Algebra MATH-310

Lecture 2

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September 15, 2025

Plan of the course

1. Integers: 1 lecture
2. Groups: 6 lectures
3. Rings and fields: 5 lectures
4. Review: 1 lecture

Today: Groups-1

(a) Definition and first examples

(b) Subgroups

(c) Cosets and Lagrange's theorem

(d) Application: Euler's and Fermat's theorems

(e) Application: RSA

Groups: definition

Definition

A group is a set G with a binary operation $\cdot : G \times G \to G$ satisfying the axioms:

1. Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any $a, b, c \in G$.
2. Neutral element: $\exists\, 1 \in G$ such that $1 \cdot a = a \cdot 1 = a$ for any $a \in G$
3. Inverse: For any $a \in G$ $\exists\, a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1}a = 1$.

Definition

A group G is called finite if $|G| < \infty$. In this case $|G| \in \mathbb{N}$ is called the order of the group.

Definition

A group G is called abelian if $a \cdot b = b \cdot a$ for any $a, b \in G$.

Groups: first examples

1. The real numbers $(\mathbb{R}, +, 0)$ form an abelian group with respect to addition. The integers $(\mathbb{Z}, +, 0)$ form an abelian group with respect to addition.

infinite abelian

2. For any $n \in \mathbb{N}, n \geq 2$, the equivalence classes of integers modulo n: $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \ldots [n-1]\}$ form an abelian group with respect to addition. In $\mathbb{Z}/6\mathbb{Z}$, we have $[2] + [5] = [1]$ etc. The order $|\mathbb{Z}/n\mathbb{Z}| = n$.

finite abelian

$n = 6$

$[0]$ is neutral ; $[2]+[4] = [0] \Rightarrow [4]$ is the inverse of $[2]$ in $\mathbb{Z}/6\mathbb{Z}$

3. The group of real invertible $n \times n$ matrices $GL(n, \mathbb{R})$ is a non-abelian infinite group with respect to the matrix multiplication.

$$A \cdot B \neq B \cdot A; \quad \forall\, A \;\exists\; A^{-1} \in GL(n\mathbb{R}).$$

$A \cdot A^{-1} = A^{-1} \cdot A = Id$

$$\mathrm{Id} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

Groups: further examples

Multiplicative group modulo n

Let $n \in \mathbb{Z}_{\geq 2}$, $K = \{x \in \mathbb{N} : 1 \leq x \leq n,\ \gcd(x, n) = 1\}$. Then K is a group with respect to multiplication modulo n, and $|K| = \varphi(n)$.

Notation: $(\mathbb{Z}/n\mathbb{Z}, \cdot)^*$.

$\{1 \leq x \leq n : \gcd(x,n) = 1\} \Leftrightarrow \exists a, b \in \mathbb{Z} : ax + bn = 1 \Leftrightarrow$

