December 15, 2025

Problem Set 13

Exercise 1. Let $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ be the field of 3 elements . Let $I = ((X^2+1))$ be the ideal in $\mathbb{F}_3[X]$.

(a) Show that X^2+1 is irreducible in $\mathbb{F}_3[X]$ and deduce that the quotient ring $\mathbb{F}_3[X]/I$ is a field.

(b) How many elements are there in this field? List all the elements.

(c) Show that $g(X) = 2X+1$ is a generator of the multiplicative group of units of this field.

Exercise 2. Let $K = \mathbb{Q}[X]/(X^2+2X+3)$. Denote by α the class of X in K.

(a) Show that K is a field.

(b) Show that $2\alpha - 1$ is nonzero and compute its inverse in K.

Exercise 3. Let $K = \mathbb{F}_7[X]/(X^3-2)$

(a) Show that the polynomial $P(X) = X^3 - 2$ is irreducible over $\mathbb{F}_7$.

(b) Decompose the polynomial (X^3-2) into irreducible factors over K.

(d) Give a basis of K as a vector space over $\mathbb{F}_7$.

(e) Find the number of elements of K.

Exercise 4. (a) Find an explicit isomorphism between the rings $\mathbb{F}_2[x]/(x^2)$ and $\mathbb{F}_2[x]/(x^2+1)$. Are they also isomorphic to the ring $\mathbb{Z}/4\mathbb{Z}$? Is any of these rings an integral domain?

(b) Show that the ring $\mathbb{F}_2[x]/(x^2+x+1)$ is a field of 4 elements and therefore is not isomorphic to $\mathbb{F}_2[x]/(x^2)$ or $\mathbb{Z}/4\mathbb{Z}$.

(c) Check that the group of units of the field $\mathbb{F}_2[x]/(x^2+x+1)$ is cyclic and find a generator.

(d) The fields $K_1 = \mathbb{F}_2[x]/(x^3+x+1)$ and $K_2 = \mathbb{F}_2[x]/(x^3+x^2+1)$ are isomorphic. Find an explicit isomorphism between them.