

Introduction to Partial Differential Equations

Lecture notes for the course “Introduction to Partial Differential Equations”
(EPFL, Winter Semester 2024/25).

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Preface

These lecture notes accompany the course “Introduction to Partial Differential Equations”, held for the Bachelor’s Degree in Mathematics at EPFL. They were written by Nicola de Nitti and have small modifications and typos corrected. Nicola’s notes are also based on previous material prepared by Fabio Nobile and the textbook: [Hun14, Chapters 1–4], [Eva10, Chapters 1, 2.2, 5–6, 8, 9.4], [Jos07, Chapters 1–4, 10–14], [Joh82, Chapter 4], [Bre13, Chapters 8–9, Appendix], [Bre11, Chapters 8–9], [HL11, Chapters 1–2, 6], [GT01, Chapters 1–4, 7–8], and [FR22, Chapters 1–2].

The prerequisites for this course are “Analysis I–IV”. We also recommend familiarity with “Measure and Integration” and “Functional Analysis I”.

Exam

A detailed list of examinable topics (a subset of the lecture notes and exercise series) will be provided on Moodle.

The exam consists of a 30-minute oral examination at the blackboard. Each student will select two questions: one from each part of the course (A: classical solutions, B: weak solutions). Students will then have an additional 30 minutes to prepare their answers without external material or support before the oral examination begins. It is essential for each student to arrive on time (i.e., 30 minutes prior to their scheduled oral exam). Each student must bring a CAMIPRO card or an ID card. Paper and pen for preparation will be provided.

Additional resources

- [Bre13] A. Bressan. *Lecture notes on functional analysis. With applications to linear partial differential equations*. Vol. 143. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2013.
- [Bre11] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. New York, NY: Springer, 2011.
- [BB98] H. Brézis and F. Browder. “Partial differential equations in the 20th century”. In: *Adv. Math.* 135.1 (1998), pp. 76–144.
- [Die82] J. Dieudonné. *A panorama of pure mathematics (as seen by N. Bourbaki). Transl. from the French by I. G. Macdonald*. Vol. 97. Pure Appl. Math., Academic Press. Academic Press, New York, NY, 1982.
- [Eva10] L. C. Evans. *Partial differential equations*. 2nd ed. Vol. 19. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2010.
- [FR22] X. Fernández-Real and X. Ros-Oton. *Regularity theory for elliptic PDE*. Zur. Lect. Adv. Math. Berlin: European Mathematical Society (EMS), 2022.
- [GT01] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Reprint of the 1998 ed. Class. Math. Berlin: Springer, 2001.
- [HL11] Q. Han and F. Lin. *Elliptic partial differential equations*. 2nd ed. Vol. 1. Courant Lect. Notes Math. New York, NY: Courant Institute of Mathematical Sciences; Providence, RI: American Mathematical Society (AMS), 2011.
- [Hun14] J. K. Hunter. *Notes on partial differential equations*. 2014. URL: <http://www.math.ucdavis.edu/~hunter/pdes/pdes.html>.
- [Joh82] F. John. *Partial differential equations. 4th ed*. Vol. 1. Appl. Math. Sci. Springer, Cham, 1982.
- [Jos07] J. Jost. *Partial differential equations*. 2nd ed. Vol. 214. Grad. Texts Math. New York, NY: Springer, 2007.

- [Kla00] S. Klainerman. “PDE as a unified subject”. In: *GFA 2000. Visions in mathematics—Towards 2000. Proceedings of a meeting, Tel Aviv, Israel, August 25–September 3, 1999. Part I*. Special volume of the journal Geometric and Functional Analysis. Basel: Birkhäuser, 2000, pp. 279–315.
- [Nir94] L. Nirenberg. “Partial differential equations in the first half of the century”. In: *Development of mathematics 1900-1950. Based on a symposium organized by the Luxembourg Mathematical Society in June 1992, at Château Bourglinster, Luxembourg*. Basel: Birkhäuser, 1994, pp. 479–515.

Introduction

1. Notation for partial derivatives

Lect. 1, 10.09

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be open, $x = (x_1, \dots, x_n) \in \Omega$ and $u : \Omega \rightarrow \mathbb{R}$ be a scalar function. The *partial derivatives* of u at x , are defined as

$$\frac{\partial u}{\partial x_i}(x) := \lim_{h \rightarrow 0} \frac{u(x + he_i) - u(x)}{h} \quad (\text{if the limit exists}),$$

for $i = 1, \dots, n$, where e_i denotes the i -th standard basis vector of \mathbb{R}^n . Commonly used are also the notations $\frac{\partial u}{\partial x_i} = \partial_{x_i} u = u_{x_i}$.

The *partial derivatives of second order* are defined as

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) := \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_j} \right)(x) \quad (\text{if they exist})$$

for $i, j = 1, \dots, n$. Commonly used are also the notations $\frac{\partial^2 u}{\partial x_i \partial x_j} = \partial_{x_i x_j}^2 u = u_{x_i x_j}$.

More generally, let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ be a multi-index. Its order is defined as

$$|\alpha| := \sum_{i=1}^n \alpha_i$$

and the corresponding $|\alpha|$ -th order *partial derivatives* of u are

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u(x) \quad (\text{if they exist}).$$

Moreover, for $k \in \mathbb{N}$ we denote by

$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\}$$

the collection of all k -th order partial derivatives of u in x .

In particular, we write $D^1 u(x)$ as a column vector (the *gradient* of u at x),

$$D^1 u(x) = Du(x) = \begin{pmatrix} \partial_{x_1} u(x) \\ \vdots \\ \partial_{x_n} u(x) \end{pmatrix} = \nabla u(x),$$

and $D^2 u(x)$ as a matrix (the *Hessian* of u at x),

$$D^2 u(x) = \begin{pmatrix} \partial_{x_1 x_1}^2 u(x) & \dots & \partial_{x_1 x_n}^2 u(x) \\ \vdots & \ddots & \vdots \\ \partial_{x_n x_1}^2 u(x) & \dots & \partial_{x_n x_n}^2 u(x) \end{pmatrix}.$$

2. What is a partial differential equation?

A *partial differential equation (PDE)* is an equation for an unknown function u of several variables that involves partial derivatives of u . The order of the highest partial derivative is called the *order* of the PDE.

DEFINITION 1.1 (k -th order PDE). Let $\Omega \subset \mathbb{R}^n$ be open, $n \geq 2$ and $k \in \mathbb{N}$. An expression of the form

$$F(D^k u(x), D^{k-1} u(x), \dots, u(x), x) = 0, \quad x \in \Omega, \quad (2.1)$$

is called a k -th order PDE, where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$$

is a given function and $u : \Omega \rightarrow \mathbb{R}$ is the unknown.

DEFINITION 1.2 (Classical solution). A classical solution of the PDE (2.1) is a k -times continuously differentiable function $u : \Omega \rightarrow \mathbb{R}$ that satisfies (2.1).

Depending on the structure of the function F in (2.1), we classify PDEs as follows.

DEFINITION 1.3. The PDE (2.1) is linear if the function F is linear in u and its derivatives, i.e. if it is of the form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) + f(x) = 0,$$

for given functions a_α and f . Moreover, if $f \equiv 0$, the PDE is called homogeneous and otherwise inhomogeneous.

The PDE (2.1) is semilinear if it is linear in the highest order derivatives, i.e. if it is of the form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + a_0(D^{k-1}u(x), \dots, u(x), x) = 0,$$

for given functions a_α and a_0 .

The PDE (2.1) is quasilinear if it is of the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u(x), \dots, u(x), x) D^\alpha u(x) + a_0(D^{k-1}u(x), \dots, u(x), x) = 0,$$

for given functions a_α and a_0 .

The PDE (2.1) is fully nonlinear if F is a nonlinear function of the highest order derivatives $D^k u$.

For linear homogeneous equations the superposition principle holds, i.e., if u and v are both solutions of the PDE, then the same applies to $\alpha u + \beta v$, for all $\alpha, \beta \in \mathbb{R}$. More generally, if u_1, \dots, u_m are solutions, then so is any linear combination of these solutions.

Typically, the difficulty of the analysis of a PDE increases with the degree of nonlinearity.

Instead of scalar equations we can also look at systems of PDEs which arise in many applications. Here, several unknown functions $u_1, \dots, u_m, m \geq 2$, have to be determined that satisfy a system of m PDEs.

DEFINITION 1.4 (k -th order system of PDEs). An expression of the form (2.1) is called a k -th order system of PDEs if $m \geq 2$ and

$$F : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \dots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^m,$$

where $u = (u_1, \dots, u_m) : \Omega \rightarrow \mathbb{R}^m$ is the unknown. Here, $D^\alpha u = (D^\alpha u_1, \dots, D^\alpha u_m)$ and $D^k u = \{D^\alpha u : |\alpha| \leq k\}$.

DEFINITION 1.5 (Classical solution). A classical solution of the system of PDEs (2.1) is a k -times continuously differentiable function $u : \Omega \rightarrow \mathbb{R}^m$ that satisfies (2.1).

EXAMPLE 1.1. Consider $u : \mathbb{R}^2 \rightarrow \mathbb{R}$. The equation

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0$$

is a first-order, linear, homogeneous PDE. One can check that any function of the type $u(x_1, x_2) = f(x_1 - x_2)$, for $f \in C^1(\mathbb{R})$ is a classical solution.

3. Type classification of linear second order PDEs

In this course, we mainly focus on linear, scalar PDEs of second order, i.e., equations of the form

$$\sum_{i,j=1}^n a_{ij}(x) \partial_{x_i x_j}^2 u(x) + \sum_{i=1}^n a_i(x) \partial_{x_i} u(x) + a_0(x) u(x) = f(x), \quad x \in \Omega, \quad (3.1)$$

that we now further classify.

By Schwarz' theorem, the Hessian matrix is symmetric if u is twice continuously differentiable; hence, when working in this regularity class, we can assume that

$$a_{ij} = a_{ji}, \quad \text{for all } i, j = 1, \dots, n,$$

i.e., that the coefficients a_{ij} form a symmetric matrix

$$A(x) = \begin{pmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{pmatrix}, \quad x \in \Omega$$

A useful type classification of the PDE (3.1) is based on the definiteness properties of A .

DEFINITION 1.6 (Elliptic, parabolic, and hyperbolic PDEs). *We call the linear second-order PDE (3.1) elliptic if $A(x)$ is positive or negative definite (no zero eigenvalue and all have the same sign), parabolic if $A(x)$ is one eigenvalue is zero and the rest have all the same sign, and hyperbolic if one eigenvalue of $A(x)$ has a different sign than all the others (where the eigenvalues are counted according to their multiplicity).*

The following three examples are the archetypes of second-order linear PDEs.

EXAMPLE 1.2 (Laplace equation).

$$\Delta u = \partial_{x_1 x_1}^2 u + \cdots + \partial_{x_n x_n}^2 u = 0 \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is open, $u : \Omega \rightarrow \mathbb{R}$ and Δ is the Laplace operator or Laplacian.

We have $A(x) = \mathbb{I} \in \mathbb{R}^{n \times n}$ (identity matrix) and, thus, the PDE is elliptic.

EXAMPLE 1.3 (Heat equation).

$$\partial_t u - \Delta u = 0 \quad \text{in } \Omega = I \times U,$$

where $t \in I$ denotes time, $x \in U$ space, $I \subset \mathbb{R}$ is an open interval, $U \subset \mathbb{R}^n$ is open, $u : I \times U \rightarrow \mathbb{R}$, and $\Delta u = \Delta_x u$ is the Laplace operator with respect to x .

We obtain a singular matrix

$$A(t, x) = \begin{pmatrix} 0 & 0 \\ 0 & -\mathbb{I} \end{pmatrix},$$

$\mathbb{I} \in \mathbb{R}^{n \times n}$, and, thus, the PDE is parabolic.

EXAMPLE 1.4 (Wave equation).

$$\partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega = I \times U,$$

where we use the same notation as for the heat equation.

In this case, we have

$$A(t, x) = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{I} \end{pmatrix},$$

and thus, the PDE hyperbolic.

4. Studying PDEs

A *classical solution* of a k -th order PDE is a k -times continuously differentiable function that satisfies the PDE pointwise in $\Omega \subset \mathbb{R}^n$.

Often, a PDE possesses families of solutions, but the solution u is uniquely determined if values of u and/or its derivatives are specified on the boundary $\partial\Omega$ of Ω . A PDE together with these boundary conditions is called a *boundary-value problem*. In applications that involve time (see, e.g., Examples 1.3–1.4), we typically consider sets of the form $\Omega := I \times U$ where $I := (t_0, t_1) \subset \mathbb{R}$ is an open interval and $U \subset \mathbb{R}^n$ is open. In this special case, the values of u and/or its derivatives specified at the initial time t_0 are called *initial conditions* and the values specified on ∂U *boundary conditions*. A PDE together with initial and boundary conditions is called an *initial-boundary-value problem*.

In the ideal case, we find explicit solutions for a given PDE, but this is only possible in few particularly simple cases. This classical approach to PDEs that dominated the 19th century was to develop methods for deriving explicit representation formulas for solutions.

If such formulas cannot be found, we aim at proving the existence and studying qualitative properties of solutions. In particular, we say that a problem is *well-posed*¹ if the following properties hold:

¹ Or *well-posed in the sense of Hadamard*, after [Had02].

- (1) there exists a solution;
- (2) the solution is unique;
- (3) the solution depends continuously on the given data (e.g., parameters, boundary or initial values).

For many PDEs the notion of classical solutions is too restrictive and such solutions do not exist. However, one can weaken the concept of solutions and consider so-called weak solutions or distributional solutions which are less regular and satisfy the PDE in a generalized sense. For instance, PDEs describing the occurrence of shocks (essentially, the appearance of discontinuities in the derivatives), require this notion. Moreover, even if classical solutions exist, it is often easier to prove the existence of weak solutions first and then to show that the solutions have a higher regularity and are, in fact, classical solutions of the problem.

5. Objectives of this course

In this course, we will focus on elliptic partial differential equations. In particular, we will cover the following topics:

- I. Laplacian operator; Laplace equation; mean-value property; maximum principles; Harnack's inequality; Weyl's lemma; fundamental solution; Green function and solutions to the Dirichlet problem for the Laplace equation; Newtonian potential.
- II. Theory of distributions; Sobolev spaces; weak derivatives and their properties; density results; extension results; traces; embedding theorems; Poincaré inequalities.
- III. Weak solutions of several elliptic PDEs involving the Laplace operator via variational methods.
- IV. General second-order linear elliptic PDEs: classical solutions (maximum principles, a priori bounds) and weak solutions theory.

Laplace equation and harmonic functions

1. Harmonic functions

Let $\Omega \subset \mathbb{R}^n$ (with $n \geq 1$) be an open set. The first equation we will study is the *Laplace equation*:¹

$$-\Delta u(x) = 0, \quad x \in \Omega. \quad (1.1)$$

REMARK 2.1. *We recall that*

$$\Delta u(x) = \operatorname{div} \nabla u(x) = \partial_{x_1 x_1}^2 u + \dots + \partial_{x_n x_n}^2 u.$$

DEFINITION 2.1 (Harmonic, sub-harmonic, and super-harmonic functions). *A function $u \in C^2(\Omega)$ is called*

- (1) *harmonic in Ω if it satisfies $\Delta u(x) = 0$ for all $x \in \Omega$;*
- (2) *sub-harmonic in Ω if it satisfies $-\Delta u(x) \leq 0$ for all $x \in \Omega$;*
- (3) *super-harmonic in Ω if it satisfies $-\Delta u(x) \geq 0$, for all $x \in \Omega$.*

As we will see, to recover a unique solution to this equation, suitable boundary conditions should be provided. The most common are the following.

Dirichlet² problem:

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega \\ u(x) = g(x), & x \in \partial\Omega \end{cases}$$

Neumann³ problem:

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega \\ \partial_\nu u(x) = h(x), & x \in \partial\Omega \end{cases}$$

Robin⁴ problem:

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega \\ \partial_\nu u(x) + \alpha u(x) = h(x), & x \in \partial\Omega \end{cases} \quad (\text{with } \alpha > 0 \text{ given})$$

The non homogeneous version of (1.1) is called the *Poisson⁵ equation*:⁶

$$-\Delta u(x) = f(x), \quad x \in \Omega \quad (1.2)$$

for some given continuous function $f : \Omega \rightarrow \mathbb{R}$.

The Poisson equation appears in many different fields of physics. For instance, it may describe the distribution of temperature in a region Ω , in the presence of a heat source f , knowing the temperature at the boundary (Dirichlet problem) or the heat flux through the boundary (Neumann problem).

EXAMPLE 2.1. *Any affine linear function in \mathbb{R}^n is harmonic in \mathbb{R}^n since*

$$\Delta \left(a + \sum_j b_j x_j \right) = 0 \quad \text{in } \mathbb{R}^n.$$

¹ Named after Pierre-Simon Laplace.

⁵ Named after Siméon Denis Poisson

⁶ The minus sign in the above equations could, of course, be removed (upon changing the sign of f). However, we prefer to keep it and always think of the operator in the Laplace or Poisson equation as “ $-\Delta$ ”.

EXAMPLE 2.2. The function $\ln \sqrt{x^2 + y^2} = \frac{1}{2} \ln(x^2 + y^2)$ is harmonic in $\mathbb{R}^2 \setminus \{(0, 0)\}$. Indeed, assuming $(0, 0) \neq (x, y) \in \mathbb{R}^2$, we compute

$$\begin{aligned} \nabla \left(\ln \sqrt{x^2 + y^2} \right) &= \nabla \frac{1}{2} \ln(x^2 + y^2) = \frac{(x, y)}{x^2 + y^2}; \\ \Delta \left(\ln \sqrt{x^2 + y^2} \right) &= \operatorname{div} \left(\nabla \left(\frac{1}{2} \ln(x^2 + y^2) \right) \right) \\ &= \operatorname{div} \frac{(x, y)}{x^2 + y^2} = \frac{\partial}{\partial x} \frac{x}{x^2 + y^2} + \frac{\partial}{\partial y} \frac{y}{x^2 + y^2} \\ &= \frac{(y^2 - x^2) + (x^2 - y^2)}{(x^2 + y^2)^2} \\ &= 0. \end{aligned}$$

EXAMPLE 2.3. The function $(x^2 + y^2 + z^2)^{-1/2}$ is harmonic in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Indeed, assuming $(0, 0, 0) \neq (x, y, z) \in \mathbb{R}^3$, we compute

$$\begin{aligned} \nabla \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) &= \frac{(-x, -y, -z)}{(x^2 + y^2 + z^2)^{3/2}}; \\ \Delta \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) &= -\operatorname{div} \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} \\ &= -\frac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{x^2 + z^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} \\ &= 0. \end{aligned}$$

EXAMPLE 2.4. Let $n \geq 2$. We compute

$$\Delta |x|^\alpha = (n\alpha + \alpha(\alpha - 2))|x|^{\alpha-2}.$$

Thus $|x|^\alpha$ is harmonic for $\alpha = 2 - n$ and sub-harmonic for $\alpha \geq 2 - n$.