$ax \equiv 1 \pmod{n} \Leftrightarrow [a] \cdot [x] = [1]$

$\Rightarrow [x]$ is invertible in $(\mathbb{Z}/n\mathbb{Z}, \cdot)^*$

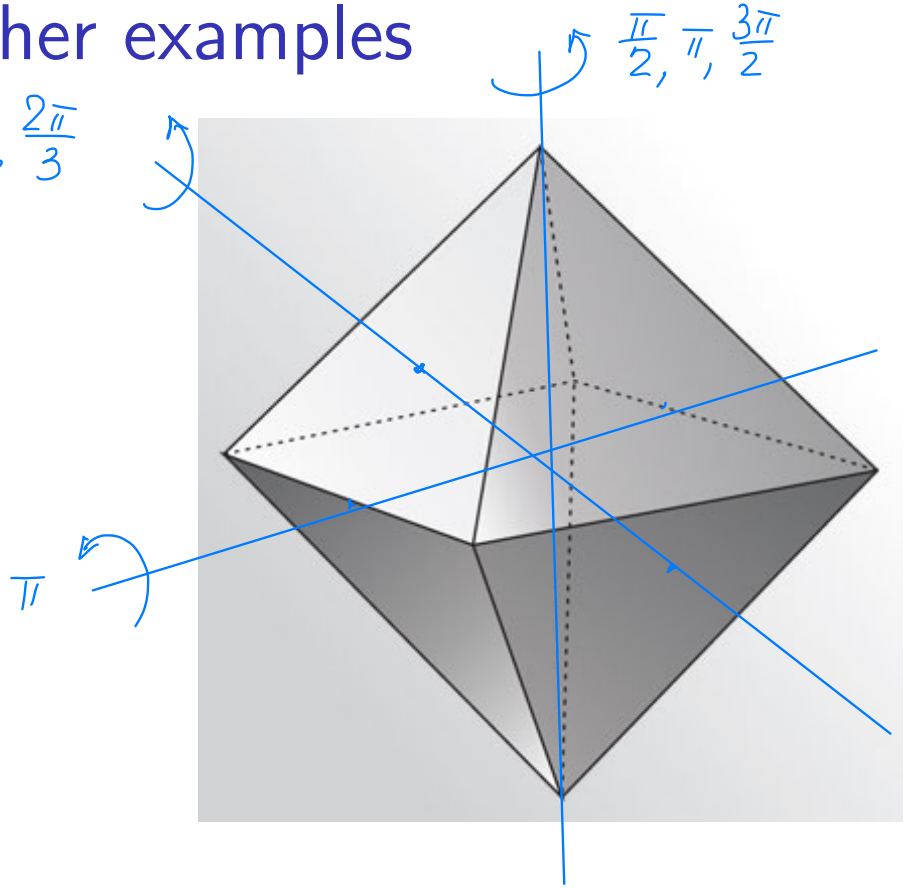
Let $n = 5$ Then $K = \{[1], [2], [3], [4]\}$

$[1] \cdot [1] = [1]$; $[2] \cdot [3] = [1]$, $[4] \cdot [4] = [1] \Rightarrow$

$[2]^{-1} = [3]$, $[3]^{-1} = [2]$, $[4]^{-1} = [4]$

$[1]$ is the neutral element in K.

Groups: further examples



vertex to vertex:

$3 \times 3 = 9$

nontrivial rotations

mid-edge to mid-edge

$1 \times 6 = 6$

nontrial rotations

midface to midface

$2 \times 4 = 8$

+ neutral elt 1

24

Poll: The order of the group of rotational symmetries of the regular octahedron is:

A: 20

B: 12

C: 16

D: 24

Conclusions

1. There are groups with respect to addition, multiplication, or another binary operation satisfying the axioms.
2. Important example: additive and multiplicative groups of integers modulo n: $(\mathbb{Z}/n\mathbb{Z}, +) \neq (\mathbb{Z}/n\mathbb{Z}, \cdot)^*$.
   In particular, $|(\mathbb{Z}/n\mathbb{Z}, +)| = n$ and $|(\mathbb{Z}/n\mathbb{Z}, \cdot)^*| = \varphi(n)$.

see Appendix B groups_Math 310.pdf for more examples of unusual groups (Appendices A and B are not part of the exam)

Subgroups

Definition

A subgroup $H \subset G$ is a subset in G such that $1 \in H$ and H is closed with respect to the multiplication and taking inverses.

Example: $\{0, \pm 3, \pm 6, \pm 9, \ldots\} \subset (\mathbb{Z}, +, 0)$ is a subgroup of integers with respect to addition.

$3n + 3k = 3(n+k) \in \{0, \pm 3, \pm 6, \ldots\}$

$3n + 3(-n) = 0$ inverse element

0 neutral element

Subgroup generated by a single element

Suppose $g \in G$. Consider the subset $\langle g \rangle = \{1, g^{\pm 1}, g^{\pm 2}, g^{\pm 3}, \ldots\} \subset G$. Then $g^i \cdot g^k = g^{i+k} \in \langle g \rangle$, and for each g^i the inverse $g^{-i} \in \langle g \rangle$. Therefore $\langle g \rangle \subset G$ is a subgroup by construction. It is called the subgroup in G generated by the element g. It is the smallest subgroup of G containing g.

Example: $(\mathbb{Z}, +, 0) = G$, $g = 4 \Rightarrow \langle g \rangle = \{0, \pm 4, \pm 8, \pm 12, \ldots\} = 4\mathbb{Z}$

$4\mathbb{Z} \subset \mathbb{Z}$ is a subgroup with respect to addition generated by $4 \in \mathbb{Z}$.

