Algebra MATH-310

Lecture 7

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Plan of the course

1. Integers: 1 lecture
2. Groups: 6 lectures
3. Rings and fields: 5 lectures
4. Review: 1 lecture

Today: Groups: lecture 6

last lecture on groups!

a) Groups: general picture
b) Classification of simple finite abelian groups
c) Direct product of groups
d) Classification of finite abelian groups
e) Elementary divisors and invariant factors: examples

Finite groups

	Abelian	Non-abelian
Definition	$ab = ba \ \forall a, b \in G$	$\exists a, b \in G : \quad ab \neq ba$
Normal subgroups	All subgroups	$H \trianglelefteq G : \ gHg^{-1} \in H \ \forall g \in G$
Conjugacy classes	$\lvert C_i \rvert = 1 \ \ \forall C_i$	$\exists C_i : \ \lvert C_i \rvert > 1$
Class equation	$\lvert G \rvert = \lvert Z(G) \rvert$	$\lvert G \rvert = \lvert Z(G) \rvert + \sum_{i, \, \lvert C_i \rvert > 1} \lvert C_i \rvert,$
Examples	Cyclic groups C_n Others ?	Dihedral groups D_n Symmetric groups $S_n \supset A_n$ alternating

Classification of finite abelian simple groups

Definition

A group G is simple if G has no proper nontrivial normal subgroups.

Theorem

Recall from last time

A group or order divisible by a prime p contains an element of order p.

Proposition

If G is a simple finite abelian group, then G is isomorphic to a cyclic group C_p of prime order.

nontrivial

Proof: $|G| = 1 \Rightarrow G = \{1\}$; otherwise $\exists$ prime $p : p \mid |G|$

$\Rightarrow$ By Theorem $\exists$ an element $t \in G$ of order p

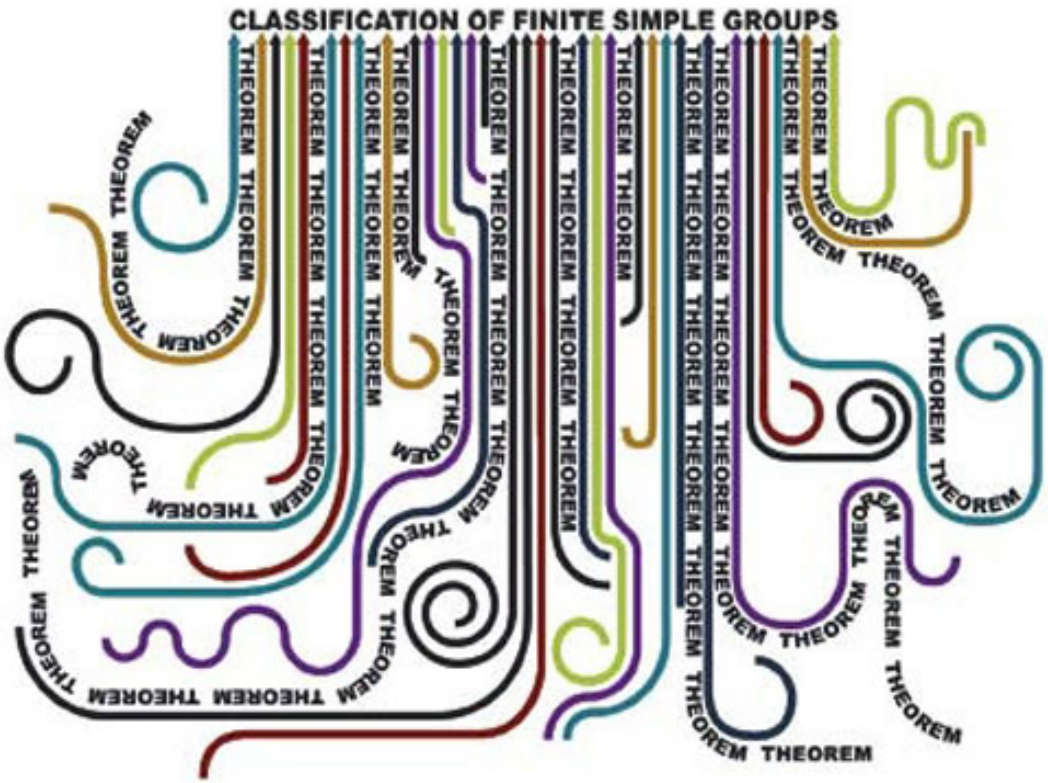
$\Rightarrow \langle t \rangle \subset G$ is a subgroup $\langle t \rangle \simeq C_p$

Any subgp is normal in an abelian $G \Rightarrow G = C_p$.

