (3)  $\Rightarrow$  (2), it is highly instructive for anyone who thinks about studying more commutative algebra/algebraic geometry, to think through the (3)  $\Rightarrow$  (2) implication using Exercise 7.*]

**Exercise 3.** Let  $R = F[x]$ , where  $F$  is a field.

- (1) If  $F$  is algebraically closed, then show that for every prime ideal  $\mathfrak{p}$  of  $R$ , either  $R_{\mathfrak{p}} \cong F(x)$  or  $R_{\mathfrak{p}} \cong F[x]_{(x)}$ , where these isomorphisms are isomorphisms of  $F$ -algebras. Show that the above two cases are not isomorphic.
- (2) If  $F = \mathbb{R}$ , then show that up to ring isomorphism there are three possibilities for  $R_{\mathfrak{p}}$ , where  $\mathfrak{p}$  is a prime ideal of  $F[x]$ .

[*Hint: To tell the three cases apart, consider the residue field, to show that there are only three cases, apply linear transformations to  $x$ .*]

- (3) Show that if  $F$  is algebraically closed, then  $F[x, y]$  has infinitely many prime ideals  $\mathfrak{p}$  for which  $F[x, y]_{\mathfrak{p}}$  are pairwise non-isomorphic  $F$ -algebras. For this, you can use the following theorem of algebraic geometry:

**Theorem.** *There exists a sequence of irreducible polynomials  $(f_d)_{d \in \mathbb{N} \setminus \{0,2\}}$  in  $F[x, y]$  such that  $f_d$  is of degree  $d$  and such that the fields  $\text{Frac}(F[x, y]/(f_d))$  are pairwise non-isomorphic as  $F$ -algebras.*

**Exercise 4.** Let  $F$  be an algebraically closed field.

- (1) List the prime ideals of  $R = F[x, y]/(xy)$ .  
 [Hint: Consider the implications of a containment  $xy \in \mathfrak{p}$ , for a prime ideal  $\mathfrak{p}$ . Consider the projections  $R \rightarrow R/(x)$  and  $R \rightarrow R/(y)$  and use that you know the prime ideals of  $F[y]$  and  $F[x]$ .]
- (2) Show that for all prime ideals  $\mathfrak{p}$  of  $R$ ,  $R_{\mathfrak{p}}$  falls into three cases up to  $F$ -algebra isomorphism, one which is a field, one which is a domain but not a field and one which is not a domain.

**Exercise 5.** Let  $R$  be a commutative ring.

- (1) Let  $T \subseteq R$  a multiplicatively closed subset of  $R$ . Let  $\mathfrak{q}$  be a prime ideal of  $T^{-1}R$ . Let  $\mathfrak{q}^c$  be the contraction of  $\mathfrak{q}$  under  $R \rightarrow T^{-1}R$ . Prove that  $\text{ht}(\mathfrak{q}) = \text{ht}(\mathfrak{q}^c)$ .
- (2) Let  $\mathfrak{p}$  be a prime ideal of  $R$ . Prove that  $\text{ht}(\mathfrak{p}) = \dim R_{\mathfrak{p}}$ .

**Exercise 6.** Let  $S \rightarrow R$  be a morphism of rings. Show that a prime ideal  $\mathfrak{p}$  of  $S$  is the contraction of a prime ideal of  $R$  if and only if  $\mathfrak{p}^{ec} = \mathfrak{p}$ .

[Hint: For one direction use ideas from the proof of Going-Up Theorem (Proposition 9.4.2 of the lecture notes).]

**Exercise 7.** Let  $R$  be a ring, let  $M$  be an  $R$ -module and let  $T, S \subseteq R$  be two multiplicatively closed subsets of  $R$ . Define  $ST := \{st \mid s \in S, t \in T\}$  and  $\tilde{S} := \{s/1 \mid s \in S\} \subseteq T^{-1}R$ .

- (1) Show that  $ST$  and  $\tilde{S}$  are multiplicatively closed subsets of  $R$  resp.  $T^{-1}R$ .
- (2) Show that there exists a ring morphism  $\tilde{S}^{-1}(T^{-1}R) \rightarrow (ST)^{-1}R$  sending  $(r/t)/(s/1) \in \tilde{S}^{-1}(T^{-1}R)$  to  $r/(st) \in (ST)^{-1}R$ . Show further that this is an isomorphism.
- (3) Show that  $\tilde{S}^{-1}(T^{-1}M)$  and  $(ST)^{-1}M$  are isomorphic as  $(ST)^{-1}R$ -modules, where the  $(ST)^{-1}R$ -module structure of  $\tilde{S}^{-1}(T^{-1}M)$  is provided via the isomorphism of the previous point.
- (4) Show that if  $T \subseteq S$  then  $ST = S$ , and formulate the results of points (2) and (3) in this case.

**Exercise 8.** In Exercise 6 of sheet 10, we saw how to construct the tensor product of two  $R$ -algebras. The goal is to show the following result:

**Proposition 0.1.** *Let  $k$  be an algebraically closed field, and let  $R, S$  two finitely generated  $k$ -algebras which are domains. Then  $R \otimes_k S$  is again a domain.*

Recall the following important facts from the course:

- Nullstellensatz (Theorem 6.5.4 from the notes)
- For any finitely generated  $k$ -algebra  $T$  and any maximal ideal  $\mathfrak{m}$ , the composition  $k \rightarrow T \rightarrow T/\mathfrak{m}$  is an isomorphism (see the proof of the weak Nullstellensatz, which is Theorem 6.2.2 in the notes).

Proceed as follows:

- (1) Let  $T$  be a finitely generated  $k$ -algebra which is a domain, and let  $a_1, \dots, a_s \in T$  be non-zero. Show that there is a maximal ideal  $\mathfrak{m}$  of  $T$  such that  $a_i \notin \mathfrak{m}$  for all  $i$ .  
 [Hint: write  $T$  as a quotient of a polynomial ring, and use Nullstellensatz.]
- (2) Show that any element in  $R \otimes_k S$  can be written as

$$\sum_i a_i \otimes b_i$$

with the  $b_i$ 's linearly independent over  $k$ .

(3) Assume that

$$\left( \sum_i a_i \otimes b_i \right) \cdot \left( \sum_j a'_j \otimes b'_j \right) = 0$$

where both families  $(b_i)_i$  and  $(b'_j)_j$  are linearly independent. Let  $\mathfrak{m}$  be a maximal ideal not containing any of the  $a_i, a'_j$ .

Show by applying the ring map

$$R \otimes_k S \rightarrow R/\mathfrak{m} \otimes_k S \cong S$$

that one of the factors must be zero, and hence conclude that  $R \otimes_k S$  is a domain.

*Remark 0.2.* By the first exercise, the algebraic closedness condition is crucial in the above.