EXAMPLE 2.5 (Holomorphic functions). Let us recall from complex analysis that a holomorphic (= complex differentiable) function $f(z) : \Omega \rightarrow \mathbb{C}$, with $\Omega \subset \mathbb{C}$, satisfies the Cauchy-Riemann equations. If we decompose f in real and imaginary parts, $f(z) = u(z) + iv(z)$, with $u(z), v(z) \in \mathbb{R}$, and denote $z = x + iy$, with $x, y \in \mathbb{R}$ the Cauchy-Riemann conditions are given by:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We know from complex analysis that holomorphic functions are smooth, in particular, we can compute second derivatives, and see that

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial}{\partial y} \frac{-\partial v}{\partial x} = 0$$

Similarly, one can check that $\Delta v = 0$. Hence, the real and imaginary parts of holomorphic functions are harmonic. For example, since the complex logarithm $\log(z) = \log \sqrt{x^2 + y^2} + i \arctan(y/x)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_+$, this gives a proof that $\log \sqrt{x^2 + y^2}$ is harmonic on $\mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$. One can combine this with the holomorphicity on other domains, such as $\mathbb{C} \setminus \mathbb{R}_-$, to conclude that $\log \sqrt{x^2 + y^2}$ is actually harmonic in the whole $\mathbb{R}^2 \setminus \{(0, 0)\}$.⁷

Since $z^2 = x^2 - y^2 + 2ixy$ is holomorphic, we can also see that $x^2 - y^2$ or $2xy$ are harmonic functions on \mathbb{R}^2 .

Lect. 2, 17.09

EXAMPLE 2.6 (Harmonic polynomials). Let us find all harmonic polynomials of degree n in two variables,

$$P_n(x, y) = \sum_{k=0}^n c_k x^{n-k} y^k,$$

⁷In contrast, the imaginary part $\arctan(y/x)$ is only harmonic on a domain of the type $\mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$ since the angle experiences a discontinuity when going from $2\pi - \varepsilon$ to 0.

that are harmonic. To this end, we compute

$$\begin{aligned}\Delta P_n(x, y) &= \sum_{k=0}^{n-2} c_k(n-k)(n-k-1)x^{n-k-2}y^k + \sum_{h=2}^n c_h h(h-1)x^{n-h}y^{h-2} \\ &= \sum_{k=0}^{n-2} c_k(n-k)(n-k-1)x^{n-k-2}y^k + \sum_{k=0}^{n-2} c_{k+2}(k+2)(k+1)x^{n-k-2}y^k \\ &= \sum_{k=0}^{n-2} [c_k(n-k)(n-k-1) + c_{k+2}(k+2)(k+1)]x^{n-k-2}y^k\end{aligned}$$

and notice that, for P_n to be harmonic, each summand must necessarily have zero coefficient. That is, for any k , we need

$$c_k(n-k)(n-k-1) + c_{k+2}(k+2)(k+1) = 0.$$

This condition yields

$$c_{k+2} = -\frac{(n-k)(n-k-1)}{(k+2)(k+1)}c_k.$$

Therefore the even coefficients depend on the choice of C_c , while the odd ones depend on c_1 . Let us consider the former ones:

$$\begin{aligned}c_2 &= -\frac{n(n-1)}{2 \cdot 1}C_c = -\frac{n(n-1)(n-2)!}{2! \cdot (n-2)!}C_c = -\binom{n}{2}C_c \\ c_4 &= -\frac{(n-2)(n-3)}{4 \cdot 3}c_2 = \frac{n(n-1)(n-2)(n-3)}{4 \cdot 3 \cdot 2 \cdot 1}C_c = \\ &= \frac{n(n-1)(n-2)(n-3)(n-4)!}{4! \cdot (n-4)!}C_c = \binom{n}{4}C_c.\end{aligned}$$

By induction, we can prove that

$$c_{2h} = (-1)^h \binom{n}{2h} C_c, \quad \text{and, analogously,} \quad c_{2h+1} = (-1)^h \binom{n}{2h+1} \frac{c_1}{n}.$$

In conclusion

$$P_n(x, y) = \sum_{k=0}^n \tilde{c}_k \binom{n}{k} x^{n-k} y^k, \quad \text{with} \quad \tilde{c}_k = \begin{cases} (-1)^h C_c & \text{for } k = 2h \\ (-1)^h \frac{c_1}{n} & \text{for } k = 2h + 1 \end{cases}$$

and C_c, c_1 arbitrary.

EXAMPLE 2.7 (Alternative approach). Since the function z^n is holomorphic, we can write $z = (x + iy)$ and the functions $\operatorname{Re}(z^n)$ and $\operatorname{Im}(z^n)$ will be harmonic polynomials of degree n . Expanding the binomial, we have that

$$z^n = \sum_{k=0}^n \binom{n}{k} i^k y^k x^{n-k} = \underbrace{\sum_{\substack{k=0 \\ \text{even}}}^n (-1)^{k/2} \binom{n}{k} x^{n-k} y^k}_{P_1(x, y)} + i \underbrace{\sum_{\substack{k=1 \\ \text{odd}}}^n (-1)^{\frac{k-1}{2}} \binom{n}{k} x^{n-k} y^k}_{P_2(x, y)}$$

Hence, both sums give us harmonic polynomials of degree n , and we since the linear combination of harmonic polynomials of degree n is a harmonic polynomial of degree n , we can obtain all the solutions from our previous deduction just as a linear combination of $P_1(x, y)$ and $P_2(x, y)$

EXAMPLE 2.8. Let $u : \Omega \rightarrow \mathbb{R}$ be harmonic and positive and $\beta \geq 1$. Then u^β is sub-harmonic. Indeed, we compute

$$\begin{aligned}\Delta u^\beta &= \sum_{i=1}^d (\beta u^{\beta-1} \partial_{x_i}^2 u + \beta(\beta-1) u^{\beta-2} \partial_{x_i} u \partial_{x_i} u) \\ &= \sum_{i=1}^d \beta(\beta-1) u^{\beta-2} \partial_{x_i} u \partial_{x_i} u = \sum_{i=1}^d \beta(\beta-1) u^{\beta-2} (\partial_{x_i} u)^2.\end{aligned}$$

EXAMPLE 2.9. From [Example 2.8](#), we note that, if $u \in C^2(\mathbb{R}^n)$ is a positive function and $\beta \in \mathbb{R}$, then

$$\Delta(u^\beta) = \beta u^{\beta-1} \Delta u + \beta(\beta-1)u^{\beta-2} |\nabla u|^2.$$

EXAMPLE 2.10. Let $u : \Omega \rightarrow \mathbb{R}$ be harmonic and positive. Then

$$\Delta \log u = \sum_{i=1}^n \left(\frac{\partial_{x_i}^2 u}{u} - \frac{\partial_{x_i} u \partial_{x_i} u}{u^2} \right) = - \sum_{i=1}^n \frac{\partial_{x_i} u \partial_{x_i} u}{u^2} = - \sum_{i=1}^n \frac{(\partial_{x_i} u)^2}{u^2}.$$

Thus, $\log u$ is super-harmonic and $-\log u$ is sub-harmonic.

EXAMPLE 2.11. Let $u : \Omega \rightarrow \mathbb{R}$ be harmonic, $f : u(\Omega) \rightarrow \mathbb{R}$ be a C^2 convex function. Then $f \circ u$ is sub-harmonic. We compute

$$\begin{aligned} \Delta f(u(x)) &= \sum_{i=1}^n (f'(u(x)) \partial_{x_i}^2 u + f''(u(x)) \partial_{x_i} u \partial_{x_i} u) \\ &= \sum_{i=1}^n f''(u(x)) (\partial_{x_i} u)^2 \quad (\text{since } u \text{ is harmonic}) \\ &\geq 0 \end{aligned}$$

since for a convex C^2 -function $f'' \geq 0$.

REMARK 2.2 (Chain rule). From [Example 2.11](#), we note that, if $u \in C^2(\mathbb{R}^n)$ and $\varphi \in C^2(\mathbb{R})$,

$$\Delta \varphi(u) = \varphi''(u) |\nabla u|^2 + \varphi'(u) \Delta u$$

(which is just the chain rule). If we assume that φ is also convex, then $\varphi'' \geq 0$, and we obtain

$$(-\Delta) \varphi(u) \leq \varphi'(u) (-\Delta) u.$$

REMARK 2.3 (Product rule). If $u, v \in C^2(\mathbb{R}^n)$, we compute

$$\Delta(uv) = u \Delta v + 2 \nabla u \cdot \nabla v + v \Delta u.$$

In particular, we notice that, if u and v are harmonic, then uv is harmonic if and only if $\nabla u \cdot \nabla v \equiv 0$.

EXAMPLE 2.12. If $u, v \in C^2(\Omega)$ with u positive, then

$$\begin{aligned} \Delta(u^v) &= v u^{v-1} \Delta u + u^v (\log u) \Delta v + v(v-1) u^{v-2} |\nabla u|^2 \\ &\quad + u^v (\log u)^2 |\nabla v|^2 + 2u^{v-1} (1 + v \log u) \nabla u \cdot \nabla v. \end{aligned}$$

EXAMPLE 2.13 (Laplacian in polar coordinates). Let $u \in C^2(\mathbb{R}^2)$ and define a function U by $U(r, \theta) = u(r \cos \theta, r \sin \theta)$. We compute

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2}.$$

EXAMPLE 2.14 (Laplacian in spherical coordinates). Let $u \in C^2(\mathbb{R}^3)$ and define a function U by $U(\rho, \theta, \varphi) = u(\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi)$. We compute

$$\Delta u = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial U}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \varphi} \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial U}{\partial \varphi} \right) + \frac{1}{\rho^2 \sin^2 \varphi} \frac{\partial^2 U}{\partial \theta^2}.$$

2. Properties of harmonic functions

Before addressing the question of the existence and uniqueness of solutions of the above-mentioned boundary value problems, we first focus on some important properties of harmonic functions.

2.1. Mean-value formula.

REMARK 2.4. *Heuristically, we can think of $\frac{\partial u}{\partial x_i}(x) \approx \frac{u(x) - u(x - he_i)}{h}$ for small h . Using a finite difference formula again, we see that:*

$$\frac{\partial^2 u}{\partial x_i^2} \approx \frac{\frac{u(x+he_i) - u(x-he_i+he_i)}{h} - \frac{u(x) - u(x-he_i)}{h}}{h} = \frac{u(x+he_i) + u(x-he_i) - 2u(x)}{h^2}$$

Hence, one can think of the laplacian of a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$\Delta u \approx \frac{1}{h^2} \sum_{i=1}^n \sum_{\pm} (u(x \pm he_i) - u(x))$$

The way to read this equality is that $x \pm he_i$ are the "neighbours" of x . The value of h is small, and the choices for \pm and e_i correspond to the possible $2n$ cartesian you can move in \mathbb{R}^n (in 3D: up/down, left/right, front/back). The laplacian is simply taking u at the neighbours of x and subtracting $u(x)$ and averaging/summing among all possible such directions.

Therefore, **the Laplacian $\Delta u(x)$ measures the average value of u at neighbouring points of x minus $u(x)$** . For example, local minima will have $\Delta u(x) \geq 0$ (u is higher at neighbours) and local maxima will have $\Delta u(x) \leq 0$.⁸ As we can see, this is a very general property that goes much beyond this heuristic, and has profound implications in the study of elliptic PDE.

That interpretation is also meaningful relevant for PDE. If we think that $\Delta u = 0$ corresponds to the equilibrium temperature of a body with some boundary conditions, what we are actually asking is that the temperature at any given point x is simply equal to the average temperature of its neighbours.

In what follows, we will use the notation $\omega_n := |\partial B_1(0)|$ for the surface of the unit sphere in \mathbb{R}^n . For instance, $\omega_2 = 2\pi$, $\omega_3 = 4\pi$, and the general expression is $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. We will use $\alpha_n = |B_1(0)|$ to denote the volume of the unit ball in \mathbb{R}^n . We can compute the relationship between α_n and ω_n as follows:

$$\alpha_n = \int_{B_1(0)} 1 \, dx = \int_0^1 |\partial B_r(0)| \, dr = \omega_n \int_0^1 r^{n-1} \, dr = \frac{\omega_n}{n}.$$

Finally, by a simple scaling argument, we can check that $|\partial B_r(x)| = r^{n-1}\omega_n$ and $|B_r(x)| = r^n\alpha_n$, for any $r > 0$ and $x \in \mathbb{R}^n$. We also use the symbol \int for the average integral, that is:

$$\int_{\Omega} f(x) dx = \frac{1}{|\Omega|} \int_{\Omega} f(x) dx.$$

Let us now explore the properties of the Laplacian regarding averages in a more rigorous way. Suppose $u \in C^2(\Omega)$. We define the spherical average of u around a point x as a function of the radius r :

$$\begin{aligned} \phi(r) &:= \int_{\partial B_r(x)} u(y) \, dS(y) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) \, dS(y), \quad \text{for } r > 0, \\ \phi(0) &:= u(x). \end{aligned}$$

Since u is continuous, ϕ is also continuous at $r = 0$. Our goal is to find a relationship between $\phi(r)$ and $\Delta u(x)$. To do this, we compute the derivatives of ϕ .

By changing variables to the unit sphere (letting $y = x + rz$ for $z \in \partial B_1(0)$), we can write $\phi(r)$ as:

$$\phi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_1(0)} u(x + rz) r^{n-1} \, dS(z) = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x + rz) \, dS(z).$$

⁸One can also deduce those facts from the eigenvalue test on the Hessian for local extrema. One just need to realize that the Laplacian is the trace of the Hessian and hence the sum of eigenvalues of the Hessian. If all eigenvalues are ≤ 0 (case of a local maximum), of course their sum is ≤ 0 .

This form is convenient for differentiation. Taking the derivative with respect to r gives:

$$\begin{aligned}\phi'(r) &= \frac{1}{\omega_n} \int_{\partial B_1(0)} \frac{d}{dr} u(x + rz) dS(z) \\ &= \frac{1}{\omega_n} \int_{\partial B_1(0)} \nabla u(x + rz) \cdot z dS(z).\end{aligned}$$

Changing variables back to an integral over $\partial B_r(x)$, we note that $z = \frac{y-x}{r}$ is the outward unit normal $\nu(y)$. This gives:

$$\phi'(r) = \frac{1}{\omega_n} \int_{\partial B_r(x)} \nabla u(y) \cdot \nu(y) \frac{dS(y)}{r^{n-1}} = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} \partial_\nu u(y) dS(y).$$

Now, we use the Gauss-Ostrogradsky theorem (Divergence Theorem) to convert the boundary integral into a volume integral:

$$\begin{aligned}\phi'(r) &= \frac{1}{\omega_n r^{n-1}} \int_{B_r(x)} \operatorname{div}(\nabla u)(y) dy \\ &= \frac{1}{\omega_n r^{n-1}} \int_{B_r(x)} \Delta u(y) dy \\ &= \frac{|B_r(x)|}{\omega_n r^{n-1}} \int_{B_r(x)} \Delta u(y) dy = \frac{\alpha_n r^n}{\omega_n r^{n-1}} \int_{B_r(x)} \Delta u(y) dy \\ &= \frac{\alpha_n r}{\omega_n} \int_{B_r(x)} \Delta u(y) dy.\end{aligned}\tag{2.1}$$

From this formula, we can see that $\lim_{r \rightarrow 0^+} \phi'(r) = 0$, so we define $\phi'(0) = 0$. To find the second derivative at 0, we compute:

$$\phi''(0) = \lim_{r \rightarrow 0^+} \frac{\phi'(r) - \phi'(0)}{r} = \lim_{r \rightarrow 0^+} \frac{\phi'(r)}{r} = \lim_{r \rightarrow 0^+} \frac{\alpha_n}{\omega_n} \int_{B_r(x)} \Delta u(y) dy.$$

Since Δu is continuous, the average value converges to the point value, so:

$$\phi''(0) = \frac{\alpha_n}{\omega_n} \Delta u(x).$$

Finally, using Taylor's expansion for $\phi(r)$ around $r = 0$, we have:

$$\phi(r) = \phi(0) + \phi'(0)r + \frac{\phi''(0)}{2}r^2 + o(r^2) = u(x) + 0 + \frac{\alpha_n}{2\omega_n} \Delta u(x)r^2 + o(r^2).$$

Rearranging this expression gives us the rigorous formula we were looking for, which connects the Laplacian to the average value of the function on a small sphere:

$$\Delta u(x) = \frac{2\omega_n}{\alpha_n} \lim_{r \rightarrow 0^+} \frac{\phi(r) - u(x)}{r^2} = 2n \lim_{r \rightarrow 0^+} \frac{\int_{\partial B_r(x)} u(y) dS(y) - u(x)}{r^2}.\tag{2.2}$$

This confirms our heuristic: the Laplacian at x is proportional to the difference between the average of u on a small sphere around x and the value of u at x itself.

Now, from (2.1) we can conclude some Theorems for harmonic and sub/superharmonic functions

THEOREM 2.1 (Mean-value formula). *Let Ω be a domain and $u \in C^2(\Omega)$ a harmonic function. Then, for any closed ball $\overline{B_r(x)} \subset \Omega$, it holds*

$$u(x) = \int_{\partial B_r(x)} u(y) dS(y) = \int_{B_r(x)} u(y) dy.\tag{2.3}$$

PROOF. We go back to (2.1) and notice that since $\Delta u = 0$, we have $\phi' = 0$. Hence, we have that $\phi(r) = \phi(0) = u(x)$. Then, the first equality then follows from the definition of ϕ .

To prove the identity for the average over the ball we integrate the spherical averages over the radius. Using the first equality, we compute:

$$\begin{aligned}
\int_{B_r(x)} u(y) dy &= \int_0^r \left(\int_{\partial B_t(x)} u(y) dS(y) \right) dt \\
&= \int_0^r \left(|\partial B_t(x)| \int_{\partial B_t(x)} u(y) dS(y) \right) dt \\
&= \int_0^r (t^{n-1} \omega_n) u(x) dt \\
&= u(x) \omega_n \int_0^r t^{n-1} dt \\
&= u(x) \frac{r^n \omega_n}{n} \\
&= u(x) |B_r(x)|.
\end{aligned}$$

Dividing by $|B_r(x)|$ yields the desired result. \square

Following the proof of [Theorem 2.1](#), we can actually show a mean-value property for sub/super-harmonic functions as well.

THEOREM 2.2 (Mean-value property for sub/super-harmonic functions). *Let $u \in C^2(\Omega)$ be a sub-harmonic (resp., super-harmonic) function. Then, for any closed ball $\overline{B_r(x)} \subset \Omega$ it holds*

$$\begin{aligned}
u(x) &\leq (\text{resp. } \geq) \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) dy, \\
u(x) &\leq (\text{resp. } \geq) \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) dS(y).
\end{aligned}$$

PROOF. The proof is totally analogous to the one above, with the only difference that in this case, we have that either $\Delta u \geq 0$ and $\phi' \geq 0$ (subharmonic) or $\Delta u \leq 0$ and $\phi' \leq 0$ (superharmonic). Clearly, when $\phi' \geq 0$, we will have that $\phi(s) \geq \phi(0) = u(x)$ and similarly when $\phi' \leq 0$ we have $\phi(s) \leq \phi(0) = u(x)$. \square

It follows from these inequalities in [Theorem 2.2](#) that the value of a sub-harmonic (or super-harmonic) function at the center of a ball is less (or greater) than or equal to the value of a harmonic function with the same values on the boundary. Thus, the graphs of sub-harmonic functions lie below the graphs of harmonic functions and the graphs of super-harmonic functions lie above, which explains the terminology.

