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Problem Set 5 Solutions

Exercise 1. Let G be a group and $H \subset G$ a subgroup of index 2. Show that H is a normal subgroup.

Solution 1. Let $[G : H] = 2$. Then there are two left cosets of H in G: $\{H, xH\}$. If $g \in H$, then $ghg^{-1} \in H$ for any $h \in H$. If $g \in G \setminus H$, then $g = xh_1$, and for any $h \in H$ $ghg^{-1} = xh_1hh_1^{-1}x^{-1}$. If this is in xH, then $h_1hh_1^{-1}x^{-1} \in H$, and therefore $x \in H$, contradiction. Therefore, $ghg^{-1} \in H$ for any $g \in G$ and $h \in H$, and H is normal in G.

Exercise 2. Let G be a group, and $f : G \to G$ a map defined by $f(g) = g^2$ for any $g \in G$. Find the conditions on G for f to be a group homomorphism.

Solution 2. Assume f is a homomorphism then f satisfies the property:

$$f(gh) = f(g)f(h)$$

where $g, h \in G$
Notice that since f is a homomorphism:

$$f(gh) = (gh)^2 = ghgh$$
$$f(g)f(h) = g^2h^2 = gghh$$

Since $f(gh) = f(g)f(h)$ then by substitution:

$$ghgh = gghh$$

Multiplying the right by h^{-1} and the left by g^{-1} we obtain that:

$$hg = gh$$

which means G is abelian. Also if G is abelian, then $f(gh) = ghgh = g^2h^2 = f(g)f(h)$, so f is indeed a homomorphism.

Exercise 3. Construct an injective (with trivial kernel) group homomorphism from the cyclic group C_4 to the symmetric group S_4. Describe its image in S_4 in terms of the cycle notation. How many different injective homomorphisms from C_4 to S_4 can you define?

Solution 3. To construct an injective homomorphism from the cyclic group C_4, it suffices to map the generator $s \in C_4$ to an element of order 4 in S_4. There are 6 such elements, namely, all 4-cycles: $\{(1234), (1243), (1324), (1342), (1423), (1432)\}$. The map sending s to any of these elements defines a homomorphism of groups with trivial kernel, and all of these form a complete list of the distinct injective homomorphisms from C_4 to S_4.

Exercise 4. (a) Write the permutations $(2\ 3\ 4\ 5)(4\ 1\ 2)$ and $(3\ 5\ 4)(3\ 6\ 1)(5\ 3)(1\ 2\ 4\ 6)(4\ 3\ 5\ 1)(7\ 3)(1\ 3\ 6)$ as a product of disjoint cycles and then as a product of transpositions.

(b) Determine if the following subsets $H = \{(12)(34); (13)(24); (14)(23); (1)\}$ and $K = \{(13)(34); (13); (34); (1)\}$ are subgroups in $G = S_4$.

(c) What is the order of the element $a = (1\ 3\ 5)(2\ 4\ 6)$ and of the element $b = (1\ 3\ 5)(2\ 5\ 6)$ in S_6? Find an element of order 6 in S_5. *Hint:* Recall that if $ab = ba$ for group elements $a, b \in G$, and the orders $o(a)$ and $o(b)$ are mutually prime, then ab is of order $o(ab) = o(a)o(b)$ (See PS4, Ex. 3(c)).

Solution 4. (a) The permutations can be decomposed in the product of the following disjoint cycles:

- $(2\ 3\ 4\ 5)(4\ 1\ 2) = (1\ 3\ 4)(2\ 5) = (1\ 4)(1\ 3)(2\ 5)$.
- $(3\ 5\ 4)(3\ 6\ 1)(5\ 3)(1\ 2\ 4\ 6)(4\ 3\ 5\ 1)(7\ 3)(1\ 3\ 6) = (1\ 7\ 6)(2\ 3\ 5)(4) = (1\ 6)(1\ 7)(2\ 5)(2\ 3)$.

(b) Notice that for every element $a, b \in H$, $ab \in H$. Moreover for every $a \in H$, $a^{-1} \in H$ and $e \in H$. Therefore, H is a subgroup of S_4. On the other hand, K is not a subgroup of S_4 since $(143) = (34)(13)$ doesn't belong to K.

(c) Note that the element $a = (1\ 3\ 5)(2\ 4\ 6)$ is the product of two disjoint 3-cycles. Therefore it has order 3. Observe that $(1\ 3\ 5)(2\ 5\ 6) = (1\ 3\ 5\ 6\ 2)$ is a 5-cycle, hence it's order is 5. Finally, the order of the element (123)(45) is 6, because it is a product of disjoint cycles of coprime orders 3 and 2.

Exercise 5. It is known that the symmetric group S_n is generated by all transpositions $\{(ik)\}_{1\leq i<k\leq n}$. Show that S_n is also generated by the following sets of elements:

(a) All transpositions of the form $\{(1, i)\}$, $2 \leq i \leq n$.

(b) The transposition (12) and the n-cycle $(123\dots n)$.

Hint: For any $\pi, \rho \in S_n$, the cycle decomposition of $\pi\rho\pi^{-1}$ is obtained by replacing each integer i in the cycle decomposition of ρ with the integer $\pi(i)$.

Solution 5. (a) Using the hint, we have $(1i)(1k)(1i) = (ik)$, so we can obtain all transpositions as products of transpositions of the form $(1i)$.

(b) Note that $(123\dots n)^{-1} = (123\dots n)^{n-1}$. Then we have,

$$(123\dots n)(12)(123\dots n)^{n-1} = (23),$$

Conjugating further by the n-cycle, we can obtain transpositions $(i, i+1)$ for all $1 \leq i \leq n-1$. Then

$$(23)(12)(23) = (13), \quad (34)(13)(34) = (14), \dots$$

so we can obtain all transpositions of the form $(1i)$, $2 \leq i \leq n$. By (a), they generate S_n.

Exercise 6. Let G be a group and H a subgroup in G. We say that H is *proper* in G if H is not equal to G, and *maximal proper* in G if H is not equal to G and no other proper subgroup of G contains H.

(a) Let H be a subgroup of $(\mathbb{Z}, +)$. Show that H is maximal proper if and only if $H = p\mathbb{Z}$ for a prime number p.

(b) Find all subgroups in $(\mathbb{Z}, +)$ that contain $72\mathbb{Z}$ as a proper subgroup. Which of them are maximal proper subgroups?

Solution 6. (a) We know that any subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for $n \in \mathbb{N}$. Since $n\mathbb{Z} \subseteq m\mathbb{Z}$ if and only if m divides n. Then for any prime number p, the subgroup $p\mathbb{Z}$ cannot be contained in any other proper subgroup of $\mathbb{Z}$. Therefore it's maximal.

(b) Observe that $72 = 2^3 \cdot 3^2$. Therefore $72\mathbb{Z}$ is contained in $2\mathbb{Z}$, $3\mathbb{Z}$, $4\mathbb{Z}$, $6\mathbb{Z}$, $8\mathbb{Z}$, $9\mathbb{Z}$, $12\mathbb{Z}$, $18\mathbb{Z}$, $24\mathbb{Z}$, $36\mathbb{Z}$ and $72\mathbb{Z}$. From the previous point the maximal subgroups are $2\mathbb{Z}$ and $3\mathbb{Z}$.