October 27, 2025

Problem Set 6

Exercise 1. Consider the permutation with the cycle decomposition $g = (12)(345)$ in the symmetric group S_5. Let $H = \langle g \rangle \subset S_5$ be the subgroup generated by this element.

(a) Find the order of H.

(b) Find the stabilizer subgroup in H of the element 1, and the orbit of 1 under the action of H.

(c) Find the stabilizer subgroup in H of the element 4, and the orbit of 4 under the action of H.

(d) In both cases, check the formula

$$[H : \mathrm{Stab}(x)] = |\mathrm{Orb}(x)|.$$

Exercise 2. (a) Recall that for any $\pi \in S_n$ and any cycle $c \in S_n$, the element $\pi c \pi^{-1}$ is the cycle obtained by replacing each integer i in the cycle c with the integer $\pi(i)$. Use this property to show that $g_1, g_2 \in S_n$ are conjugate if and only if they decompose as a product of disjoint cycles of the same lengths.

(b) The orbits under the conjugation action are called the conjugacy classes of a group. Describe the conjugacy classes in the group S_5.

(c) Recall that the class equation of a finite group is $|G| = |Z| + \sum_{i=1}^{m} |C_i|$, where Z is the center of G (the set of all one-element conjugacy classes), and $\{C_i\}_{i=1}^{m}$ is the set of all nontrivial conjugacy classes of G. Count the number of elements in each conjugacy class and write the class equation for S_5.

Exercise 3. Let G be a group, and $x \in G$ an element. The orbit of the element x under the conjugation action of G on itself, $C_x = \{gxg^{-1}\}_{g \in G}$ is called the conjugacy class of x in G. The subgroup $G_x = \{g \in G : gxg^{-1} = x\}$ is called the centralizer of x in G. Let G be the dihedral group $D_4 = \langle r, s \mid s^2 = 1, r^4 = 1, srs = r^{-1} \rangle$.

(a) For each $x \in D_4$ find its centralizer subgroup.

(b) Describe all the conjugacy classes in $G = D_4$.

(c) Check the Orbit-Stabilizer formula $|C_x| = [G : G_x]$ for each conjugacy class in D_4 and write the class equation for D_4.

Exercise 4. Consider the dihedral group D_4 acting as the group of symmetries of a square in $\mathbb{R}^2$ centered at the origin. Then each element of D_4 acts on the set of vertices by permutations. This defines an injective (with kernel containing only the identity element) group homomorphism

$$\phi : \; D_4 \to S_4.$$

Find the subgroup $H = \mathrm{Im}(\phi) \subset S_4$, and write the elements of H as products of disjoint cycles in S_4.