EXAMPLE 2.15. *The function $u(x) = |x|^4$ is sub-harmonic in \mathbb{R}^n since $\Delta u = 4(n+2)|x|^2 \geq 0$. The function is equal to the constant harmonic function $U(x) = 1$ on the sphere $|x| = 1$, and $u(x) \leq U(x)$ when $|x| \leq 1$.*

The converse of the Mean-Value property for harmonic functions also holds, i.e. if a function $u \in C^2(\Omega)$ satisfies the mean-value property, then it is harmonic.

THEOREM 2.3 (Mean-value property implies harmonicity). *If $u \in C^2(\Omega)$ satisfies the mean-value property*

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) dS(y)$$

for any closed ball $\overline{B_r(x)} \subset \Omega$, then u is harmonic in Ω .

PROOF. We argue by contradiction. If u is not harmonic, then there exists $x \in \Omega$ such that $\Delta u(x) \neq 0$. Assume without loss of generality $\Delta u(x) > 0$. Since $u \in C^2(\Omega)$, then there exists $s > 0$ such that $\Delta u(y) > 0$, for all $y \in \overline{B_s(x)}$ and $\overline{B_s(x)} \subset \Omega$. On the one hand, since u satisfies the mean value property, we have that $\phi(r)$ is constant (and therefore equal to $\phi(0) = u(x)$). On the other hand, we still have [\(2.1\)](#). Combining those two facts, we get

$$0 = \phi'(r) = \frac{1}{r^{n-1} \omega_n} \int_{B_r(x)} \Delta u(y) dy > 0,$$

which is a contradiction. \square

In [Theorem 2.3](#), we can drop the hypothesis $u \in C^2(\Omega)$ and the claim remains true for any $u \in C^0(\Omega)$, i.e., if a function $u \in C^0(\Omega)$ satisfies the mean-value formula [\(2.3\)](#) for any closed ball contained in Ω , then it is automatically $C^2(\Omega)$ and harmonic. More than that, it is of class $C^\infty(\Omega)$. To prove this result we recall first the notion of mollifier⁹.

DEFINITION 2.2 (Standard mollifier). *Let us consider the C^∞ function*

$$\phi : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad \phi(r) := \begin{cases} C \exp\left(-\frac{1}{r^2-1}\right), & \text{if } 0 \leq r < 1, \\ 0, & \text{if } r \geq 1, \end{cases}$$

with $C > 0$ chosen such that $\int_{\mathbb{R}^n} \phi(\|x\|) dx = 1$. For any $\varepsilon > 0$ we call standard ε -mollifier the function

$$\eta_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}, \quad \eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \phi\left(\frac{\|x\|}{\varepsilon}\right)$$

We can show that the ε -mollifier satisfies $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \eta_\varepsilon(x) dx = 1$, and $\text{supp}(\eta_\varepsilon) = \overline{B_\varepsilon(0)}$ where, for a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we recall that $\text{supp}(f) = \{x \in \mathbb{R}^n : f(x) \neq 0\}$.

DEFINITION 2.3 (Mollification). *Let $\Omega \subset \mathbb{R}^n$ be a domain and, for all $\varepsilon > 0$, define the subdomain $\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$. For $f : \Omega \rightarrow \mathbb{R}$ locally integrable, we call ε -mollification of f the function $f_\varepsilon : \Omega_\varepsilon \rightarrow \mathbb{R}$,*

$$f_\varepsilon(x) = (\eta_\varepsilon * f)(x) = \int_{\Omega} \eta_\varepsilon(x-y) f(y) dy = \int_{B_\varepsilon(0)} \eta_\varepsilon(y) f(x-y) dy$$

It can be proven that $f_\varepsilon \in C^\infty(\Omega_\varepsilon)$ (for all $\varepsilon > 0$) and that the following properties hold:

- (1) $f_\varepsilon \rightarrow f$ a.e. in Ω as $\varepsilon \rightarrow 0$;
- (2) if $f \in C^0(\Omega)$ then $f_\varepsilon \rightarrow f$ uniformly on any compact subset of Ω ;
- (3) if $f \in L^p_{\text{loc}}(\Omega)$ then $f_\varepsilon \rightarrow f$ in $L^p_{\text{loc}}(\Omega)$.

THEOREM 2.4. *If $u \in C^0(\Omega)$ satisfies the mean-value property*

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) dS(y)$$

for any closed ball $\overline{B_r(x)} \subset \Omega$, then $u \in C^\infty(\Omega)$ and is harmonic in Ω .

PROOF. Let us consider an ε -mollification of u , $u_\varepsilon = \eta_\varepsilon * u : \Omega_\varepsilon \rightarrow \mathbb{R}$ and recall that η_ε is a radial function, i.e. $\eta_\varepsilon(x) = \varepsilon^{-n} \phi(\|x\|/\varepsilon)$ and $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$. Then

$$\begin{aligned} u_\varepsilon(x) &= \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y) u(y) dy = \int_{B_\varepsilon(x)} \frac{1}{\varepsilon^n} \phi\left(\frac{\|x-y\|}{\varepsilon}\right) u(y) dy \\ &= \int_0^\varepsilon \left(\int_{\partial B_t(x)} \frac{1}{\varepsilon^n} \phi\left(\frac{t}{\varepsilon}\right) u(y) dS(y) \right) dt \\ &= \frac{1}{\varepsilon^n} \int_0^\varepsilon \phi\left(\frac{t}{\varepsilon}\right) \left(\int_{\partial B_t(x)} u(y) dS(y) \right) dt \\ &= \frac{1}{\varepsilon^n} \int_0^\varepsilon \phi\left(\frac{t}{\varepsilon}\right) |\partial B_t(x)| u(x) dt \quad (\text{by mean-value property}) \\ &= \frac{u(x)}{\varepsilon^n} \int_0^\varepsilon \phi\left(\frac{t}{\varepsilon}\right) |\partial B_t(x)| dt \\ &= u(x) \int_{B_\varepsilon(0)} \eta_\varepsilon(y) dy \\ &= u(x). \end{aligned}$$

Hence $u(x) = u_\varepsilon(x)$ for all $x \in \Omega_\varepsilon$ and we conclude that $u \in C^\infty(\Omega_\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, it follows that $u \in C^\infty(\Omega)$, hence, in particular, $u \in C^2(\Omega)$ and, by [Theorem 2.3](#), u is harmonic in Ω . \square

⁹ Also known as *Friedrichs mollifier*, after Kurt Otto Friedrichs [[Fri44](#)], or *approximations of the identity*

REMARK 2.5. *The stronger result of Theorem 2.4 does not extend to sub/super harmonic functions, in general. If a function $u \in C^0(\Omega)$ satisfies $u(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) dy$ for any closed $\overline{B_r(x)} \subset \Omega$, it is not true, in general, that $u \in C^\infty(\Omega)$, nor that $-\Delta u \leq 0$ (the Laplacian of u might not even exist).*

We stress that the C^∞ regularity result for harmonic functions contained in Theorem 2.4 says nothing about the behavior of u at the boundary of Ω .

EXAMPLE 2.16. *We can check that the functions*

$$u(x, y) = \frac{x}{x^2 + y^2}, \quad v(x, y) = -\frac{y}{x^2 + y^2}$$

are harmonic and C^∞ in the open unit disc

$$\Omega = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 1\}.$$

However, both are unbounded as $(x, y) \rightarrow (0, 0) \in \partial\Omega$.

2.2. Derivative estimates. An important feature of the Laplace equation is that we can estimate the derivatives of a solution in a ball in terms of the solution on a larger ball.

THEOREM 2.5 (Gradient estimate). *Let us suppose that $u \in C^2(\Omega)$ is harmonic in the open set Ω and $B_r(x) \Subset \Omega$. Then, for any $1 \leq i \leq n$,*

$$|\partial_i u(x)| \leq \frac{n}{r} \max_{x \in \overline{B_r(x)}} |u(x)|.$$

PROOF. Since u is smooth, differentiation of Laplace's equation with respect to x_i shows that $\partial_i u$ is harmonic. Therefore, by the mean-value property for balls and the divergence theorem,

$$\partial_i u = \int_{B_r(x)} \partial_i u(x) dx = \frac{1}{\alpha_n r^n} \int_{\partial B_r(x)} u(y) \nu_i(y) dS(y).$$

Taking the absolute value of this equation and using the estimate

$$\left| \int_{\partial B_r(x)} u \nu_i dS \right| \leq n \alpha_n r^{n-1} \max_{B_r(x)} |u|$$

we get the result. \square

One consequence of Theorem 2.5 is that a bounded harmonic function on \mathbb{R}^n is constant.¹⁰

COROLLARY 2.1 (Liouville's theorem). *If $u \in C^2(\mathbb{R}^n)$ is bounded and harmonic in \mathbb{R}^n , then u is constant.*

PROOF. If $|u| \leq M$ on \mathbb{R}^n , then Theorem 2.5 implies that

$$|\partial_i u(x)| \leq \frac{Mn}{r}$$

for any $r > 0$. Taking the limit as $r \rightarrow \infty$, we conclude that $\nabla u = 0$, so u is constant. \square

We will also present an alternative proof of Corollary 2.1, due to [Nel61].

ALTERNATIVE PROOF OF COROLLARY 2.1. Let $r > 0$ and consider $x, y \in \mathbb{R}^n$. The mean-value property yields:

$$\begin{aligned} |u(x) - u(y)| &= \frac{1}{|B_r|} \left| \int_{B_r(x)} u(z) dz - \int_{B_r(y)} u(z) dz \right| \\ &\leq M \frac{|B_r(x) \Delta B_r(y)|}{|B_r|} \\ &\longrightarrow 0 \quad \text{as } r \rightarrow +\infty. \end{aligned}$$

Thus $u(x) = u(y)$ and we conclude that u is a constant from the arbitrariness of $x, y \in \mathbb{R}^n$. \square

¹⁰ This result is called Liouville's theorem, after Joseph Liouville. Actually, this statement first appeared in [Bôc03] and was later rediscovered in [Pic24]. On the other hand, Liouville, in [Lio80], proved that "A doubly periodic function without poles is identically constant". Finally, the statement of Liouville's theorem about holomorphic functions is actually due to Augustin-Louis Cauchy [Cau44], in response to [Lio80].

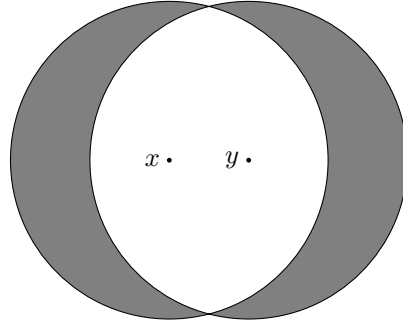


FIGURE 1. Illustration of the argument in the alternative proof of Corollary 2.1.

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We can also prove a derivative estimate in terms of the L^1 -norm of u .

THEOREM 2.6. *Let us suppose that $u \in C^2(\Omega)$ is harmonic in the open set $\Omega \subset \mathbb{R}^n$ and $B_r(x) \subset \Omega$. Then, for any $1 \leq j \leq n$,*

$$|u(x)| \leq \frac{1}{\alpha_n r^n} \|u\|_{L^1(B_r(x))}, \quad (2.4)$$

$$|\partial_{x_j} u(x)| \leq \frac{2^{n+1}n}{\alpha_n r^{n+1}} \|u\|_{L^1(B_r(x))}. \quad (2.5)$$

PROOF. By the mean-value property,

$$|u(x)| = \left| \frac{1}{\alpha_n r^n} \int_{B_r(x)} u(y) \, dy \right| \leq \frac{1}{\alpha_n r^n} \int_{B_r(x)} |u(y)| \, dy = \frac{1}{\alpha_n r^n} \|u\|_{L^1(B_r(x))},$$

which yields (2.4).

To show (2.5), note that $\partial_{x_j} u$ is harmonic. Thus, by the divergence theorem, we have

$$\begin{aligned} |\partial_{x_j} u(x)| &= \frac{1}{\alpha_n \left(\frac{r}{2}\right)^n} \left| \int_{B_{r/2}(x)} \partial_{x_j} u(y) \, dy \right| \\ &= \frac{2^n}{\alpha_n r^n} \left| \int_{\partial B_{r/2}(x)} u(y) \nu_j \, dS(y) \right| \\ &\leq \frac{2^n}{\alpha_n r^n} \int_{\partial B_{r/2}(x)} |u(y)| \, dS(y). \end{aligned}$$

Within the integral, we can apply (2.4) over $B_{r/2}(y)$, obtaining

$$\begin{aligned} |\partial_{x_j} u(x)| &\leq \frac{2^n}{\alpha_n r^n} \int_{\partial B_{r/2}(x)} \frac{2^n}{\alpha_n r^n} \|u\|_{L^1(B_{r/2}(y))} \, dS(y) \\ &= \frac{2^{2n}}{\alpha_n^2 r^{2n}} \int_{\partial B_{r/2}(x)} \|u\|_{L^1(B_{r/2}(y))} \, dS(y). \end{aligned}$$

Since $B_{r/2}(y) \subset B_r(x)$, we have $\|u\|_{L^1(B_{r/2}(y))} \leq \|u\|_{L^1(B_r(x))}$, which leads to

$$|\partial_{x_j} u(x)| \leq \frac{2^{2n}}{\alpha_n^2 r^{2n}} \int_{\partial B_{r/2}(x)} \|u\|_{L^1(B_r(x))} \, dS(y) = \frac{2^{2n}}{\alpha_n^2 r^{2n}} \|u\|_{L^1(B_r(x))} \int_{\partial B_{r/2}(x)} dy.$$

The claim follows by observing that

$$\int_{\partial B_{r/2}(x)} dy = \omega_n \left(\frac{r}{2}\right)^{n-1} = n\alpha_n \left(\frac{r}{2}\right)^{n-1}.$$

□

We can extend the estimate in Theorem 2.5 to higher-order derivatives.

THEOREM 2.7 (Estimate on higher derivatives). *Let us suppose that $u \in C^2(\Omega)$ is harmonic in the open set Ω and $B_r(x) \Subset \Omega$. Then for any multi-index $\alpha \in \mathbb{N}_0^n$ of order $k = |\alpha|$,*

$$|\partial^\alpha u(x)| \leq \frac{n^k e^{k-1} k!}{r^k} \max_{B_r(x)} |u|.$$

PROOF. We prove the result by induction on $|\alpha| = k$.

Base case: From [Theorem 2.5](#), the result is true when $k = 1$.

Induction step: Let us suppose that the result is true when $|\alpha| = k$. We will prove it for $|\alpha| = k + 1$.

If $|\alpha| = k + 1$, we may write $\partial^\alpha = \partial_{x_i} \partial^\beta$ where $1 \leq i \leq n$ and $|\beta| = k$. For $0 < \theta < 1$, let $\rho := (1 - \theta)r$. Then, since $\partial^\beta u$ is harmonic and $B_\rho(x) \Subset \Omega$, [Theorem 2.5](#) implies that

$$|\partial^\alpha u(x)| \leq \frac{n}{\rho} \max_{x \in B_\rho(x)} |\partial^\beta u(x)|.$$

Let $y \in B_\rho(x)$. Then $B_{r-\rho}(y) \subset B_r(x)$, and, using the induction hypothesis, we get

$$|\partial^\beta u(y)| \leq \frac{n^k e^{k-1} k!}{(r - \rho)^k} \max_{B_{r-\rho}(y)} |u| \leq \frac{n^k e^{k-1} k!}{r^k \theta^k} \max_{B_r(x)} |u|.$$

We then deduce that

$$|\partial^\alpha u(x)| \leq \frac{n^{k+1} e^{k-1} k!}{r^{k+1} \theta^k (1 - \theta)} \max_{B_r(x)} |u|.$$

Choosing $\theta = k/(k + 1)$ and, using the inequality

$$\frac{1}{\theta^k (1 - \theta)} = \left(1 + \frac{1}{k}\right)^k (k + 1) \leq e(k + 1),$$

we get

$$|\partial^\alpha u(x)| \leq \frac{n^{k+1} e^k (k + 1)!}{r^{k+1}} \max_{B_r(x)} |u|.$$

□

A consequence of this estimate is that the Taylor series of u converges to u near any point.

THEOREM 2.8 (Harmonic functions are analytic). *If $u \in C^2(\Omega)$ is harmonic in an open set Ω then u is real-analytic in Ω .*

PROOF. Suppose that $x \in \Omega$ and choose $r > 0$ such that $B_{2r}(x) \Subset \Omega$. Since $u \in C^\infty(\Omega)$, we may expand it in a Taylor series with remainder of any order $k \in \mathbb{N}$ to get

$$u(x + h) = \sum_{|\alpha| \leq k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x, h),$$

where we assume that $|h| < r$.

The remainder¹¹ is given by

$$R_k(x, h) = \sum_{|\alpha|=k} \frac{\partial^\alpha u(x + \theta h)}{\alpha!} h^\alpha,$$

for some $0 < \theta < 1$. We have to show that $R_k \rightarrow 0$ as $k \rightarrow +\infty$ (for a sufficiently small $|h|$). To estimate the remainder, we use [Theorem 2.7](#):

$$|\partial^\alpha u(x + \theta h)| \leq \frac{n^k e^{k-1} k!}{r^k} \max_{B_{r(x+\theta h)}} |u|.$$

¹¹ Let us recall Taylor's theorem: If $u \in C^k(B_r(x))$ and $h \in B_r(0)$, then

$$u(x + h) = \sum_{|\alpha| \leq k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x, h),$$

where the remainder is given by

$$R_k(x, h) = \sum_{|\alpha|=k} \frac{\partial^\alpha u(x + \theta h)}{\alpha!} h^\alpha$$

for some $0 < \theta < 1$. We also recall that, for a multi-index $\alpha \in \mathbb{N}_0^n$, the factorial is defined as $\alpha! = \alpha_1! \dots \alpha_k!$

Since $|h| < r$, we have $B_r(x + \theta h) \subset B_{2r}(x)$; so, for any $0 < \theta < 1$, we have

$$\max_{B_r(x+\theta h)} |u| \leq M, \quad M = \max_{B_{2r}(x)} |u|$$

and then

$$|\partial^\alpha u(x + \theta h)| \leq \frac{M n^k e^{k-1} k!}{r^k}$$

Since $|h^\alpha| \leq |h|^k$ when $|\alpha| = k$, we deduce

$$|R_k(x, h)| \leq \frac{M n^k e^{k-1} |h|^k k!}{r^k} \left(\sum_{|\alpha|=k} \frac{1}{\alpha!} \right)$$

The multinomial expansion

$$n^k = (1 + 1 + \cdots + 1)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} = \sum_{|\alpha|=k} \frac{k!}{\alpha!}$$

shows that

$$\sum_{|\alpha|=k} \frac{1}{\alpha!} = \frac{n^k}{k!}.$$

In conclusion, we have

$$|R_k(x, h)| \leq \frac{M}{e} \left(\frac{n^2 e |h|}{r} \right)^k$$

and, thus, $R_k(x, h) \rightarrow 0$ as $k \rightarrow \infty$ provided that $|h| < \frac{r}{n^2 e}$. \square

2.3. Maximum principles. We present now a second important property of harmonic and sub/super-harmonic functions, the *maximum principle*.

