November 3, 2025

Problem Set 7

Exercise 1. (a) Let G_1, G_2 be two groups. Show that the group $G_1 \times G_2$ is abelian if and only if both G_1 and G_2 are abelian.

(b) If G_1 and G_2 are both cyclic groups, find the conditions for $G_1 \times G_2$ to be cyclic.

Exercise 2. We proved in class that in any abelian group G such that its order $|G|$ is divisible by a prime p, there exists an element of order p. Use this to prove the same statement for non-abelian groups. *Hint:* Use the class equation for G and consider two cases: the order of the center $|Z(G)|$ is divisible by p and the order of $|Z(G)|$ is not divisible by p.
As a conclusion, we obtain Cauchy's theorem: any group G such that $|G|$ is divisible by a prime p, contains an element of order p.

Exercise 3. In each case provide the list of elementary divisors and invariant factors for each group.

(a) List all abelian groups of order 8 (up to an isomorphism).

(b) List all abelian groups of order 120 (up to an isomorphism).

Exercise 4. (a) Find all abelian groups of order 180 up to an isomorphism and for each group list the elementary divisors (the prime powers in the decomposition of an abelian group as a product of cyclic p-groups) and the invariant factors (the integers $d_1, d_2, \dots d_k$ such that $d_k | d_{k-1} | \dots | d_2 | d_1$ and $d_1 d_2 \dots d_k$ equals the order of the group).

(b) Find the prime power divisors and invariant factors for the group $C_3 \times C_{15} \times C_{20}$.

(c) Are the groups $G_1 = C_{16} \times C_{12} \times C_5$ and $G_2 = C_{10} \times C_{24} \times C_4$ isomorphic?

Exercise 5. Let $G = (\mathbb{Z}/315\mathbb{Z})^*$, the group of units in $\mathbb{Z}/315\mathbb{Z}$ with respect to the multiplication.

(a) Find the order of G. *Hint:* You can use the multiplicative property of the Euler's totient function: $\varphi(mn) = \varphi(m)\varphi(n)$ if $\gcd(m, n) = 1$.

(b) Show that for all $x \in G$, we have $x^{288} = 1$.

(c) Show that for all $m \in \mathbb{Z}$ such that $(m, 630) = 1$, we have $m^{144} \equiv 1 \pmod{315}$.

(d) Find all solutions modulo 315 of the equation $x^{25} = 1$. (Hint: Note that such x is invertible, and therefore an element in $(\mathbb{Z}/315\mathbb{Z})^*$.

Exercise 6. Use Lagrange's theorem to classify all groups of order 6 up to isomorphism. Let $|G| = 6$.

(a) Consider possible orders of elements in G. Consider the case when G contains an element of order 6.

(b) Show that G cannot contain only elements of order 1 and 2.

(c) Show that G cannot contain only elements of order 1 and 3.

(d) Identify the group of 6 elements that contains elements of order 1, 2 and 3, and does not contain an element of order 6.