November 24, 2025

Problem Set 10

Exercise 1. Recall that a ring homomorphism between two commutative rings $f : A \to B$ is a map that satisfies the conditions: (1) $f(x+y) = f(x) + f(y)$, (2) $f(xy) = f(x)f(y)$ for any $x, y \in A$, and (3) $f(1_A) = 1_B$.

(a) Show that the map $F : \mathbb{C} \to M_2(\mathbb{R})$ from complex numbers to 2×2 real matrices defined by

$$F(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

is a ring homomorphism. Find its image and kernel.

(b) Let A be a commutative ring and $I \subset A$ an ideal. Prove that the map $\psi : A \to A/I$ sending $\psi(a) = a + I$ is a ring homomorphism. Find its image and kernel.

Exercise 2. Find the characteristic of the following rings :

(a) $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$.

(b) $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$.

(c) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$.

(d) Let A and B be rings and c_A, c_B their characteristics. What is the characteristic of the ring $A \times B$?

(e) $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$

Exercise 3.
Solve the system of congruences:

$$\begin{cases} x \equiv 2 \pmod 3 \\ x \equiv 4 \pmod 5 \\ x \equiv 3 \pmod 7. \end{cases}$$

Use the Chinese remainder theorem to show that a solution exists. Then find the complete set of solutions by solving the congruences consecutively. Solve the congruences (mod 3) and (mod 5) first, then use the obtained solution (mod 15) together with the last congruence (mod 7).

Exercise 4. Is the ring $\mathbb{Z}/180\mathbb{Z}$ isomorphic to

$$\begin{aligned} &(a) \quad \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3Z \times \mathbb{Z}/15\mathbb{Z}, \\ &(b) \quad \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/4Z \times \mathbb{Z}/5\mathbb{Z}, \\ &(c) \quad \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6Z \times \mathbb{Z}/5\mathbb{Z}? \end{aligned}$$

Hint: To prove that two rings are not isomorphic you can compare the number of invertible elements or the characteristics of the rings.