Definition

If there exists $n \in \mathbb{N}_+$ minimal such that $g^n = 1$ in G, then n is called the order of element g. In this case $\langle g \rangle = \{1, g, g^2, \ldots g^{n-1}\}$ and $|\langle g \rangle| = n$.

Cosets

Definition

Let $H \subset G$ be a subgroup and $g \in G$ an element. The left coset gH is the set of group elements of the form $gH = \{gh,\ h \in H\}$.

Proposition

1. *Two cosets xH and yH are either equal or disjoint: $xH = yH$ or $xH \cap yH = \emptyset$.*
2. *Any element $g \in G$ belongs to a left H-coset*
3. *If H is finite, then $|xH| = |H|$ for any $x \in G$.*

Proof: (1) If $xH \cap yH \neq \emptyset \Rightarrow \exists h_1, h_2 \in H : xh_1 = yh_2 \Rightarrow x = y\underbrace{h_2 h_1^{-1}}_{\in H} = yh_3 \in yH$

Then $\forall h \in H \Rightarrow xh = y\underbrace{h_3 h}_{\in H} \in yH \Rightarrow xH \subset yH$ similarly $yH \subset xH \Rightarrow xH = yH$.

(2) Take the coset of g : $gH = \{g, gh_1, gh_2 \dots\}$

(3) Let $f : H \to xH$, $f(h) = xh$ $\forall h \in H$ $\Rightarrow$ f is surjective: $xH = \{xh\}_{h \in H}$

f is injective: if $xh_1 = xh_2 \Rightarrow x^{-1}xh_1 = x^{-1}xh_2 \Rightarrow h_1 = h_2$

Example of cosets

$(\mathbb{Z}, +, 0) = G$; $H = 4\mathbb{Z} \subset \mathbb{Z}$ subgroup.

Coset of 0 wrt $4\mathbb{Z}$ in $\mathbb{Z}$:

$$\{0+4k\}_{k\in\mathbb{Z}} = 4\mathbb{Z} = H = \{0, \pm 4, \pm 8, \dots\}$$

Coset of 1 wrt $4\mathbb{Z}$ in $\mathbb{Z}$:

$$\{1+4k\}_{k\in\mathbb{Z}} = \{1, 5, 9, -3, -7, \dots\}$$ left coset in $\mathbb{Z}$

Coset of 5 wrt $4\mathbb{Z}$ in $\mathbb{Z}$:

$$\{5+4k\}_{k\in\mathbb{Z}} = \{1, 5, 9, -3, -7, \dots\} = \{1+4k\}_{k\in\mathbb{Z}}$$ same coset

Coset of 2 wrt $4\mathbb{Z}$ in $\mathbb{Z}$:

$$\{2+4k\}_{k\in\mathbb{Z}} = \{\pm 2, \pm 6, \pm 10, \dots\}$$ left coset in $\mathbb{Z}$

Note that $\mathbb{Z} = \{0+4k\}_{k\in\mathbb{Z}} \cup \{1+4k\}_{k\in\mathbb{Z}} \cup \{2+4k\}_{k\in\mathbb{Z}} \cup \{3+4k\}_{k\in\mathbb{Z}}$

Lagrange's theorem

Theorem

Let G be a finite group and $H \subset G$ a subgroup. Then $|H|$ divides $|G|$.

Proof: Each $g \in G$ belongs to a left H-coset, and $\begin{cases} xH = yH, \text{ or} \\ xH \cap yH = \emptyset \end{cases}$

$\Rightarrow G = \bigcup_{i=1}^{r} x_i H$ disjoint union of finitely many left H-cosets

$|G| = \sum_{i=1}^{r} |x_i H| \quad ; \quad |x_i H| = |H| \quad \forall x_i$

$\Rightarrow |G| = \sum_{i=1}^{r} |H| = r \cdot |H| \Rightarrow |H|$ divides $|G|$ ▨

Definition

The number of left H-cosets in G is called the index of H in G. Notation: $[G : H] = \frac{|G|}{|H|} \in \mathbb{N}_+$.

Order of an element divides order of the group

Corollary

1. *Let G be a finite group, and $g \in G$. Then the order of g divides $|G|$.*
2. $g^{|G|} = 1$.