2.3.1. *Strong maximum principle.* First, we present the *strong maximum principle*, which states that a sub/super-harmonic function which attains an interior maximum/minimum is a trivial constant function.

THEOREM 2.9 (Strong maximum principle for sub-harmonic functions). *Let us suppose that $\Omega \subset \mathbb{R}^n$ is a connected open set and $u \in C^2(\Omega)$. If u is sub-harmonic and attains a global maximum value in Ω , then u is constant in Ω .*

PROOF. By assumption, u is bounded from above and attains its maximum in Ω . Let

$$M := \max_{\Omega} u$$

and consider

$$F := \{x \in \Omega : u(x) = M\}.$$

Then F is non-empty (by assumption, since the maximum is attained) and relatively closed in Ω (since u is continuous). If $x \in F$ and $B_r(x) \Subset \Omega$, then the mean value inequality for sub-harmonic functions implies that

$$\int_{B_r(x)} [u(y) - u(x)] dy = \int_{B_r(x)} u(y) dy - u(x) \geq 0.$$

Since u attains its maximum at x , we have $u(y) - u(x) \leq 0$ for all $y \in \Omega$, and it follows that $u(y) = u(x)$ in $B_r(x)$. Therefore, F is open as well as closed. Since Ω is connected, and F is non-empty, we must have $F = \Omega$, so u is constant in Ω .

If Ω is not connected, then u is constant in any connected component of Ω that contains an interior point where u attains a maximum value. \square

EXAMPLE 2.17. *The function $u(x) = |x|^2$ is sub-harmonic in \mathbb{R}^n . It attains a global minimum in \mathbb{R}^n at the origin, but it does not attain a global maximum in any open set $\Omega \subset \mathbb{R}^n$. It does, of course, attain a maximum in any bounded closed set $\bar{\Omega}$, but the attainment of a maximum at a boundary point instead of an interior point does not imply that a sub-harmonic function is constant.*

It follows immediately that super-harmonic functions satisfy a minimum principle, and harmonic functions satisfy a maximum and minimum principle.

THEOREM 2.10 (Strong maximum principle for harmonic functions). *Let us suppose that Ω is a connected open set and $u \in C^2(\Omega)$. If u is harmonic and attains either a global minimum or maximum in Ω , then u is constant.*

PROOF. Any super-harmonic function u that attains a minimum in Ω is constant, since $-u$ is sub-harmonic and attains a maximum. A harmonic function is both sub-harmonic and super-harmonic. \square

EXAMPLE 2.18. *The function*

$$u(x, y) = x^2 - y^2$$

is harmonic in \mathbb{R}^2 . It has a critical point at 0, meaning that $Du(0) = 0$. This critical point is a saddle-point, however, not an extreme value. Note also that

$$\int_{B_r(0)} u \, dx \, dy = \frac{1}{2\pi} \int_0^{2\pi} (\cos^2 \theta - \sin^2 \theta) \, d\theta = 0,$$

as required by the mean-value property.

2.3.2. Weak maximum principle. **Theorem 2.10** leads to a *weak maximum principle* for harmonic functions, which states that the function is bounded inside a domain by its values on the boundary. In physical terms, this means for example that the interior of a bounded region which contains no heat sources or sinks cannot be hotter than the maximum temperature on the boundary or colder than the minimum temperature on the boundary.

THEOREM 2.11 (Weak maximum principle). *Let Ω be bounded and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $-\Delta u \leq 0$ (resp. $-\Delta u \geq 0$) in Ω . Then*

$$\max_{x \in \Omega} u(x) = \max_{x \in \partial\Omega} u(x) \quad \left(\text{resp. } \min_{x \in \Omega} u(x) = \min_{x \in \partial\Omega} u(x) \right)$$

In particular, if u is harmonic in Ω , then

$$\min_{y \in \partial\Omega} u(y) \leq u(x) \leq \max_{y \in \partial\Omega} u(y), \quad \forall x \in \Omega$$

PROOF. Suppose u sub-harmonic in Ω and let $M = \max_{x \in \bar{\Omega}} u(x)$ which is finite since $u \in C^0(\bar{\Omega})$. If there exists $y \in \Omega$ such that $u(y) = M$, then u is constant by strong maximum principle and the thesis follows. If there is no such $y \in \Omega$, then $u(y) < M = \max_{x \in \partial\Omega} u(x)$ for all $y \in \Omega$. \square

EXAMPLE 2.19. *The harmonic function $u(x) = x_1$ on the half-space $\{x \in \mathbb{R}^n : x_1 > 0\}$ is equal to zero on the boundary, but is positive in the domain. We cannot apply the maximum principle because the domain is unbounded.*

REMARK 2.6. *The maximum principle is a second-order phenomenon. The function $u : [0, 1] \rightarrow \mathbb{R}$ defined as $u(x) = 3x^2 - 4x^3$ satisfies $\frac{d^4}{dx^4} u(x) = 0$ but $x = 1/2$ is an interior maximum.*

COROLLARY 2.2. *Let u and v be harmonic and sub-harmonic in a bounded domain Ω , respectively. If $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω .*

PROOF. We observe that $v - u$ is sub-harmonic in Ω , and that $v - u \leq 0$ on $\partial\Omega$. By the weak maximum principle, $\max_{\Omega} (v - u) \leq 0$, and the claim follows. \square

2.3.3. Alternative proof of the weak maximum principle. We present also a proof by contradiction, which does not use the strong maximum principle and generalizes easily to more general second-order elliptic equations.

ALTERNATIVE PROOF OF THEOREM 2.11. We will split the proof into two cases.

Case 1: $-\Delta u < 0$ (with strict inequality). Let us consider the case $-\Delta u < 0$. Let us assume, for the sake of finding a contradiction, that

$$\max_{x \in \Omega} u(x) > \max_{x \in \partial\Omega} u(x).$$

Then there exists $y \in \Omega$ such that $u(y) = M = \max_{x \in \bar{\Omega}} u(x)$ and, since $u \in C^2(\Omega)$, we have $\partial_{x_i}^2 u(y) \leq 0$ for $i = 1, \dots, n$, which contradicts $-\Delta u(y) < 0$.

Case 2: $-\Delta u \leq 0$. In this case, we introduce an auxiliary function $v \in C^2(\bar{\Omega})$ which satisfies $-\Delta v(x) \leq \alpha < 0$ for all $x \in \Omega$ and some $\alpha > 0$. For instance, we can take

$$v(x) := |x|^2 = \sum_{i=1}^n x_i^2.$$

Then, for any $\varepsilon > 0$, the function $u_\varepsilon := u + \varepsilon v$ satisfies

$$-\Delta u_\varepsilon \leq -2n\varepsilon < 0 \quad \text{in } \Omega$$

and (by the analysis in Case 1)

$$\max_{x \in \bar{\Omega}} u_\varepsilon(x) = \max_{x \in \partial\Omega} u_\varepsilon(x).$$

Since Ω is bounded, there exists $R > 0$ such that $|x| \leq R$ for all $x \in \Omega$. Then

$$\max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} u_\varepsilon \leq \max_{\partial\Omega} u_\varepsilon \leq \max_{\partial\Omega} u + \varepsilon R^2,$$

and, letting $\varepsilon \searrow 0$, we conclude. Finally, we note that the statement for super-harmonic functions, $-\Delta u \geq 0$ is proved in the same way (considering that, if u is sub-harmonic, then $-u$ is super-harmonic).

In particular, for harmonic functions, we conclude

$$\min_{y \in \partial\Omega} u(y) = \min_{y \in \bar{\Omega}} u(y) \leq u(x) \leq \max_{y \in \bar{\Omega}} u(y) = \max_{y \in \partial\Omega} u(y), \quad \text{for all } x \in \Omega.$$

□

2.3.4. *Application to the study of uniqueness for the Laplace equation.* The weak maximum principle can be used to prove uniqueness of solutions of the Dirichlet problem in bounded domains. Indeed, let us assume that two solutions u_1 and u_2 exist. Then, $v = u_1 - u_2$ satisfies the problem

$$\begin{cases} -\Delta v = 0, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

By the weak maximum principle, we have $v = 0$, hence $u_1 = u_2$.

The same principle can be used to establish stability estimates for the solution. Indeed, if u is a solution of the Dirichlet problem with boundary data $g \in C^0(\partial\Omega)$, then

$$\min_{y \in \partial\Omega} g(y) \leq u(x) \leq \max_{y \in \partial\Omega} g(y), \quad \text{for all } x \in \Omega,$$

which implies

$$\|u\|_{C^0(\bar{\Omega})} \leq \|g\|_{C^0(\partial\Omega)},$$

i.e., the C^0 -norm of the solution is controlled by the C^0 -norm of the data.

Similarly, if u_{g_1} and u_{g_2} solve the problems

$$\begin{cases} -\Delta u_{g_1} = 0, & x \in \Omega, \\ u_{g_1} = g_1, & x \in \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_{g_2} = 0, & x \in \Omega, \\ u_{g_2} = g_2, & x \in \partial\Omega, \end{cases}$$

with $g_1, g_2 \in C^0(\partial\Omega)$ and $\|g_1 - g_2\|_{C^0(\partial\Omega)} \leq \varepsilon$, then

$$\|u_{g_1} - u_{g_2}\|_{C^0(\bar{\Omega})} \leq \|g_1 - g_2\|_{C^0(\partial\Omega)} \leq \varepsilon,$$

i.e., small perturbations on the data imply small perturbations on the solution. Equivalently, the solution depends continuously on the data (the map $g \mapsto u_g$ is continuous in the C^0 topology).

2.3.5. *Uniqueness via energy methods.* We mention also an alternative way, based on the so called *energy estimates*, to establish uniqueness of solutions for either the Dirichlet, the Neumann, or the Robin problems.

Energy estimates are obtained by multiplying the equation $-\Delta u = 0$ by u on both sides, integrating over the domain and integrating by parts. If $u_1, u_2 \in C^2(\bar{\Omega})$ are two classical solutions of the boundary value problem and we apply the procedure to their difference, $w = u_1 - u_2$, which also satisfies the equation $-\Delta w = 0$, we obtain

$$0 = \int_{\Omega} w \Delta w \, dx = - \int_{\Omega} \nabla w \cdot \nabla w \, dx + \int_{\partial\Omega} w \partial_\nu w \, dS \quad (2.6)$$

(1) **Dirichlet problem:** Since $w = 0$ on $\partial\Omega$, from (2.6) yields

$$\int_{\Omega} |\nabla w|^2 = \int_{\partial\Omega} w \partial_{\nu} w = 0$$

Since Ω is connected, this implies that w is constant in Ω and being $w = 0$ on $\partial\Omega$ we conclude $w = u_1 - u_2 = 0$ in $\bar{\Omega}$.

(2) **Neumann problem:** Since $\partial w = 0$ on $\partial\Omega$,

$$\int_{\Omega} |\nabla w|^2 = \int_{\partial\Omega} w \partial_{\nu} w = 0.$$

hence, again, w is constant in Ω . However, this time the value of the solution on the boundary is not fixed, so the only conclusion that we can draw is that any two solutions u_1, u_2 differ for a constant value or, equivalently, the solution to the Neumann problem is unique up to an additive constant. We note, on the other hand, that the boundary datum h cannot be chosen arbitrarily for the solution to exist and has to satisfy a compatibility condition

$$\int_{\partial\Omega} h = \int_{\partial\Omega} \partial_{\nu} u = \int_{\Omega} \Delta u = 0.$$

(3) **Robin problem:** Since $\partial_{\nu} w + \alpha w = 0$ on $\partial\Omega$,

$$\int_{\Omega} |\nabla w|^2 + \alpha \int_{\partial\Omega} w^2 = 0,$$

which yields $|\nabla w| = 0$ in Ω and $w = 0$ on $\partial\Omega$, hence again $w = u_1 - u_2 = 0$ in $\bar{\Omega}$ and uniqueness of solutions.

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2.3.6. *Alternative proof of the strong maximum principle.* Next we present another proof, based on a key lemma due to¹² Hopf¹³ [Hop27] and Oleĭnik¹⁴ [Ole52] (see also [AN16; Naz12; PS04b; PS04a] for further discussion) which does not rely on the mean-value property.

LEMMA 2.1 (Zaremba–Hopf–Oleĭnik’s boundary point lemma). *Let us suppose that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is sub-harmonic in an open set Ω and $u(x) < M$ for every $x \in \Omega$. If $u(\bar{x}) = M$ for some $\bar{x} \in \partial\Omega$ and Ω satisfies the interior sphere condition¹⁵ at \bar{x} , then $\partial_{\nu} u(\bar{x}) > 0$, where ∂_{ν} is the derivative in the outward unit normal direction to a sphere that touches $\partial\Omega$ at \bar{x} .*

PROOF. By the interior sphere condition, there is a ball $B_R(x) \subset \Omega$ with $\bar{x} \in \partial B_R(x)$. Let $M' := \max_{\bar{B}_{R/2}(x)} u < M$ and define $\varepsilon := M - M' > 0$. We consider a perturbation

$$w = u + \varepsilon v - M,$$

of u , where $v \in C^2(\mathbb{R}^n)$ to have the following properties:

- (1) $v = 0$ on $\partial B_R(x)$;
- (2) $v = 1$ on $\partial B_{R/2}(x)$;
- (3) $\partial_{\nu} v < 0$ on $\partial B_R(x)$;
- (4) $-\Delta v \leq 0$ in $B_R(x) \setminus \bar{B}_{R/2}(x)$.

Then $w \leq 0$ on $\partial B_R(x)$ and $\partial B_{R/2}(x)$, and $-\Delta w \leq 0$ in $B_R(x) \setminus \bar{B}_{R/2}(x)$. The weak maximum principle for sub-harmonic functions in Theorem 2.11 implies that $w \leq 0$ in $B_R(x) \setminus \bar{B}_{R/2}(x)$. Since $w(\bar{x}) = 0$, it follows that $\partial_{\nu} w(\bar{x}) \geq 0$. Therefore,

$$\partial_{\nu} u(\bar{x}) = \partial_{\nu} w(\bar{x}) - \varepsilon \partial_{\nu} v(\bar{x}) > 0,$$

which proves the result.

¹² A particular case, for the Laplace equation in a 3-dimensional domain, of this lemma is due to Stanisław Zaremba [Zar10].

¹³ Eberhard Hopf (known for the Hopf maximum principle, the Hopf bifurcation theorem, the Wiener–Hopf method in integral equations, and the Cole–Hopf transformation for solving the viscous Burgers equation), not to be confused with Heinz Hopf (known for the Hopf–Rinow theorem, the Hopf fibration, and Hopf algebras).

¹⁴ Olga Oleĭnik.

¹⁵ We say that an open set Ω satisfies the *interior sphere condition* at $\bar{x} \in \partial\Omega$ if there is an open ball $B_r(x)$ contained in Ω such that $\bar{x} \in \partial B_r(x)$. Note that the interior sphere condition is satisfied by open sets with a C^2 -boundary, but it need not be satisfied by open sets with a C^1 -boundary, and in that case the conclusion of the Hopf lemma may not hold.

Let us give an explicit example of the perturbation v (considering $B_R(0)$, without loss of generality):

$$v(x) = c \left[e^{-\alpha|x|^2} - e^{-\alpha R^2} \right],$$

where c, α are suitable positive constants. We have $v(x) = 0$ on $|x| = R$, and by choosing

$$c = \frac{1}{e^{-\alpha R^2/4} - e^{-\alpha R^2}},$$

we ensure that $v(R/2) = 1$. We compute

$$\partial_\nu v(x) = -2c\alpha|x|e^{-\alpha|x|^2} < 0 \quad \text{on } |x| = R.$$

and

$$\Delta v(x) = 2c\alpha [2\alpha|x|^2 - n] e^{-\alpha|x|^2}.$$

Thus, by choosing $\alpha \geq \frac{2n}{R^2}$, we obtain $-\Delta v < 0$ for $R/2 < |x| < R$. □

ALTERNATIVE PROOF OF [THEOREM 2.10](#). As before, let

$$M := \max_{\Omega} u$$

and define

$$F := \{x \in \Omega : u(x) = M\}.$$

Then F is non-empty by assumption, and it is relatively closed in Ω since u is continuous.

Let us suppose, for the sake of finding a contradiction, that $F \neq \Omega$. Then

$$G = \Omega \setminus F$$

is non-empty and open, and the boundary $\partial F \cap \Omega = \partial G \cap \Omega$ is non-empty (otherwise F and G are open and Ω is not connected).

Choose $y \in \partial G \cap \Omega$ and let $d = \text{dist}(y, \partial\Omega) > 0$. Then choose $x \in G$ such that $|x - y| < \frac{d}{2}$ and let $r = \text{dist}(x, F)$, which satisfies $0 < r < \frac{d}{2}$, so $\overline{B}_r(x) \subset G$. Moreover, there exists at least one point $\bar{x} \in \partial B_r(x) \cap \partial G$ such that $u(\bar{x}) = M$.

We therefore have the following situation: u is sub-harmonic in an open set G where $u < M$, the ball $B_r(x)$ is contained in G , and $u(\bar{x}) = M$ for some point $\bar{x} \in \partial B_r(x) \cap \partial G$. [Lemma 2.1](#) then implies that

$$\partial_\nu u(\bar{x}) > 0$$

where ∂_ν is the outward unit normal derivative to the sphere $\partial B_r(x)$.

However, since \bar{x} is an interior point of Ω and u attains its maximum value M there, we have $\nabla u(\bar{x}) = 0$, so $\partial_\nu u(\bar{x}) = \nabla u(\bar{x}) \cdot \nu = 0$, which is a contradiction.

If Ω is not connected, then u is constant in any connected component of Ω that contains an interior point where u attains a maximum value. □

2.4. Harnack's inequality. The maximum principle gives a basic pointwise estimate for solutions of Laplace's equation. Harnack's inequality¹⁶ is another useful pointwise estimate. It states that if a function is nonnegative and harmonic in a domain, then the ratio of the maximum and minimum of the function on a compactly supported subdomain is bounded by a constant that depends only on the domains. This inequality controls, for example, the amount by which a harmonic function can oscillate inside a domain in terms of the size of the function.

THEOREM 2.12 (Harnack's inequality). *Suppose that $\Omega' \subset\subset \Omega$ is a connected open set that is compactly contained in an open set Ω . There exists a constant C , depending only on Ω and Ω' , such that if $u \in C(\Omega)$ is a non-negative function with the mean-value property (i.e., a non-negative harmonic function), then*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u. \tag{2.7}$$

¹⁶ Named after Carl Gustav Axel Harnack.

