

In this exercise sheet, all rings are commutative.

General definitions around the notion of a functor

Let us see how the construction of the Ext-modules can be adapted to an additive right exact covariant functor.

Definition 1. Given R a commutative ring, we set Mod_R to be the category of R -modules.

A covariant functor $F : \text{Mod}_R \rightarrow \text{Mod}_S$ is said to be *additive* if $F(0) = 0$ (for both the 0 map and the 0 object) and for all R -modules M, N , the natural map

$$F(M \oplus N) \rightarrow F(M) \oplus F(N)$$

(induced by the projections) is an isomorphism.

An additive covariant functor is said to be *right exact* if for any short exact sequence

$$M \rightarrow N \rightarrow P \rightarrow 0$$

the sequence

$$F(M) \rightarrow F(N) \rightarrow F(P) \rightarrow 0$$

is exact.

Definition 2. Let F be an additive right exact covariant functor. For an R -module M , perform the following construction.

- (1) Take a projective resolution $P_\bullet \rightarrow M$ of M .
- (2) Apply F to the projective resolution to obtain a chain complex $F(P_\bullet)$ given by

$$\cdots \rightarrow F(P_1) \rightarrow F(P_0) \rightarrow 0.$$

- (3) Define $L_i F(M) = H_i(F(P_\bullet))$, i.e. $L_i F(M)$ is the i -th homology group of the chain complex $F(P_\bullet)$.

One can show that $L_i F(M)$ doesn't depend on the projective resolution, so that it gives a well defined object. In a similar way as for the Ext-modules, one can also construct $L_i F(f) : L_i F(M) \rightarrow L_i F(M')$ for any R -module homomorphism $f : M \rightarrow M'$, and show that this turns $L_i F$ into a covariant functor. It is called the i -th left derived functor of F .

As mentioned in Remark 9.1.7 of the printed course notes, the Tor-functors are the left derived functors of $- \otimes_R N$. In formulas, $\text{Tor}_i^R(M, N) = L_i(- \otimes_R N)(M)$.

One can prove the corresponding statement of Theorem 5.5.6 also for the left derived functors of a covariant functor:

Theorem. Let F be a right exact covariant functor and let $0 \rightarrow K \rightarrow M \rightarrow L \rightarrow 0$ be a short exact sequence. Then there exists a long exact sequence

$$\cdots \rightarrow L_2F(K) \rightarrow L_1F(K) \rightarrow L_1F(M) \rightarrow L_1F(L) \rightarrow FK \rightarrow FM \rightarrow FL \rightarrow 0$$

Exercise 1. Let R be a ring and let M, N be R -modules. Prove that $\mathrm{Tor}_0^R(M, N) \cong M \otimes_R N$. [Hint: Try to adapt the proof of Proposition 5.3.8 in the printed course notes.]

Exercise 2. Let R be a ring and N an R -module. We say that N is *flat* if for every short exact sequence of R -modules

$$0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$$

the sequence

$$0 \rightarrow M \otimes_R N \rightarrow M' \otimes_R N \rightarrow M'' \otimes_R N \rightarrow 0$$

is exact. Prove that the following are equivalent:

- (1) N is flat,
- (2) $\mathrm{Tor}_i^R(M, N) = 0$ for every R -module M and every $i > 0$,
- (3) $\mathrm{Tor}_1^R(M, N) = 0$ for every R -module M .

[Hint: For (1) \Rightarrow (2) take a free resolution of M and tensor it with N to compute the Tor-functors. For (3) \Rightarrow (1) use the long exact sequence for left derived functors.]

Exercise 3. Let $R = k[x, y]$ where k is a field. Consider the R -modules $M := (x, y)$ (i.e. the ideal generated by x and y) and $N := R/M$.

- (1) Compute $\mathrm{Tor}_i^R(M, N)$ for all integers $i \geq 0$.
[Hint: Use the definition.]
- (2) Is N flat?
- (3) Compute $\mathrm{Tor}_i^R(N, N)$ for all integers $i \geq 0$.
[Hint: Use the long exact sequence.]

Exercise 4. Let R be a ring.

- (1) Prove that free R -modules are flat.
- (2) Prove that projective R -modules are flat.
[Hint: Use the characterization of projective modules as direct summands of free modules.]
- (3) Assume that R is an integral domain. Determine for which ideals I of R the R -module R/I is flat.

Remark 0.1. There exists a partial converse of (2): a flat finitely generated module over a Noetherian ring is projective.