Proof: (1) Let $H = \langle g \rangle = \{1, g, g^2 \dots g^{k-1}\}$ where k is the order of the element g in G.

$\Rightarrow \langle g \rangle = H \subset G$ subgroup $\Rightarrow$ by Lagrange $\Rightarrow |\langle g \rangle| = k$ divides $|G|$.

(2) We have $g^k = 1$, also $|G| = kt$ where $t \in \mathbb{N}$

$$\Rightarrow g^{|G|} = g^{kt} = (g^k)^t = 1^t = 1 \in G$$

Applications of Lagrange's theorem

Theorem

(Euler's theorem)

Let $a, n \in \mathbb{Z}_+$ such that $\gcd(a, n) = 1$. Then $a^{\varphi(n)} \equiv 1 \pmod{n}$.

Proof: Let $G = ((\mathbb{Z}/n\mathbb{Z})^*, \cdot, 1)$. Then $|G| = \varphi(n)$
if $\gcd(a,n) = 1 \Rightarrow [a] \in G \Rightarrow$ by Corollary $[a]^{\varphi(n)} = [1] \in G$
$\Leftrightarrow$ $a^{\varphi(n)} \equiv 1 \pmod{n}$

Theorem

(Fermat's little theorem)

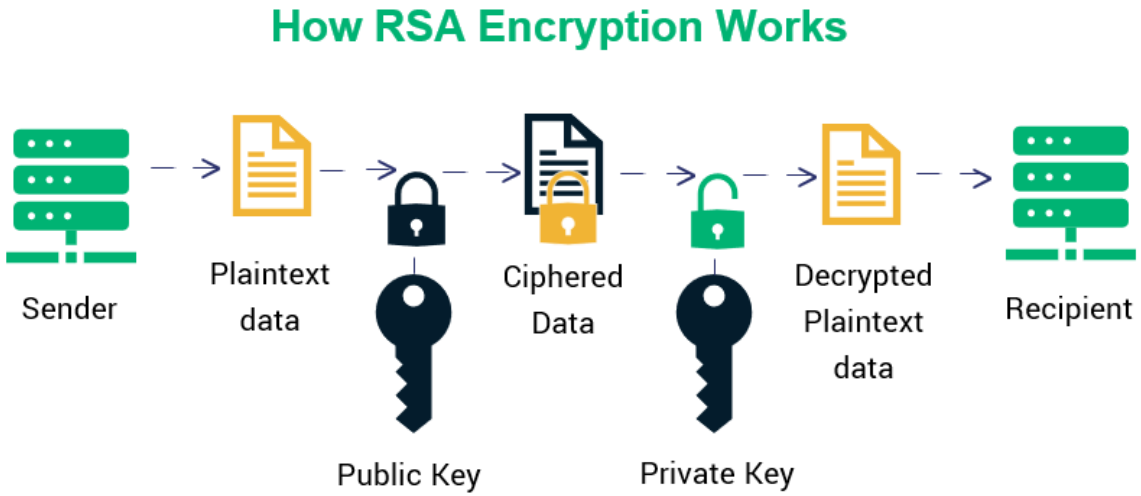
Let $a \in \mathbb{Z}_+$ and p a prime that does not divide a. Then $a^{p-1} \equiv 1 \pmod{p}$.

Proof: $\gcd(a,p) = 1$, $\varphi(p) = p-1 \Rightarrow$ By Euler's thm $\Rightarrow a^{\varphi(p)} = a^{p-1} \equiv 1 \pmod{p}$

Conclusions

1. If G is a finite group, and $H \subset G$ a subgroup, then $|H|$ divides $|G|$.
2. If G is a finite group, and $g \in G$, then the order of the element g divides $|G|$.
3. Let $a, n \in \mathbb{Z}_+$ such that $\gcd(a, n) = 1$. Then $a^{\varphi(n)} \equiv 1 \pmod{n}$.
   Why do we care?