PROOF. Step 1. First, we establish the inequality for a compactly contained open ball. Suppose that $x \in \Omega$ and $B_{4R}(x) \subset\subset \Omega$, and let u be any non-negative function with the mean-value property in Ω . If $y \in B_R(x)$, then

$$u(y) = \int_{B_R(y)} u \, dx \leq 2^n \int_{B_{2R}(x)} u \, dx,$$

since $B_R(y) \subset B_{2R}(x)$ and u is non-negative. Similarly, if $z \in B_R(x)$, then

$$u(z) = \int_{B_{3R}(z)} u \, dx \geq \left(\frac{2}{3}\right)^n \int_{B_{2R}(x)} u \, dx,$$

since $B_{3R}(z) \supset B_{2R}(x)$. It follows that

$$\sup_{B_R(x)} u \leq 3^n \inf_{B_R(x)} u.$$

Step 2. Suppose that $\Omega' \subset\subset \Omega$ and $0 < 4R < \text{dist}(\Omega', \partial\Omega)$. Since $\bar{\Omega}'$ is compact, we may cover $\bar{\Omega}'$ by a finite number of open balls of radius R , where the number N of such balls depends only on Ω' and Ω . Moreover, since Ω' is connected, for any $x, y \in \Omega'$ there is a sequence of at most N overlapping balls $\{B_1, B_2, \dots, B_k\}$ such that $B_i \cap B_{i+1} \neq \emptyset$ and $x \in B_1, y \in B_k$. Applying the above estimate to each ball and combining the results, we obtain that

$$\sup_{\Omega'} u \leq 3^{nN} \inf_{\Omega'} u.$$

□

Harnack's inequality leads to an important convergence theorem for harmonic functions known as Harnack's principle or Harnack's convergence theorem. Consider a monotone sequence of continuous functions on Ω . The pointwise limit of such a sequence need not behave well—it could be infinite at some points and finite at other points. Even if it is finite everywhere, there is no reason to expect that our sequence converges uniformly on every compact subset of Ω . Harnack's principle shows that this kind of pathologies cannot occur for a monotone sequence of harmonic functions.

THEOREM 2.13 (Harnack's convergence theorem). *Suppose Ω is connected and $\{u_m\}_{m \in \mathbb{N}}$ is a pointwise increasing sequence of harmonic functions on Ω . Then either $\{u_m\}_{m \in \mathbb{N}}$ converges uniformly on compact subsets of Ω to a function harmonic on Ω , or $u_m(x) \rightarrow \infty$ for every $x \in \Omega$.*

PROOF. Replacing u_m by $u_m - u_1 + 1$, we can assume that each u_m is positive on Ω (since $\{u_m\}_{m \in \mathbb{N}}$ is monotone increasing). Set $u(x) = \lim_{m \rightarrow \infty} u_m(x)$ for each $x \in \Omega$.

Case 1. First suppose u is finite everywhere on Ω . Let K be a compact subset of Ω . Fix $x \in K$. Harnack's inequality shows there is a constant $C \in (1, \infty)$ such that

$$u_m(y) - u_k(y) \leq C(u_m(x) - u_k(x))$$

for all $y \in K$, whenever $m > k$. This implies (u_m) is uniformly Cauchy on K , and thus $u_m \rightarrow u$ uniformly on K , as desired. Thanks to Problem 1 in Exercise Sheet 2, we have that the limit function u is harmonic on Ω .

Case 2. Now suppose $u(x) = \infty$ for some $x \in \Omega$. Let $y \in \Omega$. Then Harnack's inequality, applied to the compact set $K = \{x, y\}$, shows that there is a constant $C \in (1, \infty)$ such that $u_m(x) \leq C u_m(y)$ for every m . Because $u_m(x) \rightarrow \infty$, we also have $u_m(y) \rightarrow \infty$, and so $u(y) = \infty$. This implies that u is identically ∞ on Ω . □

3. An interlude about distributions

The theory of distributions¹⁷ is a powerful theory that allows one to extend the definition of derivative of a function beyond the classical sense.

Let us consider a function $f \in C^1(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$, where $C_c^\infty(\Omega)$ denotes the space of C^∞ functions compactly supported in Ω . The integration by parts formula gives in this case

$$\int_{\Omega} \frac{\partial f}{\partial x_i} \varphi \, dx = - \int_{\Omega} f \frac{\partial \varphi}{\partial x_i} \, dx \quad (3.1)$$

¹⁷ Laurent Schwartz was awarded the Fields Medal for his work on distributions in 1950 [Sch50; Sch51; Sch66].

where the boundary term disappears since φ vanishes on $\partial\Omega$. If now $f \notin C^1(\Omega)$, the idea is to define $\frac{\partial f}{\partial x_i}$ via (3.1), i.e., we define $\frac{\partial f}{\partial x_i}$ as an object that, when integrated (tested) against smooth functions with compact support, returns

$$\left\langle \frac{\partial f}{\partial x_i}, \varphi \right\rangle := - \int_{\Omega} f \frac{\partial \varphi}{\partial x_i} dx.$$

The symbol $\langle \cdot, \cdot \rangle$ is used here to denote the action of the derivative $\frac{\partial f}{\partial x_i}$ on the test function φ .

DEFINITION 2.4 (Space of test functions). *We denote by $\mathcal{D}(\Omega)$, called the space of test functions, the space $C_c^\infty(\Omega)$ of infinitely smooth functions with compact support in Ω , endowed with the following notion of convergence: given $\{\varphi_k\}_{k \in \mathbb{N}} \subset \mathcal{D}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, we say that $\varphi_k \rightarrow \varphi$ in $\mathcal{D}(\Omega)$ as $k \rightarrow \infty$ if*

- (1) *there exists a compact subset $K \subset \Omega$ such that $\text{supp}(\varphi_k) \subset K$ for all $k \in \mathbb{N}$;*
- (2) *$D^\alpha \varphi_k \rightarrow D^\alpha \varphi$ uniformly in K , for all $\alpha \in \mathbb{N}^n$.*

One can construct a topology \mathcal{T} on $\mathcal{D}(\Omega)$ which is consistent with the notion of convergence given in Definition 2.4. The topological space $(\mathcal{D}(\Omega), \mathcal{T})$ is (sequentially) complete, however non-metrizable.

DEFINITION 2.5 (Distribution). *A distribution T in Ω is a functional $T : \mathcal{D}(\Omega) \rightarrow \mathbb{R}$ that satisfies the following properties:*

- *linearity: $\langle T, \alpha\varphi + \beta\psi \rangle = \alpha\langle T, \varphi \rangle + \beta\langle T, \psi \rangle$, for all $\varphi, \psi \in \mathcal{D}(\Omega)$, $\alpha, \beta \in \mathbb{R}$;*
- *sequential continuity: $\langle T, \varphi_k \rangle \rightarrow \langle T, \varphi \rangle$ whenever $\varphi_k \rightarrow \varphi$ in $\mathcal{D}(\Omega)$.*

The set of all distributions is denoted by $\mathcal{D}'(\Omega)$.

We say that two distributions $T_1, T_2 \in \mathcal{D}'(\Omega)$ are equal if

$$\langle T_1, \varphi \rangle = \langle T_2, \varphi \rangle, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

One can show that linear functionals on $(\mathcal{D}(\Omega), \mathcal{T})$ are continuous if and only if they are sequentially continuous. The space $\mathcal{D}'(\Omega)$ thus coincides then with the topological dual of $\mathcal{D}(\Omega)$.

EXAMPLE 2.20. *As a first example of distribution, let us consider a locally integrable function $f \in L^1_{loc}(\Omega)$. To it, we can associate a distribution $T_f : \mathcal{D}(\Omega) \rightarrow \mathbb{R}$ defined as*

$$\langle T_f, \varphi \rangle = \int_{\Omega} f \varphi dx, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

The distribution T_f is clearly a linear functional. It is also sequentially continuous in $\mathcal{D}(\Omega)$. Indeed, let $\varphi_k \rightarrow \varphi$ in $\mathcal{D}(\Omega)$; in particular, there exists $K \subset\subset \Omega$ such that $\text{supp } \varphi_k \subset K$, for all k . (Here $A \subset\subset B$ means that A has compact closure in B) and $\varphi_k \rightarrow \varphi$ uniformly in K . It follows that

$$|\langle T_f, \varphi_k \rangle - \langle T_f, \varphi \rangle| = \left| \int_{\Omega} f(\varphi_k - \varphi) dx \right| \leq \|f\|_{L^1(K)} \|\varphi_k - \varphi\|_{L^\infty(K)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

With some abuse of notation, we will denote by f both the function in $L^1_{loc}(\Omega)$ and the corresponding distribution $T_f \in \mathcal{D}'(\Omega)$.

EXAMPLE 2.21. *As a second example of distribution, we define the Dirac mass or Dirac distribution at y , denoted δ_y : it is the distribution that satisfies*

$$\langle \delta_y, \varphi \rangle = \varphi(y), \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Again, the functional $\delta_y : \mathcal{D}(\Omega) \rightarrow \mathbb{R}$ is clearly linear. It is also sequentially continuous in $\mathcal{D}(\Omega)$ since, for $\varphi_k \rightarrow \varphi$ in $\mathcal{D}(\Omega)$, it holds

$$\langle \delta_y, \varphi_k \rangle = \varphi_k(y) \rightarrow \varphi(y) = \langle \delta_y, \varphi \rangle.$$

We conclude then that $\delta_y \in \mathcal{D}'(\Omega)$.

On the space of distributions, we can introduce the following notion of (weak) convergence.

DEFINITION 2.6 (Convergence in distribution). *Let $\{T_k\}_{k \in \mathbb{N}} \subset \mathcal{D}'(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. We say that $T_k \rightarrow T$ in $\mathcal{D}'(\Omega)$ as $k \rightarrow \infty$ if*

$$\langle T_k, \varphi \rangle \rightarrow \langle T, \varphi \rangle, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

We now introduce the notion of *distributional derivative*.

DEFINITION 2.7 (Distributional derivatives). *Given a distribution T , for $\alpha \in \mathbb{N}^n$, the α -distributional derivative $D^\alpha T \in \mathcal{D}'(\Omega)$ is defined as*

$$\langle D^\alpha T, \varphi \rangle := (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

EXAMPLE 2.22. *In the particular case $f \in L^1_{\text{loc}}(\Omega)$, the distributional partial derivative $\partial_{x_i} f \in \mathcal{D}'(\Omega)$ is defined as*

$$\langle \partial_{x_i} f, \varphi \rangle := - \int_{\Omega} f \partial_{x_i} \varphi \, dx, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

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PROPOSITION 2.1. *Let $(T_k)_{k \geq 0}$ be a sequence in $\mathcal{D}'(\Omega)$ that converges to $T \in \mathcal{D}'(\Omega)$. Then, for all $\alpha \in \mathbb{N}^n$, $(\partial^\alpha T_k)_{k \geq 0}$ converges to $\partial^\alpha T$ in $\mathcal{D}'(\Omega)$.*

PROOF. Let $\varphi \in C_c^\infty(\Omega)$. For all $k \in \mathbb{N}$,

$$\langle \partial^\alpha T_k, \varphi \rangle = (-1)^{|\alpha|} \langle T_k, \partial^\alpha \varphi \rangle \longrightarrow (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle = \langle \partial^\alpha T, \varphi \rangle \quad \text{as } k \rightarrow \infty.$$

□

EXAMPLE 2.23. *The distributional derivative of the Heaviside function¹⁸*

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

is given by the Delta distribution δ_0 . Indeed,

$$\langle H', \varphi \rangle = - \int_{\mathbb{R}} H(x) \varphi'(x) \, dx = - \int_0^{\infty} \varphi'(x) \, dx = \varphi(0), \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

EXAMPLE 2.24. *The function defined for $x \neq 0$ by $f(x) = \log|x|$ and assigned any arbitrary value at 0 belongs to $L^1_{\text{loc}}(\mathbb{R})$. Therefore, it can be associated with a distribution $T_f \in \mathcal{D}'(\mathbb{R})$. Let us compute its distributional derivative. For $\varphi \in C_c^\infty(\mathbb{R})$, we have*

$$\langle f', \varphi \rangle = - \langle f, \varphi' \rangle = - \int_{\mathbb{R}} \log|x| \cdot \varphi'(x) \, dx$$

Due to the integrability of the logarithm at 0, we have

$$\begin{aligned} - \int_{\mathbb{R}} \log|x| \cdot \varphi'(x) \, dx &= - \lim_{\varepsilon \rightarrow 0} \int_{|x| \geq \varepsilon} \log|x| \cdot \varphi'(x) \, dx \\ &= - \lim_{\varepsilon \rightarrow 0} \left(\int_{-\infty}^{-\varepsilon} \log(-x) \cdot \varphi'(x) \, dx + \int_{\varepsilon}^{\infty} \log(x) \cdot \varphi'(x) \, dx \right) \end{aligned}$$

Integrating by parts yields

$$\begin{aligned} &\int_{|x| \geq \varepsilon} \log|x| \cdot \varphi'(x) \, dx \\ &= - \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} \, dx + \varphi(-\varepsilon) \log(\varepsilon) - \varphi(\varepsilon) \log(\varepsilon) \\ &= - \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} \, dx - 2\varepsilon \log(\varepsilon) \left(\frac{\varphi(-\varepsilon) + \varphi(\varepsilon)}{2\varepsilon} \right). \end{aligned}$$

Since $\varepsilon \log(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and φ' is bounded, we conclude that

$$\begin{aligned} \langle f', \varphi \rangle &= - \lim_{\varepsilon \rightarrow 0} \int_{|x| \geq \varepsilon} \log|x| \cdot \varphi'(x) \, dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} \, dx =: \left\langle \text{p.v.} \left(\frac{1}{x} \right), \varphi \right\rangle. \end{aligned}$$

¹⁸ Named after Oliver Heaviside.

The expression p.v. stands for principal value. If f has an isolated singularity at the origin but is C^∞ away from it, then the principal-value distribution of f is defined by

$$\langle \text{p.v.}(f), \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} f(x) \varphi(x) \, dx, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Such a limit may not be well-defined, or, being well-defined, it may not necessarily define a distribution. It is, however, well-defined if f is a continuous homogeneous function of degree $-n$.

EXAMPLE 2.25. Let $u \in L^1_{\text{loc}}(\mathbb{R})$, and define, for $x \in \mathbb{R}$, $v(x) = \int_0^x u(t) \, dt$. Then v is a continuous function on \mathbb{R} and $v' = u$ in the sense of distributions.

Let us first show the continuity of v . Let $x_0 \in \mathbb{R}$ and let $\{x_k\}_{n \geq 0}$ be a sequence converging to x_0 . We have, for all $n \geq 0$,

$$v(x_k) = \int_{\mathbb{R}} \chi_{(0, x_k)}(t) u(t) \, dt$$

By Lebesgue's dominated convergence theorem, the sequence $(v(x_k))_{n \geq 0}$ converges to $\int_{\mathbb{R}} \chi_{(0, x_0)}(t) u(t) \, dt = v(x_0)$, proving the continuity of v at x_0 , and thus on \mathbb{R} .

Let $\varphi \in C_c^\infty(\mathbb{R})$ and assume that $\text{supp}(\varphi) \subset [-A, A]$. Using Fubini's theorem, we have:

$$\begin{aligned} \langle v', \varphi \rangle &= -\langle v, \varphi' \rangle = - \int_{-A}^A \left(\int_0^x u(t) \, dt \right) \varphi'(x) \, dx \\ &= - \int_0^A \int_0^x u(t) \varphi'(x) \, dt \, dx + \int_{-A}^0 \int_x^0 u(t) \varphi'(x) \, dt \, dx \\ &= - \int_0^A u(t) \left(\int_t^A \varphi'(x) \, dx \right) \, dt + \int_{-A}^0 u(t) \left(\int_{-A}^t \varphi'(x) \, dx \right) \, dt \\ &= \int_0^A u(t) \varphi(t) \, dt + \int_{-A}^0 u(t) \varphi(t) \, dt = \int_{\mathbb{R}} u(t) \varphi(t) \, dt = \langle u, \varphi \rangle. \end{aligned}$$

Thus, $v' = u$ in $\mathcal{D}'(\mathbb{R})$.

PROPOSITION 2.2. Let $a \in C^\infty(\Omega)$ and let $T \in \mathcal{D}'(\Omega)$. Then, defining the product aT as

$$\langle aT, \varphi \rangle := \langle T, a\varphi \rangle, \quad \text{for all } \varphi \in \mathcal{D}(\Omega),$$

we have $\partial_{x_i}(aT) = (\partial_{x_i}a)T + a\partial_{x_i}T$.

EXAMPLE 2.26. The i -th partial derivative of a distribution T_f with $f \in C^1(\mathbb{R}^n)$ is the distribution $T_{\partial_{x_i}f}$.

EXAMPLE 2.27. Let f be a piecewise C^1 function on $[a, b]$. This means that there exists a subdivision of $[a, b]$ into intervals $[a_i, a_{i+1}]$ such that f is C^1 on $[a_i, a_{i+1}]$. Suppose that, at every point where it is not continuous, f admits a right limit and a left limit. To fix notations, we let $a_0 = a$, $a_{n+1} = b$, and $a_1, \dots, a_n \in (a, b)$. We denote by $f(a_i^+)$ and $f(a_i^-)$ the right and left limits of f at the point a_i , respectively. As a convention, we let $f(a_0^-) = f(a_{n+1}^+) = 0$.

Then, we claim that

$$(T_f)' = T_{f'} + \sum_{i=1}^{n+1} (f(a_i^+) - f(a_i^-)) \delta_{a_i}.$$

First, we note that the function f defines a distribution, of which we calculate the derivative, which we denote by $(T_f)'$. By definition, for $\varphi \in C_c^\infty(\mathbb{R})$, we have

$$\langle (T_f)', \varphi \rangle = -\langle T_f, \varphi' \rangle = - \int_a^b f(x) \varphi'(x) \, dx$$

Thus,

$$\int_a^b f(x) \varphi'(x) \, dx = \sum_{i=0}^n \int_{a_i}^{a_{i+1}} f(x) \varphi'(x) \, dx.$$

Integrating by parts, we get

$$\int_{a_i}^{a_{i+1}} f(x) \varphi'(x) \, dx = [f(x) \varphi(x)]_{a_i}^{a_{i+1}} - \int_{a_i}^{a_{i+1}} f'(x) \varphi(x) \, dx,$$

which yields

$$-\int_a^b f(x)\varphi'(x) dx = \sum_{i=0}^n [f(a_i^+)\varphi(a_i^+) - f(a_{i+1}^-)\varphi(a_{i+1}^-)] + \int_{a_i}^{a_{i+1}} f'(x)\varphi(x) dx.$$

Hence,

$$\langle (T_f)', \varphi \rangle = \langle T_{f'}, \varphi \rangle + f(a_n^+)\varphi(a_n^+) - f(a_0^-)\varphi(a_0^-) + \sum_{i=1}^n (f(a_i^+) - f(a_i^-))\varphi(a_i),$$

which implies our claim.