Classification of finite abelian and non-abelian simple groups 1832 - 2012

180 years of work by more than 30 mathematicians.

Answer: 18 infinite series and 27 exceptional groups. The order of the biggest exceptional simple group, The Monster, is about $8 \cdot 10^{53}$.



Our goal today: classification of all finite abelian groups

Direct product of groups (AICC II)

Definition

Let G, H be groups. The direct product $G \times H$ is the set of pairs $G \times H = \{(g, h),\ g \in G, h \in H\}$ with multiplication $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$, the neutral element $(1_G, 1_H)$ and the inverse $(g^{-1}, h^{-1})(g, h) = (1_G, 1_H)$.

Example: $G = C_2 \times C_3$. $= \{(g,h),\ g \in C_2, h \in C_3\}$ $C_2 = \langle a : a^2 = 1 \rangle$, $C_3 = \langle b : b^3 = 1 \rangle$

$\{(1,1), (1,b), (1,b^2), (a,1), (a,b), (a,b^2)\} = C_2 \times C_3$ Let $t = (a,b)$

$\Rightarrow t^2 = (1,b^2)$, $t^3 = (a,1)$, $t^4 = (1,b)$, $t^5 = (a,b^2)$, $t^6 = (1,1)$

$t \in C_2 \times C_3$ has order 6, $|C_2 \times C_3| = 6$

$\Rightarrow C_2 \times C_3 \simeq C_6$ is cyclic

Question: Is $C_n \times C_m \simeq C_{nm}$ always? — No.

Direct product of groups

Example: $G = \underset{a^2=1}{C_2} \times \underset{b^2=1}{C_2}$. $= \{(1,1), \overset{t}{(1,b)}, \overset{g}{(a,1)}, \overset{tg}{(a,b)}\} \not\cong C_4$

each elt in $C_2 \times C_2$ has order 2 : $t^2=1$, $g^2=1$, $(tg)^2=1$

$\Rightarrow$ $C_2 \times C_2$ does not have an elt of order 4.

$C_2 \times C_2 \not\cong C_4$

Remark

Suppose $(a, b) \in C_n \times C_m$ such that $o(a) = n$, $o(b) = m$. then $(a, b)^s = (a^s, b^s) = (1, 1)$ implies $o(a)$ divides s and $o(b)$ divides s. Therefore, the order of (a, b) is $\operatorname{lcm}(o(a), o(b)) = \operatorname{lcm}(n, m)$.

Direct product of cyclic groups

Proposition

$C_n \times C_m \simeq C_{nm}$ if and only if $\gcd(n, m) = 1$.

Proof: PS 7. $lcm(n,m) = nm \iff gcd(n,m) = 1$

Corollary

Let C_n be a cyclic group such that $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ is the prime factorization of n. Then $C_n \simeq C_{p_1^{k_1}} \times C_{p_2^{k_2}} \times \dots \times C_{p_r^{k_r}}$.

Proof:

$C_n \simeq C_{p_1^{k_1}} \times C_m$ since $gcd(p_1^{k_1}, m = p_2^{k_2} \dots p_r^{k_r}) = 1$

Repeat with C_m, and so on ...

$\Rightarrow$ get the decomposition

Properties of the direct product of groups

1. $G \times H \simeq H \times G$, $|G \times H| = |G| \cdot |H|$. $\varphi: (g,h) \to (h,g)$ isomorphism
2. $H \subset G \times H$, $G \subset G \times H$ are subgroups. $\{(1,h), h \in H\} \simeq H \subset G$
3. $G \times H$ is abelian if and only if G and H are abelian.
4. If $H, K \subset G$ are subgroups such that
   (a) $H \cap K = \{1\}$
   (b) $\forall h \in H, \forall k \in K,\ hk = kh$
   (c) $HK = \{hk\}_{h \in H, k \in K} = G$

   Then $G \simeq H \times K$. The isomorphism is given by $\phi : H \times K \to G$, $\phi(h,k) = hk$.