Rivest-Shamir-Adleman 1977



RSA encryption system

Setting

1. Choose two large distinct primes p, q. $\sim 2^{1030}$
2. Let $m = pq$. Then $\varphi(m) = (p-1)(q-1)$. ← factorization is HARD
3. Choose $1 < e < \varphi(m)$ such that $\gcd(e, \varphi(m)) = 1$.
4. Use the Euclidean algorithm to find $d \in \mathbb{Z}$ such that $ed - k\varphi(m) = 1$ for some $k \in \mathbb{Z}$.
5. Publish the encryption key (m, e).
6. Keep secret the decryption key (m, d).

Send a message

1. A publishes the encryption key (m, e).
2. B wants to send message x, $0 < x < m$ to A. Then B computes $y \equiv x^e \pmod{m}$ and sends y publicly to A. ← HARD
3. A computes $y^d = x^{ed} \equiv x \pmod{m}$.

RSA encryption system

Why does it work?

Proposition

Let p, q be two distinct primes and $m = pq$. Let $1 < e < \varphi(m)$ be such that $\gcd(e, \varphi(m)) = 1$, and $d \in \mathbb{Z}$ such that $ed - k\varphi(m) = 1$ for some $k \in \mathbb{Z}$. Then for any $0 < x < m$, $x^{ed} \equiv x \pmod{m}$.

Proof: (1) If $x = pt \Rightarrow x^{ed} \equiv 0 \pmod{p} \Rightarrow (x^{ed} - x) \equiv 0 \pmod{p}$.

(2) If x is not divisible by p. $\Rightarrow$ Fermat's thm $\Rightarrow x^{p-1} \equiv 1 \pmod{p}$

$$x^{ed} = x^{k\varphi(m)+1} = x^{k(p-1)(q-1)+1} = \underbrace{(x^{p-1})^{k(q-1)}}_{\equiv 1 \pmod{p}} \cdot x \equiv 1 \cdot x \pmod{p}$$

$$\Rightarrow (x^{ed} - x) \equiv 0 \pmod{p}$$

The same argument works for q $\Rightarrow (x^{ed} - x)$ divisible by p and q

$$\Rightarrow x^{ed} \equiv x \pmod{\underbrace{pq}_{m}}$$

RSA encryption system: example

Setting: Let p, q be two distinct primes and $m = pq$. Let $1 < e < \varphi(m)$ be such that $\gcd(e, \varphi(m)) = 1$, and $d \in \mathbb{Z}$ such that $ed - k\varphi(m) = 1$ for some $k \in \mathbb{Z}$. Then for any $0 < x < m$, $x^{ed} \equiv x \pmod{m}$.

Example $p = 5$, $q = 7 \Rightarrow m = pq = 35$, $\varphi(m) = (p-1)(q-1) = 24$

Let $e = 5 \Rightarrow \gcd(5, 24) = 1$ Find d: $ed - k\varphi(m) = 1$

$5 \cdot 5 - 24 \cdot 1 = 1 \Rightarrow d = 5$ $(m, e) = (35, 5)$ encoding key

(m, d) $(35, 5)$ decoding key.

Send $x = 31$. Compute $X^e \pmod{m}$

$31^5 \pmod{35} = (-4)^5 \pmod{35} = -64 \cdot 16 \pmod{35} = 6 \cdot 16 \pmod{35} =$

$= 32 \cdot 3 \pmod{35} = -3 \cdot 3 \pmod{35} = -9 \pmod{35}$

$\Rightarrow$ Send $y = -9$. or $y = 26$

RSA encryption system: example

Setting: Let p, q be two distinct primes and $m = pq$. Let $1 < e < \varphi(m)$ be such that $\gcd(e, \varphi(m)) = 1$, and $d \in \mathbb{Z}$ such that $ed - k\varphi(m) = 1$ for some $k \in \mathbb{Z}$. Then for any $0 < x < m$, $x^{ed} \equiv x \pmod{m}$.

To decode: $y^d \pmod m$ $\qquad (m, d) = (35, 5)$

$(-9)^5 \pmod{35} = 81 \cdot 81 \cdot (-9) \pmod{35} = -11 \cdot 11 \cdot 9 \pmod{35}$

$= -99 \cdot 11 \pmod{35} = -29 \cdot 11 \pmod{35} = 6 \cdot 11 \pmod{35} =$

$= 66 \pmod{35} \equiv 31 \pmod{35}$

$\Rightarrow \quad x = 31.$ ☺