REMARK 2.7. To regularize a distribution, we can use the convolution with the standard mollifier: if $T \in \mathcal{D}'(\Omega)$, then

$$\langle T * \eta_\varepsilon, \varphi \rangle = \langle T, \eta_\varepsilon(\cdot) * \varphi \rangle, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

It can be shown that the convolution of a smooth, compactly supported function and a distribution is a smooth function, namely

$$x \mapsto (T * \eta_\varepsilon)(x) = \langle T, \eta_\varepsilon(\cdot - x) \rangle.$$

Furthermore, we note that

$$T * \eta_\varepsilon \rightarrow T \quad \text{in } \mathcal{D}'(\Omega) \text{ as } \varepsilon \rightarrow 0.$$

REMARK 2.8. The standard mollifier η_ε from [Definition 2.2](#) converges in distribution to the Dirac delta as $\varepsilon \rightarrow 0$: for every $\varphi \in \mathcal{D}(\Omega)$ and $y \in \Omega$, we have

$$\varphi_\varepsilon(y) = \int_\Omega \eta_\varepsilon(x - y)\varphi(x) dx \rightarrow \varphi(y) \quad \text{as } \varepsilon \rightarrow 0;$$

hence

$$\langle \eta_\varepsilon(\cdot - y), \varphi \rangle = \varphi_\varepsilon(y) \rightarrow \varphi(y) = \langle \delta_y, \varphi \rangle \quad \text{as } \varepsilon \rightarrow 0.$$

i.e., $\eta_\varepsilon(\cdot - y) \rightarrow \delta_y$ in $\mathcal{D}'(\Omega)$.

PROPOSITION 2.3. Let $T \in \mathcal{D}'(\mathbb{R}^d)$, $\varphi \in C_c^\infty(\mathbb{R}^d)$, and $\alpha \in \mathbb{N}^d$. Then,

$$\partial^\alpha (T * \varphi) = (\partial^\alpha T) * \varphi = T * (\partial^\alpha \varphi).$$

More generally, for any decomposition of the multi-index $\alpha = \alpha_1 + \alpha_2$, we have

$$\partial^\alpha (T * \varphi) = (\partial^{\alpha_1} T) * (\partial^{\alpha_2} \varphi).$$

PROPOSITION 2.4. Let $T \in \mathcal{D}'(\mathbb{R})$. We have $T' = 0$ if and only if T is constant.

PROOF. If we assume that T is constant, it is clear that $T' = 0$ since φ has compact support. Conversely, suppose that $T' = 0$. Then, for all $\varphi \in C_c^\infty(\mathbb{R})$,

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle = 0.$$

Thus, T vanishes on all functions of the form φ' , where $\varphi \in C_c^\infty(\mathbb{R})$. Let us characterize these functions. We show that

$$(\psi = \varphi', \text{ with } \varphi \in C_c^\infty(\mathbb{R})) \iff \left(\psi \in C_c^\infty(\mathbb{R}) \text{ and } \int_{\mathbb{R}} \psi(x) dx = 0 \right). \quad (3.2)$$

The direct implication is clear because φ has compact support. Conversely, we set $\varphi(x) = \int_{-\infty}^x \psi(t) dt$ with $\text{supp } \psi \in [-M, M]$. Then $\varphi \in C^\infty(\mathbb{R})$. If $x < -M$, then $\varphi(x) = 0$ (since ψ vanishes on $] -\infty, x]$ in this case). If $x > M$, then (since $\int_x^{+\infty} \psi(t) dt = 0$ and $\int_{\mathbb{R}} \psi(t) dt = 0$ by assumption),

$$\varphi(x) = \int_{-\infty}^x \psi(t) dt + 0 = \int_{-\infty}^x \psi(t) dt + \int_x^{+\infty} \psi(t) dt = \int_{\mathbb{R}} \psi(t) dt = 0.$$

Thus, $\text{supp } \varphi \subset [-M, M]$ and $\varphi \in C_c^\infty(\mathbb{R})$ and, of course, $\psi = \varphi'$.

We will use the equivalence (3.2) to conclude the proof. Let $\chi \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \chi(x) dx = 1$. Let $\varphi \in C_c^\infty(\mathbb{R})$. Let us define

$$\psi(x) = \varphi(x) - \left(\int_{\mathbb{R}} \varphi(t) dt \right) \chi(x), \quad x \in \mathbb{R}.$$

Then $\psi \in C_c^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \psi(t) dt = 0$. Consequently, there exists $\varphi \in C_c^\infty(\mathbb{R})$ such that $\psi = \varphi'$ and $\langle T, \psi \rangle = 0$. Thus, by the linearity of T ,

$$\langle T, \varphi \rangle = \langle T, \chi \rangle \cdot \int_{\mathbb{R}} \varphi(x) dx = C \cdot \langle 1, \varphi \rangle = \langle C, \varphi \rangle$$

with $C := \langle T, \chi \rangle$ being a constant. Thus, we conclude that T is constant. \square

4. Distributional Laplacian and Weyl's lemma

Thanks to [Definition 2.7](#), we can say that any distribution admits distributional derivatives of any order. In particular, the *distributional Laplacian* of $T \in \mathcal{D}'(\Omega)$ will be defined as

$$\langle \Delta T, \varphi \rangle = \langle T, \Delta \varphi \rangle, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

We will say that a distribution $u \in \mathcal{D}'(\Omega)$ is a *distributional solution of the Laplace equation* if

$$\langle \Delta u, \varphi \rangle = 0 \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

We conclude by stating the *fundamental lemma of the calculus of variations*. This tool is typically used to transform the distributional formulation of a differential problem into the strong formulation using a priori knowledge on the regularity of the distributional solution.

LEMMA 2.2 (Fundamental lemma of the calculus of variations). *Let $\Omega \subset \mathbb{R}^n$, and suppose $f \in C(\Omega)$. Suppose also that*

$$\int_{\Omega} f \varphi dx = 0$$

for every $\varphi \in C_c^\infty(\Omega)$. Then $f = 0$ on Ω .

REMARK 2.9. *One could state a more general version of the fundamental lemma of the calculus of variations without requiring continuity, e.g., assuming $f \in L_{\text{loc}}^1(\Omega)$ and reach the conclusion $f = 0$ almost everywhere. The argument is done by approximation, but we skip the details.*

PROOF. For the sake of finding a contradiction, suppose that f is strictly positive at some point $\bar{x} \in \Omega$. By continuity, there exists some neighborhood $U = \{x \mid \|x - \bar{x}\| < \delta\}$ where f is positive. Consider the function

$$\varphi(x) = \begin{cases} 0, & \text{if } x \notin U, \\ \prod_{j=1}^n e^{-1/(x_j - \bar{x}_j)^2}, & \text{if } x \in U, \end{cases}$$

which is continuous and infinitely differentiable on \mathbb{R}^n , and satisfies $\varphi(x) = 0$ for $x \in \partial\Omega$, so it satisfies the hypotheses required above. Furthermore,

$$\int_{\Omega} f \varphi dx = \int_U f \varphi dx > 0,$$

which is a contradiction. \square

EXAMPLE 2.28. *If $u \in C^\infty(\Omega)$ is a distributional solution of the Laplace equation, then, for every $\varphi \in \mathcal{D}(\Omega)$,*

$$\langle \Delta u, \varphi \rangle = \langle u, \Delta \varphi \rangle = \int_{\Omega} u \Delta \varphi dx = \int_{\Omega} \Delta u \varphi dx$$

where we first used some abuse of notation to identify the distribution u with the L_{loc}^1 function u and then we integrated by parts. By [Lemma 2.2](#), we now conclude that $\Delta u = 0$ in Ω .

We can now strengthen the regularity result for harmonic functions shown in [Section 2.1](#) and prove Weyl's lemma,¹⁹ which states that every distributional solution of Laplace's equation is smooth.

THEOREM 2.14 (Weyl's lemma). *Let $u \in \mathcal{D}'(\Omega)$ and suppose that $\langle \Delta u, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$. Then $u \in C^\infty(\Omega)$ and is harmonic in Ω .*

¹⁹ Named after Hermann Weyl [[Wey40](#)]. See also [[Str08](#)].

PROOF. Let η_ε be the standard mollifier and consider $u_\varepsilon := u * \eta_\varepsilon$. Then $u_\varepsilon \in C^\infty$ and $\Delta u_\varepsilon = (\Delta u) * \eta_\varepsilon = 0$. So u_ε , being smooth and harmonic, satisfies the mean-value property. Hence, arguing as in the proof of [Theorem 2.4](#),

$$\begin{aligned} u_\varepsilon * \eta(x) &= \int_0^\infty r^{n-1} \eta(r) dr \int_{\partial B_1(0)} u_\varepsilon(x - ry) dS(y) \\ &= \omega_n u_\varepsilon(x) \int_0^\infty r^{n-1} \eta(r) dr = u_\varepsilon(x) \int_{\mathbb{R}^n} \eta(y) dy \\ &= u_\varepsilon(x) \end{aligned}$$

(recall that here η denotes the standard mollifier with $\varepsilon = 1$). Letting $\varepsilon \rightarrow 0$, we get $u = u * \eta \in C^\infty$. Moreover, by the fundamental lemma of the calculus of variations, $-\Delta u = 0$ (i.e., u is harmonic in the classical sense). \square

ALTERNATIVE PROOF OF [THEOREM 2.14](#). Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be the standard mollifier. Recall that $\eta(x) = \eta(|x|) = \theta(|x|^2)$. Fix $\Omega' \subset \Omega$ and put $\varepsilon_0 = \text{dist}(\Omega', \partial\Omega)$. For each $x \in \Omega'$ and $\varepsilon \in (0, \varepsilon_0)$ the function

$$y \mapsto \eta_\varepsilon(x - y)$$

belongs to $\mathcal{D}(\Omega)$ and so we may consider $\langle u, \eta_\varepsilon(x - \cdot) \rangle$.

We compute

$$\begin{aligned} \frac{d}{d\varepsilon} \left(\varepsilon^{-n} \eta \left(\frac{x-y}{\varepsilon} \right) \right) &= -n \varepsilon^{-n-1} \eta \left(\frac{x-y}{\varepsilon} \right) - \varepsilon^{-n} \nabla \eta \left(\frac{x-y}{\varepsilon} \right) \cdot \frac{x-y}{\varepsilon} \\ &= -\frac{1}{\varepsilon^{n+1}} \text{div} \left(\frac{x-y}{\varepsilon} \eta \left(\frac{x-y}{\varepsilon} \right) \right) \end{aligned}$$

Defining

$$\Theta(t) := \frac{1}{2} \int_t^\infty \theta(s) ds,$$

we note that

$$-z \eta(z) = -z \theta(|z|^2) = \nabla (\Theta(|z|^2)),$$

and then we arrive at

$$\frac{d}{d\varepsilon} \left(\varepsilon^{-n} \eta \left(\frac{x-y}{\varepsilon} \right) \right) = -\frac{1}{\varepsilon^{n+1}} \text{div} \left(\frac{x-y}{\varepsilon} \eta \left(\frac{x-y}{\varepsilon} \right) \right) = \Delta_y \left(\varepsilon^{1-n} \Theta \left(\left(\frac{x-y}{\varepsilon} \right)^2 \right) \right).$$

Since, by assumption, u is harmonic in the sense of distributions, we deduce

$$\left\langle u, \Delta_y \left(\varepsilon^{1-n} \Theta \left(\left(\frac{x-y}{\varepsilon} \right)^2 \right) \right) \right\rangle = 0.$$

Now, by considering difference quotients, we see that

$$\frac{d}{d\varepsilon} \langle u, \eta_\varepsilon(x - \cdot) \rangle = \left\langle u, \frac{d}{d\varepsilon} \eta_\varepsilon(x - \cdot) \right\rangle = \left\langle u, \Delta_y \left(\varepsilon^{1-n} \Theta \left(\left(\frac{x-y}{\varepsilon} \right)^2 \right) \right) \right\rangle = 0.$$

Integrating between ε and ε_1 (such that $\varepsilon, \varepsilon_1 \in (0, \varepsilon_0)$), we deduce

$$\langle u, \eta_\varepsilon(x - \cdot) \rangle = \langle u, \eta_{\varepsilon_1}(x - \cdot) \rangle$$

For $\varphi \in \mathcal{D}(\Omega')$, we have

$$\langle u * \eta_\varepsilon, \varphi \rangle = \int_{\Omega'} \langle u, \eta_\varepsilon(x - \cdot) \rangle \varphi(x) dx = \int_{\Omega'} \langle u, \eta_{\varepsilon_1}(x - \cdot) \rangle \varphi(x) dx.$$

Hence, as $\eta_\varepsilon * \varphi \rightarrow \varphi$ in $\mathcal{D}(\Omega)$ as $\varepsilon \rightarrow 0^+$, we get

$$\langle u, \varphi \rangle = \int_{\Omega'} \langle u, \eta_{\varepsilon_1}(x - \cdot) \rangle \varphi(x) dx.$$

Consequently, $u|_{\Omega'} \in C^\infty(\Omega')$, and since Ω' was arbitrary, this concludes the proof. \square

5. Fundamental solution of the Laplace equation

Motivated by the fact that harmonic functions are invariant under rotations (Problem 4 in Exercise Sheet 1), we want to construct a radially symmetric solution $u(x) = v(|x|)$, with $v : \mathbb{R}_+ \rightarrow \mathbb{R}$, to the Laplace equation in \mathbb{R}^n .

We denote by $r = |x| = \sqrt{\sum_{i=1}^n x_i^2}$ the radial coordinate and note that

$$\partial_{x_i} r = \frac{x_i}{r}, \quad \partial_{x_i}^2 = \frac{1}{r} - \frac{x_i^2}{r^3}, \quad i = 1, \dots, n.$$

Hence,

$$\partial_{x_i} u(x) = v'(r) \frac{x_i}{r}, \quad \partial_{x_i}^2 u(x) = v''(r) \frac{x_i^2}{r^2} + v'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right).$$

Then the Laplace equation is reduced to

$$\Delta u(x) = \sum_{i=1}^n \partial_{x_i}^2 u = v''(r) + v'(r) \frac{n-1}{r} = 0. \quad (5.1)$$

If $v' \neq 0$, we can solve the ODE (5.1) by noticing that

$$\frac{v''}{v'} = \frac{1-n}{r},$$

which yields

$$\ln v' = (1-n) \ln r,$$

that is

$$v' = \frac{C}{r^{n-1}}.$$

Integrating once more, we conclude that (5.1) admits the non-constant solutions of the form

$$v(r) = \begin{cases} C_1 r + C_2 & \text{if } n = 1, \\ C_1 \log r + C_2, & \text{if } n = 2, \\ C_1 r^{2-n} + C_2, & \text{if } n \geq 3. \end{cases}$$

We define the *fundamental solution* or *free-space Green's function* $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ of the Laplace equation²⁰ by

$$\Phi(x) := \begin{cases} -\frac{1}{2}|x|, & \text{if } n = 1, \\ -\frac{1}{2\pi} \log |x|, & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha_n |x|^{n-2}}, & \text{if } n \geq 3. \end{cases} \quad (5.2)$$

The fundamental solution satisfies

$$-\Delta \Phi = \delta_0$$

in the sense of distributions in \mathbb{R}^n , as we will show in [Theorem 2.15](#).

Lect. 6, 15.10

REMARK 2.10 (Some properties of the fundamental solution). *We collect here some properties of the fundamental solution Φ of the Laplace equation.*

(1) Φ is smooth away from the origin: $\Phi \in C^\infty(\mathbb{R}^n \setminus \{0\})$. In particular, for $x \neq 0$, we compute that

$$\partial_{x_i} \Phi(x) = -\frac{1}{n\alpha_n} \frac{x_i}{|x|^n}, \quad (5.3)$$

$$\partial_{x_i x_j}^2 \Phi(x) = \frac{x_i x_j}{\alpha_n |x|^{n+2}} - \frac{\delta_{ij}}{n\alpha_n |x|^n} \quad (5.4)$$

²⁰ In general, the fundamental solution Φ of a linear partial differential operator \mathcal{L} is the distributional solution of $\mathcal{L}\Phi = \delta_0$. The existence of fundamental solutions for every linear partial differential equation with constant coefficients was first proved independently by Bernard Malgrange [[Mal56](#), Th. 1] and Leon Ehrenpreis [[Ehr54](#), Th. 6]. Before 1950, when the first edition of [[Sch50](#)] appeared, the question about the existence of a fundamental solution was not even raised, since there did not exist a generally accepted definition of a fundamental solution. By now there are several proofs of Malgrange–Ehrenpreis' theorem, which can be roughly classified into three categories: (1) non-constructive proofs using Hahn–Banach's theorem; (2) constructive proofs via explicit formulae; (3) proofs by solving a “division problem”. See [[OW96](#); [Ros91](#); [Wag09](#)] for further information.

(here

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

is the Kronecker²¹ delta).

(2) From (5.4), we deduce that

$$\Delta\Phi = 0 \quad \text{if } x \neq 0,$$

so Φ is harmonic in any open set that does not contain the origin.

(3) The function Φ is homogeneous of degree $-n + 2$, its first derivative is homogeneous of degree $-n + 1$, and its second derivative is homogeneous of degree n .

(4) Φ is unbounded as $x \rightarrow 0$ with $\Phi(x) \rightarrow \infty$ as $|x| \rightarrow 0$. Nevertheless, Φ and $\nabla\Phi$ are locally integrable. For example, the local integrability of $\partial_i\Phi$ follows from (5.3) by noticing that

$$|\partial_{x_i}\Phi(x)| \leq \frac{C_n}{|x|^{n-1}},$$

holds and that $|x|^{n-1}$ is locally integrable on \mathbb{R}^n .

(5) The second partial derivatives of Φ are not locally integrable, however, since they are of the order $|x|^{-n}$ as $x \rightarrow 0$.

(6) For $x \neq 0$,

$$\nabla\Phi \cdot \frac{x}{|x|} = -\frac{1}{n\alpha_n} \frac{1}{|x|^{n-1}}. \quad (5.5)$$

Thus we get the following surface integral over a sphere centered at the origin with normal $\nu = \frac{x}{|x|}$:

$$-\int_{\partial B_r(0)} \nabla\Phi \cdot \nu \, dS = 1. \quad (5.6)$$

(7) As follows from the divergence theorem and the fact that Φ is harmonic in $B_R(0) \setminus B_r(0)$, the integral (5.6) does not depend on r . The surface integral is not zero, however, as it would be for a function that was harmonic everywhere inside $B_r(0)$, including at the origin.

(8) The normalization of the flux integral in (5.6) to one accounts for the choice of the multiplicative constant in the definition of Φ .

THEOREM 2.15. We have

$$-\Delta\Phi = \delta_0$$

in the sense of distributions, i.e.,

$$-\langle \Phi, \Delta\varphi \rangle = \langle \delta_0, \varphi \rangle$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

We will prove [Theorem 2.15](#) for $n \geq 2$. The case $n = 1$ is left as an exercise. Indeed, it follows directly from Problem 4 of Exercise Sheet 4: after computing the distributional derivatives $(T_{|\cdot|})' = T_{\text{sign}}$ and $(T_H)' = \delta_0$ (where $H = \chi_{[0,+\infty)}$ is the Heaviside function), one only has to notice the relationship between sign and Heaviside function, namely $\text{sign} = 2H - 1$, to conclude that

$$-\Delta \left(-\frac{1}{2}|x| \right) = \delta_0$$

in the sense of distributions.