See groups.pdf

Classification of finite abelian groups

Theorem

Let G be a finite abelian group. Then G is isomorphic to a direct product of cyclic groups of prime power orders

$$G \simeq C_{p_1^{n_1}} \times C_{p_2^{n_2}} \times \ldots \times C_{p_m^{n_m}},$$

where $|G| = p_1^{n_1} p_2^{n_2} \ldots p_m^{n_m}$. Here $\{p_1, p_2, \ldots p_m\}$ are primes, not necessarily distinct, and $n_1, \ldots n_m \geq 1$.
This presentation if unique up to the order of factors. The numbers $(p_1^{n_1}, p_2^{n_2}, \ldots p_m^{n_m})$ are called the elementary divisors of G.

Examples:

$C_3 \times C_2 \simeq C_6$

$(3, 2)$

$C_2 \times C_2 = G$ not cyclic $|G| = 4$

$(2, 2)$

Proof of the classification theorem

Generators and relations $G = \langle g_1, \dots g_k \mid R_1, \dots R_l \rangle$

$$R_1 = g_1^{n_{11}} g_2^{n_{12}} \dots g_k^{n_{1k}} = 1$$
$$R_2 = g_1^{n_{21}} g_2^{n_{22}} \dots g_k^{n_{2k}} = 1$$
$$\dots$$
$$R_l = g_1^{n_{l1}} g_2^{n_{l2}} \dots g_k^{n_{lk}} = 1$$

They can be encoded in a rectangular matrix

$$\begin{pmatrix} n_{11} & n_{12} & \dots & \dots & n_{1k} \\ n_{21} & n_{22} & \dots & & \\ n_{31} & n_{32} & \dots & & \\ & & & & \end{pmatrix}$$

l rows
r columns
$n_{ij} \in \mathbb{Z}$

Which row-column operations do not change the group G?

Operations on the matrix without changing the group

1. Adding an integer multiple of one row to another row.

Ex $\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$ $\Rightarrow$ row 1 − 3(row 2) $\rightarrow$ $\begin{pmatrix} 0 & 7 \\ 1 & -2 \end{pmatrix}$

$\begin{cases} g^3 h = 1 \\ g h^{-2} = 1 \end{cases}$ $\begin{cases} g^3 h \cdot (g h^{-2})^{-3} = 1 \\ g h^{-2} = 1 \end{cases}$ $= \begin{cases} h \cdot h^6 = h^7 = 1 \\ g h^{-2} = 1 \end{cases}$ $= \begin{cases} h^7 = 1 \\ g h^{-2} = 1 \end{cases}$

2. Adding integer multiple of one column to another column.

Ex $\begin{pmatrix} g & h \\ 3 & 1 \\ 1 & -2 \end{pmatrix}$ $\Rightarrow$ column 2 + 2 column 1 $\rightarrow$ $\begin{pmatrix} 3 & 7 \\ 1 & 0 \end{pmatrix}$

$\begin{cases} g^3 h = 1 \\ g h^{-2} = 1 \end{cases}$ $(g, h) \rightarrow (\underbrace{(g h^{-2})}_{f}, h)$ $\Rightarrow \begin{cases} (g h^{-2})^3 \cdot h^6 \cdot h = 1 \\ (g h^{-2}) \cdot h^2 \cdot h^{-2} = 1 \end{cases}$ $\Rightarrow \begin{cases} f^3 \cdot h^7 = 1 \\ f = 1 \end{cases}$

$\Rightarrow C_7$

3. Swapping two columns or swapping two rows.

$\longleftrightarrow$ reordering generators or relations $\begin{pmatrix} 3 & 7 \\ 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 7 & 3 \\ 0 & 1 \end{pmatrix}$ $h^7 f^3 = 1$, $f = 1$ $\rightsquigarrow \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$

Operations on the matrix without changing the group

Applying these operations, we can get $n_{11} = \gcd$(elements of the first column and first row). Then by column and row operations we get

$$\begin{pmatrix} n_{11} & 0 & 0 & \dots & 0 \\ 0 & n_{22} & \dots & & \\ 0 & n_{32} & \dots & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}$$

Repeating with the smaller matrix, we get the diagonal matrix

$$\begin{pmatrix} n_{11} & 0 & 0 & \dots & 0 \\ 0 & n_{22} & \dots & & \\ 0 & 0 & n_{33} & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}$$

This matrix defines the same group:

$$G = \langle g_1, g_2, \dots g_r \mid g_1^{n_{11}} = 1, g_2^{n_{22}} = 1, \dots g_r^{n_{rr}} = 1 \rangle.$$

The classification theorem: end of the proof

We have $G = \langle g_1, g_2, \ldots g_r \mid g_1^{n_{11}} = 1, g_2^{n_{22}} = 1, \ldots g_r^{n_{rr}} = 1 \rangle$.