The finite generation hypothesis is very important, as the \mathbb{Z} -module \mathbb{Q} is flat but not projective. There are also counter-examples in the non-Noetherian case.

Exercise 5. Let R be a ring containing a multiplicatively closed subset T , and let M be an R -module. Show that there is an isomorphism of R -modules

$$T^{-1}M \cong T^{-1}R \otimes_R M.$$

Further show that this is an isomorphism of $T^{-1}R$ -modules.

[*Remark:* The right hand side naturally has the structure of a $T^{-1}R$ -module by point (1) of Exercise 6 on Sheet 10.]

Exercise 6. Let R be a ring with multiplicative subset T , and suppose that L , M and N are R -modules.

- (1) Show that if there is an R -module homomorphism $f : M \rightarrow N$ then there is a natural $T^{-1}R$ -module homomorphism $f_T : T^{-1}M \rightarrow T^{-1}N$.
- (2) Show that there is an isomorphism of $T^{-1}R$ -modules $T^{-1}(M \oplus N) \cong (T^{-1}M) \oplus (T^{-1}N)$.
- (3) Suppose there is an exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0.$$

Prove that the sequence

$$0 \rightarrow T^{-1}L \rightarrow T^{-1}M \rightarrow T^{-1}N \rightarrow 0$$

is also exact. Deduce that if $L \subset M$ is a sub R -module, then $T^{-1}(M/L) \cong T^{-1}M / T^{-1}L$ and that localization by T is an exact functor of R -modules and that $T^{-1}R$ is a flat R -module.

- (4) Let \mathfrak{p} be a prime ideal of R . Show that there is an isomorphism of rings $\text{Frac}(R/\mathfrak{p}) \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

[*Remark:* For a local ring A with maximal ideal \mathfrak{m} we call A/\mathfrak{m} the residue field of A .]

Exercise 7. Let R be a ring, let S be a multiplicatively closed subset, and let M and N be R -modules. Show that for all $i \geq 0$,

$$S^{-1} \text{Tor}_i^R(M, N) \cong \text{Tor}_i^{S^{-1}R}(S^{-1}M, S^{-1}N).$$

If furthermore R is Noetherian and M is finitely generated, then also

$$S^{-1} \text{Ext}_R^i(M, N) \cong \text{Ext}_{S^{-1}R}^i(S^{-1}M, S^{-1}N).$$

Exercise 8. Many algebraic geometers tend not to say “projective modules”, but more “locally free modules” or even “vector bundles” (as in differential geometry!). It would take us too far to understand the name “vector bundle” (see the course Algebraic Geometry II), but we can already understand the name “locally free”.

Let R be a commutative Noetherian ring, and let M be a finitely generated R -module. Our goal is to show that M is projective if and only if there exists a collection of elements $\{a_i\}_i$ of R such that each M_{a_i} is free as an R_{a_i} -module, and $\text{Spec } R = \bigcup_i D(a_i)$ where $D(a_i)$ is the complement of $V((a_i))$ in $\text{Spec } R$ (we call such a module a *locally free module*).

Throughout, you may freely use the following statement, which will be an exercise in the next exercise sheet.

Proposition 0.2. *For any finitely generated R -module N , we have*

$$N = 0 \iff N_{\mathfrak{p}} = 0 \forall \mathfrak{p} \in \text{Spec}(R).$$

We will prove our result in several steps:

- Show that M is locally free if and only if $M_{\mathfrak{p}}$ is free as an $R_{\mathfrak{p}}$ -module for all $\mathfrak{p} \in \text{Spec}(R)$.
Hint: It can be useful to show that for any finitely generated module N and $\mathfrak{p} \in \text{Spec}(R)$, if $N_{\mathfrak{p}} = 0$, then there exists a $\notin \mathfrak{p}$ such that $N_a = 0$.

- Show that if M is locally free, then it is projective.

Hint: Remember that projectivity can be detected by exactness of a certain functor.

- Assume that R is a *local ring* (i.e. it has a unique maximal ideal). Show that M is projective if and only if it is free.

Hint: Let \mathfrak{m} be the maximal ideal of R . A good starting idea would be to show that if $m_1, \dots, m_n \in M$ is a basis of $M/\mathfrak{m}M$, then M is in fact generated by m_1, \dots, m_n . Nakayama's lemma can help.

- Conclude that if M is projective, then it is locally free.