PROOF. Let T_Φ be the distribution associated with the fundamental solution Φ (which is an L^1_{loc} -function). That is, let $T_\Phi : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathbb{R}$ be defined as

$$\langle T_\Phi, g \rangle = \int_{\mathbb{R}^n} \Phi(x)g(x) \, dx, \quad \text{for all } g \in \mathcal{D}(\mathbb{R}^n).$$

We want to show that

$$\langle T_\Phi, \Delta g \rangle = -\langle \delta_0, g \rangle = -g(0),$$

which means $-\Delta\Phi = \delta_0$ in \mathbb{R}^n in the sense of distributions.

²¹ Named after Leopold Kronecker.

By definition,

$$\langle \Delta T_\Phi, g \rangle = \langle T_\Phi, \Delta g \rangle = \int_{\mathbb{R}^n} \Phi(x) \Delta g(x) \, dx.$$

Now we would like to apply the divergence theorem, but Φ has a singularity at $x = 0$. We get around this by breaking up the integral into two pieces:

$$\begin{aligned} (T_\Phi, \Delta g) &= \int_{\mathbb{R}^n} \Phi(x) \Delta g(x) \, dx \\ &= \int_{B_\delta(0)} \Phi(x) \Delta g(x) \, dx + \int_{\mathbb{R}^n \setminus B_\delta(0)} \Phi(x) \Delta g(x) \, dx \\ &=: I + J. \end{aligned}$$

Step 1. We look first at term I . For $n = 2$,

$$\begin{aligned} \left| - \int_{B_\delta(0)} \frac{1}{2\pi} \ln |x| \Delta g(x) \, dx \right| &\leq C \|\Delta g\|_{L^\infty(\mathbb{R}^n)} \left| \int_{B_\delta(0)} \ln |x| \, dx \right| \\ &\leq C \int_0^{2\pi} \int_0^\delta |\ln r| r \, dr \, d\theta \\ &= C \int_0^\delta |\ln r| r \, dr \\ &= C\delta^2 |\ln \delta|. \end{aligned}$$

For $n \geq 3$,

$$\begin{aligned} \left| \int_{B_\delta(0)} \frac{1}{n(n-2)\alpha_n} \frac{1}{|x|^{n-2}} \Delta g(x) \, dx \right| &\leq C \|\Delta g\|_{L^\infty(\mathbb{R}^n)} \int_{B_\delta(0)} \frac{1}{|x|^{n-2}} \, dx \\ &\leq C \int_0^\delta \left(\int_{\partial B_r(0)} \frac{1}{|y|^{n-2}} \, dS(y) \right) dr \\ &= C \int_0^\delta \frac{1}{r^{n-2}} \left(\int_{\partial B_r(0)} dS(y) \right) dr \\ &= C \int_0^\delta \frac{1}{r^{n-2}} n\alpha_n r^{n-1} \, dr \\ &= n\alpha_n C \int_0^\delta r \, dr \\ &= \frac{n\alpha_n C}{2} \delta^2. \end{aligned}$$

Therefore, as $\delta \rightarrow 0^+$, $|I| \rightarrow 0$.

Step 2. Next, we look at term J . Applying the divergence theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_\delta(0)} \Phi(x) \Delta_x g(x) \, dx &= \underbrace{\int_{\mathbb{R}^n \setminus B_\delta(0)} \Delta_x \Phi(x) g(x) \, dx}_{=0} \\ &\quad - \int_{\partial(\mathbb{R}^n \setminus B_\delta(0))} \partial_\nu \Phi(x) g(x) \, dS(x) + \int_{\partial(\mathbb{R}^n \setminus B_\delta(0))} \Phi(x) \partial_\nu g(x) \, dS(x) \\ &= - \int_{\partial(\mathbb{R}^n \setminus B_\delta(0))} \partial_\nu \Phi(x) g(x) \, dS(x) + \int_{\partial(\mathbb{R}^n \setminus B_\delta(0))} \Phi(x) \partial_\nu g(x) \, dS(x) \\ &=: J_1 + J_2, \end{aligned}$$

using the fact that $\Delta_x \Phi(x) = 0$ for $x \in \mathbb{R}^n \setminus B_\delta(0)$.

Step 2a. We first look at term J_1 . Since g vanishes as $|x| \rightarrow \infty$, we only need to calculate the integral over $\partial B_\delta(0)$ where the normal vector ν is the outer normal to $\mathbb{R}^n \setminus B_\delta(0)$. From (5.3), we have

$$\nabla_x \Phi(x) = -\frac{x}{n\alpha_n |x|^n}.$$

The outer unit normal to $\mathbb{R}^n \setminus B_\delta(0)$ on $B_\delta(0)$ is given by $\nu = -\frac{x}{|x|}$, so the normal derivative of Φ on $B_\delta(0)$ is given by

$$\partial_\nu \Phi = \left(-\frac{x}{n\alpha_n |x|^n} \right) \cdot \left(-\frac{x}{|x|} \right) = \frac{1}{n\alpha_n |x|^{n-1}}$$

(as in (5.5)). Therefore, J_1 can be written as

$$-\int_{\partial B_\delta(0)} \frac{1}{n\alpha_n |x|^{n-1}} g(x) \, dS(x) = -\frac{1}{n\alpha_n \delta^{n-1}} \int_{\partial B_\delta(0)} g(x) \, dS(x).$$

Since g is a continuous function, then we conclude

$$-\int_{\partial B_\delta(0)} g(x) \, dS(x) \rightarrow -g(0) \quad \text{as } \delta \searrow 0.$$

Step 2b. Lastly, we look at term J_2 . Using the fact that g vanishes as $|x| \rightarrow \infty$, we only need to integrate over $\partial B_\delta(0)$. Since $g \in \mathcal{D}(\mathbb{R}^n)$, we compute

$$\left| \int_{\partial B_\delta(0)} \Phi(x) \partial_\nu g(x) \, dS(x) \right| \leq \|\partial_\nu g\|_{L^\infty(\partial B_\delta(0))} \int_{\partial B_\delta(0)} |\Phi(x)| \, dS(x) \leq C \int_{\partial B_\delta(0)} |\Phi(x)| \, dS(x).$$

For $n = 2$,

$$\begin{aligned} \int_{\partial B_\delta(0)} |\Phi(x)| \, dS(x) &= C \int_{\partial B_\delta(0)} |\ln |x|| \, dS(x) \\ &\leq C |\ln \delta| \int_{\partial B_\delta(0)} dS(x) \\ &= C |\ln \delta| (2\pi\delta) \leq C\delta |\ln \delta|. \end{aligned}$$

For $n \geq 3$,

$$\begin{aligned} \int_{\partial B_\delta(0)} |\Phi(x)| \, dS(x) &= C \int_{\partial B_\delta(0)} \frac{1}{|x|^{n-2}} \, dS(x) \\ &\leq \frac{C}{\delta^{n-2}} \int_{\partial B_\delta(0)} dS(x) \\ &= \frac{C}{\delta^{n-2}} n\alpha_n \delta^{n-1} \leq C\delta. \end{aligned}$$

In both cases, we have $|J_2| \rightarrow 0$ as $\delta \searrow 0$.

Step 3. Combining these estimates, we see that

$$\int_{\mathbb{R}^n} \Phi(x) \Delta_x g(x) \, dx = \lim_{\delta \rightarrow 0^+} (I + J_1 + J_2) = -g(0),$$

which concludes the proof. \square

5.1. Solution of the Poisson equation. We can use the fundamental solution Φ to build a solution to the Poisson equation.

THEOREM 2.16. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, and let*

$$u = \Phi * f$$

where Φ is the fundamental solution (5.2). Then $u \in C^\infty(\mathbb{R}^n)$ and

$$-\Delta u = f \quad \text{in } \mathbb{R}^n.$$

PROOF. By a change of variables, we write

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy = \int_{\mathbb{R}^n} \Phi(y) f(x-y) \, dy.$$

Let $e_i := (\dots, 0, 1, 0, \dots)$ be the unit vector in \mathbb{R}^n with a 1 in the i^{th} slot. Then

$$\frac{u(x + he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \left[\frac{f(x + he_i - y) - f(x - y)}{h} \right] dy.$$

Now $f \in C^2$ implies

$$\frac{f(x + he_i - y) - f(x - y)}{h} \rightarrow \partial_{x_i} f(x - y), \quad \text{as } h \rightarrow 0, \text{ uniformly in } \mathbb{R}^n.$$

Therefore,

$$\partial_{x_i} u(x) = \int_{\mathbb{R}^n} \Phi(y) \partial_{x_i} f(x - y) \, dy.$$

Similarly, we also obtain

$$\partial_{x_i x_j}^2 u(x) = \int_{\mathbb{R}^n} \Phi(y) \partial_{x_i x_j}^2 f(x - y) \, dy.$$

In particular,

$$\Delta_x u(x) = \int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x - y) \, dy = \int_{\mathbb{R}^n} \Phi(y) \Delta_y f(x - y) \, dy.$$

Since $f \in \mathcal{D}(\mathbb{R}^n)$, we apply [Theorem 2.15](#) and conclude that

$$-\Delta u = \langle \delta_x, f \rangle = f(x).$$

□

Classical solutions to the Dirichlet problem

1. Dirichlet problem for the Laplace equation

The fundamental solution Φ of the Laplace equation studied in [Section 5](#) can be used to derive a representation formula for the point value of a $C^2(\Omega)$ function in terms of its Laplacian and boundary values.

THEOREM 3.1 (Green's representation formula). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. For any $u \in C^2(\bar{\Omega})$ and $y \in \Omega$, we have*

$$u(y) = \int_{\Omega} \Phi(x-y)(-\Delta u(x)) dx - \int_{\partial\Omega} \partial_{\nu_x} \Phi(x-y)u(x) dS(x) + \int_{\partial\Omega} \Phi(x-y)\partial_{\nu} u(x) dS(x), \quad (1.1)$$

where ∂_{ν_x} denotes the normal derivative with respect to the x variables, i.e. $\partial_{\nu_x}(\cdot) = \nabla_x(\cdot) \cdot \nu = \sum_{i=1}^n \frac{\partial}{\partial x_i}(\cdot)\nu_i$.

The starting point of the proof is Green's integration by parts identity,

$$\int_{\Omega} (v\Delta u - u\Delta v) dx = \int_{\partial\Omega} (v\partial_{\nu} u - u\partial_{\nu} v) dS. \quad (1.2)$$

which holds for every $u, v \in C^2(\bar{\Omega})$ and a bounded domain with C^1 boundary. Now fix $y \in \Omega$. To prove the [\(1.1\)](#), the idea is to take $v = \Phi(\cdot - y)$ in [\(1.2\)](#). However, $x \mapsto \Phi(x - y)$ is not $C^2(\bar{\Omega})$ and, in particular, a singularity occurs at $x = y$. So we need to be more careful: to circumvent this difficulty, we write the identity on the domain $\Omega_{\varepsilon} = \Omega \setminus B_{\varepsilon}(y)$, with ε small enough so that $B_{\varepsilon}(y) \subset \Omega$, and let $\varepsilon \rightarrow 0$.

PROOF. Exploiting the fact that $\Phi(x - y) \in C^2(\bar{\Omega}_{\varepsilon})$ and $\Delta_x \Phi(x - y) = 0$ in Ω_{ε} , we have

$$\begin{aligned} & \underbrace{\int_{\Omega_{\varepsilon}} \Delta u(x)\Phi(x-y) dx}_{=:C} - \int_{\partial\Omega} (\partial_{\nu} u(x)\Phi(x-y) - u(x)\partial_{\nu} \Phi(x-y)) dS(x) \\ &= \underbrace{\int_{\partial B_{\varepsilon}(y)} \partial_{\nu} u(x)\Phi(x-y) dS(x)}_{=:A} - \underbrace{\int_{\partial B_{\varepsilon}(y)} u(x)\partial_{\nu} \Phi(x-y) dS(x)}_{=:B}. \end{aligned}$$

We show now that $A \rightarrow 0$, $B \rightarrow u(y)$, and $C \rightarrow \int_{\Omega} \Delta u(x)\Phi(x-y) dx$ as $\varepsilon \rightarrow 0$.

For A , we estimate

$$|A| = |\Phi(\varepsilon)| \left| \int_{\partial B_{\varepsilon}(y)} \partial_{\nu} u dS \right| \leq |\Phi(\varepsilon)| \varepsilon^{n-1} \omega_n \sup_{B_{\varepsilon}(y)} |\nabla u| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

For B , we estimate the difference $|B - u(y)|$:

$$\begin{aligned}
|B - u(y)| &= \left| \int_{\partial B_\varepsilon(y)} (u(y) + \nabla u(y + \theta_x(x - y)) \cdot (x - y)) \partial_\nu \Phi(x - y) \, dS(x) - u(y) \right| \\
&\leq \underbrace{\left| u(y) \left(\int_{\partial B_\varepsilon(y)} \partial_\nu \Phi(x - y) \, dS(x) - 1 \right) \right|}_{=0} \\
&\quad + \left| \int_{\partial B_\varepsilon(y)} \nabla u(y + \theta_x(x - y)) \cdot (x - y) \partial_\nu \Phi(x - y) \, dS(x) \right| \\
&\leq \sup_{B_\varepsilon(y)} |\nabla u| \varepsilon \int_{\partial B_\varepsilon(y)} |\partial_\nu \Phi(x - y)| \, dS(x) \longrightarrow 0 \text{ as } \varepsilon \rightarrow 0..
\end{aligned}$$

Notice that, in the second line, the fact that $\int_{\partial B_\varepsilon(y)} \partial_\nu \Phi(x - y) \, dS(x) = 1$ is due to ν being the normal outgoing vector to the domain $\Omega \setminus B_\varepsilon(y)$, hence the normal *ingoing* vector to $B_\varepsilon(y)$.

Finally, for the term C , since $\Phi(x - y)$ is integrable in Ω for any $y \in \Omega$, we have

$$\begin{aligned}
\left| C - \int_{\Omega} \Delta u(x) \Phi(x - y) \, dx \right| &= \left| \int_{B_\varepsilon(y)} \Delta u(x) \Phi(x - y) \, dx \right| \\
&\leq \sup_{B_\varepsilon(y)} |\Delta u| \int_{B_\varepsilon(y)} |\Phi(x - y)| \, dx \\
&\leq \sup_{B_\varepsilon(y)} |\Delta u| \int_0^\varepsilon s^{n-1} \omega_n |\Phi(s)| \, ds \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned}$$

□

REMARK 3.1. *In these notes, we will use also the notation $\Phi(x, y) := \Phi(x - y) = \Phi(|x - y|)$, as needed.*

Applying Green's representation formula to a test function $\varphi \in \mathcal{D}(\Omega)$, we obtain

$$\varphi(y) = \int_{\Omega} \Phi(x, y) \Delta \varphi(x) \, dx$$

that is,

$$\Delta_x \Phi(x, y) = \delta_y,$$

which is consistent with [Theorem 2.15](#).

We may draw the following consequence from the Green representation formula: If one knows Δu , then u is completely determined by its values and those of its normal derivative on $\partial\Omega$. In particular, a harmonic function on Ω can be reconstructed from its values on $\partial\Omega$. One may then ask conversely whether one can construct a harmonic function for arbitrary given values on $\partial\Omega$ for the function and its normal derivative.

DEFINITION 3.1 (Green function). *A function $G = G(x, y)$, defined for $x, y \in \bar{\Omega}$, with $x \neq y$, is called a Green function for Ω if*

- (1) $G(x, y) = 0$ for $x \in \partial\Omega$,
- (2) $h(x, y) := G(x, y) - \Phi(x, y)$ is harmonic in $x \in \Omega$ (thus in particular also at the point $x = y$).

Roughly speaking, (if it exists) $G(x, y) = \Phi(x, y) + h(x, y)$, where h solves

$$\begin{cases} -\Delta_x h(x, y) = 0, & x \in \Omega, \\ h(x, y) = -\Phi(x, y), & x \in \partial\Omega. \end{cases} \quad (1.3)$$

In other words, G solves, for $y \in \Omega$,

$$\begin{cases} -\Delta_x G(x, y) = \delta_0(x - y), & x \in \Omega, \\ G(x, y) = 0, & x \in \partial\Omega. \end{cases}$$

We now assume that a Green function $G(x, y)$ for Ω exists and put $v(x) = h(x, y)$ in Green's identity (see [Appendix 2](#)) and subtract the result from [\(1.1\)](#), obtaining

$$u(y) = - \int_{\partial\Omega} u(x) \partial_{\nu_x} G(x, y) \, dS(x) - \int_{\Omega} G(x, y) \Delta u(x) \, dx. \quad (1.4)$$

We call $K(x, y) := -\partial_{\nu_x} G(x, y)$ the *Poisson kernel* of Ω .

In particular, formula [\(1.4\)](#) implies that a harmonic u is already determined by its boundary values $u|_{\partial\Omega}$. This construction now raises the converse question. If we are given functions $g : \partial\Omega \rightarrow \mathbb{R}$ and $f : \Omega \rightarrow \mathbb{R}$, can we obtain a solution of the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases} \quad (1.5)$$

by the representation formula

$$u(y) = - \int_{\partial\Omega} g(x) \partial_{\nu_x} G(x, y) \, dS(x) + \int_{\Omega} f(x) G(x, y) \, dx \quad (1.6)$$

derived above? One implication is already proven: if u is a solution, it does satisfy this formula. For the other implication (namely, if u satisfies the formula [\(1.6\)](#), then it is a solution of [\(1.5\)](#)), we will see that, essentially, the answer is yes, under suitable hypotheses.

We start our analysis by first considering the case $f \equiv 0$ and then extending it to a general f . In special domains, the Green function can be explicitly constructed.

1.1. Dirichlet problem for the Laplace operator in a ball. We start with the case $\Omega := B_R(0)$ and consider the problem

$$\begin{cases} -\Delta u(x) = 0, & x \in B_R(0), \\ u(x) = g(x), & x \in \partial B_R(0). \end{cases} \quad (1.7)$$

The idea is to construct h by a proper reflection of $\Phi(\cdot - y)$.

For $y \in \mathbb{R}^n$, we put

$$\tilde{y} := \begin{cases} \frac{R^2}{|y|^2} y & \text{if } y \neq 0, \\ \infty & \text{if } y = 0, \end{cases}$$

(\tilde{y} is the point obtained from y by reflection across $\partial B_R(0)$.) We then put

$$G(x, y) := \begin{cases} \Phi(|x - y|) - \Phi\left(\frac{|y|}{R}|x - \tilde{y}|\right), & \text{if } y \neq 0, \\ \Phi(|x|) - \Phi(R), & \text{if } y = 0. \end{cases} \quad (1.8)$$

For $x \neq y$, $G(x, y)$ is harmonic in x , since, for $y \in B_R(0)$, the point \tilde{y} lies in the exterior of $B_R(0)$. The function $G(x, y)$ has only one singularity in $B_R(0)$, namely, at $x = y$, and this singularity is the same as that of $\Phi(x, y)$. The formula

$$G(x, y) = \Phi\left(\left(|x|^2 + |y|^2 - 2x \cdot y\right)^{1/2}\right) - \Phi\left(\left(\frac{|x|^2 |y|^2}{R^2} + R^2 - 2x \cdot y\right)^{1/2}\right)$$

then shows that for $x \in \partial B_R(0)$, i.e., $|x| = R$, we have indeed $G(x, y) = 0$.