$\Rightarrow$ $G_i = \langle g_i \rangle$ are cyclic subgroups, $\langle g_i \rangle \cap \langle g_j \rangle = 1$

$g_i g_j = g_j g_i$ G abelian

$\Rightarrow$ By property (4) of the direct product

$$G \simeq C_{n_{11}} \times C_{n_{22}} \times \ldots \times C_{n_{rr}}$$

By Proposition on the structure of each $C_{n_{kk}}$.

$$C_{n_{kk}} \simeq C_{p_1^{a_1}} \times C_{p_2^{a_2}} \times \ldots \times C_{p_r^{a_r}}$$ for some distinct primes $\{p_1, p_2 \ldots p_r\}$

Finally $G \simeq C_{p_1^{n_1}} \times C_{p_2^{n_2}} \times \ldots \times C_{p_m^{n_m}}$ where the primes $\{p_1 \ldots p_m\}$ can repeat.

$$G \simeq C_{p_1^{n_1}} \times C_{p_2^{n_2}} \times \ldots \times C_{p_m^{n_m}},$$

a direct product of cyclic groups of prime power orders (not necessarily distinct primes).

Corollary: Structure of abelian groups of prime power order

Corollary

Let G be an abelian group. If $|G| = p^n$, where p a prime, then $G \simeq C_{p^{i_1}} \times C_{p^{i_2}} \times \ldots \times C_{p^{i_k}}$ such that $i_1 + i_2 + \ldots + i_k = n$.
The set of abelian groups of order p^n is in bijection with the partitions of n: $n = i_1 + i_2 + \ldots + i_k$, such that $i_1 \geq i_2 \geq \ldots \geq i_k \geq 1$.

Example: $|G| = 8$. $= 2^3$

Partitions of 3: (3), $(2,1)$, $(1,1,1)$

$G_1 \simeq C_8$ (8), $G_2 \simeq C_4 \times C_2$ $(4,2)$, $G_3 \simeq C_2 \times C_2 \times C_2$ $(2,2,2)$ elementary divisors

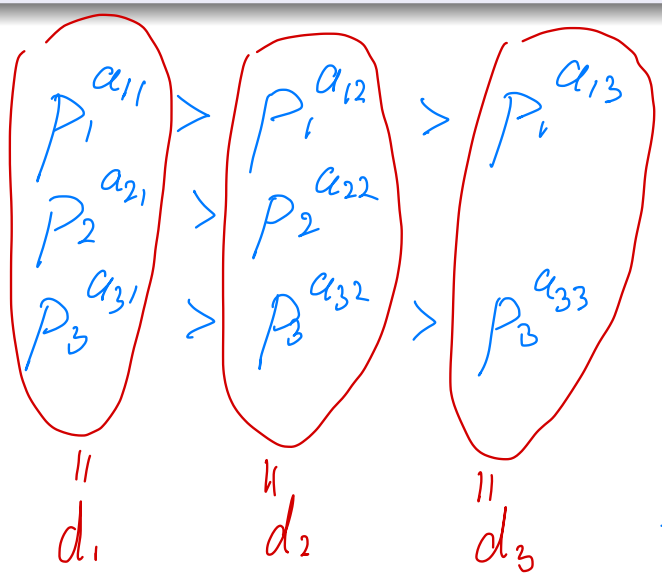
Pairwise non-isomorphic because $C_n \times C_m \simeq C_{nm} \iff \gcd(n,m) = 1$

# of abelian gps of order p^n = # of partitions of n.

Another way to encode a finite abelian group

Theorem

A finite abelian group $G \simeq C_{d_1} \times C_{d_2} \times \ldots \times C_{d_n}$, where d_n divides d_{n-1}, d_{n_1} divides d_{n-2}, etc, d_2 divides d_1, and $|G| = d_1 d_2 \ldots d_n$. The numbers $(d_1, d_2, \ldots d_n)$ are called the invariant factors of G. They determine G uniquely.



Elementary divisors of G

$G \simeq C_{p_1}^{a_{11}} \times C_{p_1}^{a_{12}} \times C_{p_1}^{a_{13}} \times C_{p_2}^{a_{21}} \times C_{p_2}^{a_{22}} \times C_{p_3}^{a_{31}} \times C_{p_3}^{a_{32}} \times C_{p_3}^{a_{33}}$

$a_{11} > a_{12} > a_{13}$

$a_{21} > a_{22}$

$a_{31} > a_{32} > a_{33}$

$\Rightarrow$ By construction $d_3 \mid d_2$ and $d_2 \mid d_1$

$G \simeq C_{d_1} \times C_{d_2} \times C_{d_3}$ $\Rightarrow$ invariant factors are (d_1, d_2, d_3)

$d_3 \mid d_2 \mid d_1$

Algorithm to classify all abelian groups of given order

1. Decompose $|G| = n = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$ (prime factorization).
2. Find partitions for each power $k_1, k_2, \dots k_n$.
3. For each partition of k_i, there is a unique group of order $p_i^{k_i}$:

$$k_i = a_1 + a_2 + \dots + a_t \quad \Longrightarrow \quad C_{p_i^{a_1}} \times C_{p_i^{a_2}} \times \dots \times C_{p_i^{a_t}}.$$

4. The possible groups of order n are the direct products of all possible groups of orders $p_i^{k_i}$. This gives a decomposition of G as a direct product of cyclic groups of prime power orders - the elementary divisors of G.
5. For each of the distinct primes p_i, write the obtained cyclic groups $C_{p_i^{n_i}}$ in the order of decreasing powers of p_i, different primes in different lines. Then in each column you will have cyclic groups of coprime orders, their direct product is a cyclic group. Thus you obtain cyclic groups of orders $(d_1, d_2, \dots d_n)$ and by construction $d_n | d_{n-1} | \dots | d_2 | d_1$. These are the invariant factors of G.

Classification of finite abelian groups: example

$|G| = 72 = 2^3 \cdot 3^2$ partitions (3) $(2 \geq 1)$ $(1 \geq 1 \geq 1)$
(2) $(1 \geq 1)$

	G_1	G_2	G_3	G_4	G_5	G_6
$p_1 = 2$	$C_{2^3} \times$	$C_{2^3} \times$	$C_{2^2} \times C_2$	$C_{2^2} \times C_2$	$C_2 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2$
$p_2 = 3$	C_{3^2}	$C_3 \times C_3$	C_{3^2}	$C_3 \times C_3$	C_{3^2}	$C_3 \times C_3$
	C_{72}	$C_{24} \times C_3$	$C_{36} \times C_2$	$C_{12} \times C_6$	$C_{18} \times C_2 \times C_2$	$C_6 \times C_6 \times C_2$

There are 6 non-isomorphic abelian groups of order 72.

The elementary divisors: $\{(2^3, 3^2)\ (2^3, 3, 3)\ (2^2, 2, 3^2)\ (2^2, 2, 3, 3),$
$(2, 2, 2, 3^2)\ (2, 2, 2, 3, 3)\}$

The invariant factors: $\{(72), (24, 3), (36, 2)\ (12, 6)\ (18, 2, 2), (6, 6, 2)\}$
G_1 G_2 G_3 G_4 G_5 G_6

$G_3 \simeq C_4 \times C_2 \times C_9 \simeq C_{36} \times C_2 \simeq C_4 \times C_{18}$

elementary divisors; invariant factors; neither: 4 does not divide 18

Poll:

Which of the statements below is false?

A: There is only one abelian group of order 77 up to isomorphism

B: If p is an odd prime, then the groups $C_{p^3} \times C_{2^3}$ and $C_{2p} \times C_{2p} \times C_{2p}$ are isomorphic

C: For any $n \in \mathbb{N}^+$ the cyclic group C_{n^5} contains an element of order n^3

D: If $p \neq q$ are distinct primes, and pq divides n, then an abelian group of order n contains an element of order pq

E: If 49 divides $|G|$, then G contains an element of order 7.