Therefore, the function G defined by [\(1.8\)](#) is the Green function of $B_R(0)$. Equation [\(1.8\)](#) also implies the symmetry

$$G(x, y) = G(y, x).$$

We compute the partial derivatives of the Green function:

$$\partial_{x_i} G(x, y) = \frac{1}{\omega_n} \left(-\frac{x_i - y_i}{|x - y|^n} + \frac{R^{n-2}}{|y|^{n-2}} \frac{x_i - \tilde{y}_i}{|x - \tilde{y}|^n} \right);$$

since $\frac{|y|}{R}|x - \tilde{y}| = |x - y|$ whenever $x \in \partial B_R(0)$, we have, in particular,

$$\partial_{x_i} G(x, y) = -\frac{\left(1 - \frac{|y|^2}{R^2}\right) x_i}{\omega_n |x - y|^n}, \quad x \in \partial B_R(0).$$

Finally, since $x \cdot \nu = R$, for $x \in \partial B_R(0)$, we have

$$\partial_{\nu_x} G(x, y) = -\frac{R^2 - |y|^2}{\omega_n R |x - y|^n}, \quad x \in \partial B_R(0), \quad y \in B_R(0).$$

In conclusion, we construct then a candidate solution to the Dirichlet problem (1.7) on the ball $B_R(0)$ with the formula

$$u(y) = \frac{R^2 - |y|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{g(x)}{|x - y|^n} dS(x). \quad (1.9)$$

The function

$$K(x, y) = -\partial_{\nu_x} G(x, y) = \frac{R^2 - |y|^2}{\omega_n R |x - y|^n}$$

is called the *Poisson kernel* for the ball $B_R(0)$ and formula (1.9) is called the Poisson integral formula for the ball $B_R(0)$.

THEOREM 3.2. *Given $g \in C^0(\partial B_R(0))$, the function*

$$u(y) = \begin{cases} \int_{\partial B_R(0)} K(x, y) g(x) dS(x), & y \in B_R(0), \\ g(y), & y \in \partial B_R(0), \end{cases}$$

belongs to $C^2(B_R(0)) \cap C^0(\overline{B_R(0)})$ and is the unique solution of (1.7).

PROOF. In the proof we use the shorthand notation $B_R = B_R(0)$. Since $G(x, y) = G(y, x)$ for all $x, y \in \overline{B_R}$, with $x \neq y$, the function $y \mapsto G(x, y)$ is harmonic in B_R for any $x \in \partial B_R$, and so is $y \mapsto \partial_{\nu_x} G(x, y)$. It follows that u is harmonic (and C^∞) in B_R .

To prove that $u \in C^0(\overline{B_R})$, we first observe that

$$\int_{\partial B_R} K(x, y) dS(x) = 1, \quad \text{for all } y \in B_R(0).$$

We want to show that, for all $y_0 \in \partial B_R$, we have $\lim_{B_R \ni y \rightarrow y_0} u(y) = g(y_0)$.

Since g is continuous at y_0 , we have: for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $|g(x) - g(y_0)| \leq \varepsilon$ for any $x \in \partial B_R$ such that $|x - y_0| \leq \delta_\varepsilon$. Then

$$\begin{aligned} |u(y) - g(y_0)| &= \left| \int_{\partial B_R} K(x, y) (g(x) - g(y_0)) dS(x) \right| \\ &\leq \int_{\partial B_R} K(x, y) |g(x) - g(y_0)| dS(x) \\ &= \underbrace{\int_{\partial B_R \cap B_{\delta_\varepsilon}(y_0)} K(x, y) |g(x) - g(y_0)| dS(x)}_{=:A} \\ &\quad + \underbrace{\int_{\partial B_R \setminus B_{\delta_\varepsilon}(y_0)} K(x, y) |g(x) - g(y_0)| dS(x)}_{=:B}, \end{aligned}$$

where we have dropped the absolute value on the Poisson kernel since $K(x, y) > 0$ for $y \in B_R$ and $x \in \partial B_R$.

The first term can be bounded as

$$A \leq \varepsilon \int_{\partial B_R \cap B_{\delta_\varepsilon}(y_0)} K(x, y) dS(x) < \varepsilon, \quad \text{for all } y \in B_R$$

For the second term, we take $|y - y_0| \leq \tilde{\delta} < \delta_\varepsilon/2$ and $|x - y_0| \geq \delta_\varepsilon$. Then $|x - y| > \delta_\varepsilon/2$ and, since $|y_0| = R$, also $R - |y| \leq |y - y_0| \leq \tilde{\delta}$ so that

$$K(x, y) = \frac{R^2 - |y|^2}{\omega_n R |x - y|^n} \leq \frac{(R + |y|)(R - |y|)2^n}{R\omega_n \delta_\varepsilon^n} < \frac{2R2^n \tilde{\delta}}{R\omega_n \delta_\varepsilon^n}, \quad \text{for all } x \notin \overline{B_{\delta_\varepsilon}(y_0)}, \quad y \in \overline{B_{\tilde{\delta}}(y_0)}$$

and, setting $M = \|g\|_{C^0(\partial B_R)}$, we have

$$B < 2M \int_{\partial B_R \setminus B_{\delta_\varepsilon}(y_0)} K(x, y) \, dS(x) \leq \frac{2^{n+2} M R^{n-1}}{\delta_\varepsilon^n} \tilde{\delta}.$$

Taking now $\tilde{\delta} < \min \left\{ \frac{\delta_\varepsilon}{2}, \frac{\varepsilon \delta_\varepsilon^n}{2^{n+2} M R^{n-1}} \right\}$ we conclude that

$$|u(y) - g(y_0)| < 2\varepsilon \quad \text{if } |y - y_0| < \tilde{\delta},$$

which proves the continuity of u in y_0 . □

1.2. Dirichlet problem for the Laplace operator in a half-plane. Let $\Omega := \mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$. We want to build the Green function of the Laplacian in \mathbb{R}_+^n .

Again, the idea is to construct the function h by suitable reflection of $\Phi(\cdot - y)$. For any $y = (y_1, \dots, y_{n-1}, y_n) \in \mathbb{R}_+^n$, we define the reflected point $\tilde{y} := (y_1, \dots, y_{n-1}, -y_n) \notin \mathbb{R}_+^n$.

We claim that $h(x, y) := -\Phi(x - \tilde{y})$ solves

$$\begin{cases} -\Delta h(x, y) = 0, & x \in \mathbb{R}_+^n, \\ h(x, y) = -\Phi(x - \tilde{y}), & x \in \partial \mathbb{R}_+^n, \end{cases}$$

and $G(x, y) := \Phi(x - y) - \Phi(x - \tilde{y})$ is the Green function of the Laplace equation in \mathbb{R}_+^n .

Moreover, since

$$\partial_{x_i} G(x, y) = \frac{1}{\omega_n} \left[-\frac{x_i - y_i}{|x - y|^n} + \frac{x_i - \tilde{y}_i}{|x - \tilde{y}|^n} \right] = \begin{cases} 0, & \text{if } i \neq n, \\ \frac{2y_n}{\omega_n |x - y|^n}, & \text{if } i = n, \end{cases}$$

the Poisson Kernel is

$$K(x, y) := -\partial_{\nu_x} G(x, y) = \partial_{x_n} G(x, y) = \frac{2y_n}{\omega_n |x - y|^n},$$

and a candidate solution to the Dirichlet problem in \mathbb{R}_+^n is given by the following Poisson integral formula

$$u(y) = \int_{\partial \mathbb{R}_+^n} K(x, y) g(x) \, dS(x).$$

We report these results in the following theorem (whose proof is left as an exercise in Exercise Sheet 6 for $n = 2$).

THEOREM 3.3. *Let $g \in C_c^0(\partial \mathbb{R}_+^n)$ and define*

$$u(y) := \begin{cases} \frac{2y_n}{\omega_n} \int_{\partial \mathbb{R}_+^n} \frac{g(x)}{|x - y|^n} \, dx, & y \in \mathbb{R}_+^n, \\ g(y), & y \in \partial \mathbb{R}_+^n. \end{cases}$$

Then, $u \in C^2(\mathbb{R}_+^n) \cap C^0(\overline{\mathbb{R}_+^n})$ is the unique solution to the Dirichlet problem

$$\begin{cases} -\Delta u = 0, & x \in \mathbb{R}_+^n, \\ u = g, & x \in \partial \mathbb{R}_+^n, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases}$$

1.3. Dirichlet problem for the Laplace operator in a general domain via Perron's method. We now consider an open bounded set $\Omega \subseteq \mathbb{R}^n$ and the Dirichlet problem

$$\begin{cases} -\Delta u = 0, & x \in \Omega, \\ u = g, & x \in \partial \Omega. \end{cases} \tag{1.10}$$

In general, we will not be able to find a closed form expression for the Green function. However, we can still ask a more fundamental question whether the solution to such Dirichlet problem exists for any $g \in C^0(\partial \Omega)$. Observe that by the weak maximum principle, if a solution of this problem exists, it is unique (see Section 2.3.4).

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NB: 22.10 is in the semester break

If the answer to the existence question is positive, this will, in particular, ensure the existence of the Green function for the domain Ω . To answer this question, we construct a solution using *Perron's method*.¹

1.3.1. *Sub-harmonic functions and harmonic lifting.* As a first step, we revisit the characterization of sub-harmonic functions via the mean-value property by suitably generalizing [Theorem 2.4](#).² The point of the following Proposition is to generalize the notion of sub-harmonicity, which will be a necessary step to use Perron's method.

PROPOSITION 3.1. *Let $u \in C(\Omega)$. Then the following properties are equivalent.*

(1) *For all $x \in \Omega$ and $B_r(x) \subset\subset \Omega$,*

$$u(x) \leq \int_{B_r(x)} u(y) \, dy$$

(2) *For all $B \subset\subset \Omega$ and for all $h : \bar{B} \rightarrow \mathbb{R}$ which satisfies*

$$\begin{cases} -\Delta h = 0, & x \in B, \\ h \geq u, & x \in \partial B, \end{cases}$$

we have $h(x) \geq u(x)$ for all $x \in \bar{B}$.

(3) *For all $x \in \Omega$ and $B_r(x) \subset\subset \Omega$,*

$$u(x) \leq \int_{\partial B_r(x)} u(y) \, dS(y).$$

(4) *For all $x \in \Omega$ and for all $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum in x , then $-\Delta\phi(x) \leq 0$.*

PROOF. (1) \implies (2). This is basically a re-proof of strong maximum principle. Let $M = \sup_{x \in B} u(x) - h(x)$, we want to show $M \leq 0$. We define $S = \{x \in B : u(x) - h(x) = M\}$, the set where the maximum is achieved. Combining (1) with the fact that h is harmonic, we have that

$$u(x) - h(x) \leq \int_{B_r(x)} (u(y) - h(y)) \, dy$$

for any ball $B_r(x) \subset\subset \Omega$. In particular, if $x \in S$, we have that $u(x) - h(x) \geq u(y) - h(y)$ and get from the equation above that $u(x) - h(x) = u(y) - h(y)$ is equal to that maximum in the whole $B_r(x)$.

Therefore, if $x \in S$, we have $B_r(x) \subset S$, so S is open. Since S is the preimage of a closed set $\{M\}$ by a continuous function $u - h$, it is also closed in the topology of B . Hence, S is open and closed in the topology of B (which is connected), so we conclude that either $S = \emptyset$ or $S = B$. In the first case the maximum of $u(x) - h(x)$ over B is achieved only at the boundary, in the second case $u(x) - h(x)$ is constant. In any case, using that $u(x) \leq h(x)$ at ∂B , we conclude $M \leq 0$.

(2) \implies (3). Let $x \in \Omega$ and $B_r(x) \subset\subset \Omega$ and let h be the solution to

$$\begin{cases} -\Delta h = 0, & y \in B_r(x), \\ h = u, & y \in \partial B_r(x). \end{cases}$$

Then by the Poisson integral formula (or by the property of spherical mean for harmonic functions) $h(x) = \int_{\partial B_r(x)} h(y) \, dS(y)$. Moreover by (2), $h(x) \geq u(x)$, so we conclude.

(3) \implies (1). Let $x \in \Omega$ and $B_r(x) \subset\subset \Omega$. Then, by the formula of integral over spheres and by (3),

$$\int_{B_r(x)} u(y) \, dy = \int_0^r \int_{\partial B_s(x)} u(y) \, dS(y) \, ds \geq \int_0^r u(x) n \omega_n s^{n-1} \, ds = u(x) \omega_n r^n,$$

which gives the conclusion.

¹ Introduced by Oskar Perron [[Per23](#)].

² Point (iv) in [Proposition 3.1](#) may be restated as “ u is sub-harmonic in Ω in the sense of viscosity solutions”, where we refer to the theory of *viscosity solutions* introduced in [[CL83](#); [CEL84](#); [CIL92](#)].

(3) \implies (4). Let $x \in \Omega$, $\phi \in C^2(\Omega)$ and $B_r(x) \subseteq \Omega$ such that $u(y) - \phi(y) \leq u(x) - \phi(x)$ for all $y \in \overline{B_r(x)}$. Combining that with (3), we get:

$$u(x) - \phi(x) \geq \int_{\partial B_r(x)} u(y) dS(y) - \int_{\partial B_r(x)} \phi(y) dS(y) \geq u(x) - \int_{\partial B_r(x)} \phi(y) dS(y). \quad (1.11)$$

which implies that $\phi(x) \leq \int_{\partial B_r(x)} \phi(y) dS(y)$. Finally, we get the desired sign from formula (2.2) for the Laplacian (average of neighbours minus $u(x)$), so that

$$\Delta\phi(x) = 2n \lim_{r \rightarrow 0^+} \frac{1}{r^2} \left(\int_{\partial B_r(x)} \phi(y) dS(y) - \phi(x) \right) \geq 0$$

(4) \implies (3). We will assume that (3) fails and construct a function $\phi \in C^2$ with $\Delta\phi < 0$ and $u - \phi$ having a maximum at x . Take a ball $B_r(x)$ where (3) fails.

First, we consider ψ solving $\Delta\psi = 0$ in $B_r(x)$ and $\psi = u$ at $\partial B_r(x)$. Therefore, we have that

$$\psi(x) = \int_{\partial B_r(x)} \psi(y) dS(y) = \int_{\partial B_r(x)} u(y) dS(y) = u(x) - \eta$$

for some $\eta > 0$ due to (3) failing on $B_r(x)$.

Now, the function $u - \psi$ is 0 at $\partial B_r(x)$ and $\eta > 0$ at x , so it achieves a maximum in the interior of $B_r(x)$. However, it is not still a counterexample of (4) since $\Delta\psi = 0$.

In order to solve that we consider $\varphi(y) = -c|y - x|^2$ for c sufficiently small so that $cr^2 < \eta$. Therefore, $u - \psi - \varphi$ has value cr^2 in $\partial B_r(x)$ and η at x , so that it achieves its maximum in the interior. Moreover, $\Delta(\varphi + \psi) = \Delta\varphi = -2c < 0$, so we see that $\phi = \varphi + \psi$ gives a valid counterexample to (4). □

We now define, given a sub-harmonic function u in Ω , its *harmonic lifting* in $B \subset \Omega$.

REMARK 3.2 (If $u \in C^2$, the previous Proposition agrees with the definition of sub-harmonic-ity). Recall that in the last part of (3) \implies (4) we proved that if $\phi \in C^2$ satisfies property (3), that is, $\phi(x) \leq \int_{\partial B_r(x)} \phi(y) dS(y)$, then it holds that $\Delta\phi \geq 0$.

In particular, if u satisfies any of the properties above **and moreover** is C^2 , then we have that $\Delta u \geq 0$ using that argument. Hence, the Proposition above is a natural generalization of sub-harmonic functions to functions that are just continuous, and we will call sub-harmonic function to a function satisfying any of the properties above. If moreover, $u \in C^2$, we can define the Laplacian and we recover our previous definition $\Delta u \geq 0$.

REMARK 3.3 (Strong maximum principle holds with our new definition of sub-harmonic). Notice that in our implication of (1) \implies (2) we are only using that B is a open connected domain (not necessarily a ball). We are proving that the maximum of a sub-harmonic function over a open connected domain is either achieved only at the boundary or u is constant, which is exactly the strong maximum principle. Thus, strong maximum principle holds for our generalization of sub-harmonicity.

DEFINITION 3.2 (Harmonic lifting). Let u be a sub-harmonic function in Ω and $B \subset\subset \Omega$. Then the harmonic lifting of u in B is the function U which coincides with u in $\Omega \setminus B$ and in B solves the Dirichlet problem

$$\begin{cases} -\Delta U = 0, & x \in B, \\ U = u, & x \in \partial B. \end{cases}$$

By maximum principle, we have that $u \leq U$ in Ω .

REMARK 3.4 (The harmonic lifting is sub-harmonic). Let U be the harmonic lifting of u in B , then U is a sub-harmonic function. We do it showing property (4). Property (4) trivially holds for $x \notin \partial B$, because in that case U will either agree locally with u (if x is outside of \overline{B}) or will be harmonic inside B (if $x \in B$). Therefore, it suffices to show (4) for $x \in \partial B$. Assuming $U(x) - \phi(x)$ achieves a local maximum in $B_\delta(x)$, we have

$$u(x) - \phi(x) = U(x) - \phi(x) \geq U(y) - \phi(y) \geq u(y) - \phi(y)$$

for all $y \in B_\delta(x)$. Hence, $u(x) - \phi(x)$ achieves a local maximum as well. Since u is sub-harmonic, property (4) holds for u , and therefore $-\Delta\phi(x) \leq 0$.

1.3.2. *Existence result.* Let Ω be a bounded open set and $g \in L^\infty(\partial\Omega)$. Let us define

$$S_g := \{v \in C(\bar{\Omega}) : v \text{ sub-harmonic in } \Omega, \\ v(x) \leq g(x), \text{ in } \partial\Omega, \text{ and } \inf_{\partial\Omega} g \leq v(x) \text{ for all } x \in \Omega\}.$$

REMARK 3.5 (Non-empty). *The set S_g is not empty and bounded from above. In fact the constant function $v := \inf_{\partial\Omega} g$ is in S_g . Moreover, by the weak maximum principle, we get $v \leq \sup_{\partial\Omega} g$ for all $v \in S_g$.*

REMARK 3.6 (Closer under supremum). *If $\emptyset \neq \mathcal{V} \subset S_g$ then we have that $M_{\mathcal{V}}(x) = \sup_{v \in \mathcal{V}} v(x)$ is an element of S_g as well. The only nontrivial condition is to check that $M_{\mathcal{V}}$ is sub-harmonic, which we do via property (2) of Proposition 3.1.*

Assume h is harmonic on $B \subset\subset \Omega$ and $h(x) \geq M_{\mathcal{V}}(x)$ at ∂B . Then, for every $v \in \mathcal{V}$, we have $h(x) \geq v(x)$, and using property (2) for v we get $h(x) \geq v(x)$ for all $x \in \bar{B}$. Taking supremum over v , we recover $h(x) \geq M_{\mathcal{V}}(x)$, showing that $M_{\mathcal{V}}(x)$ satisfies property (2) of Proposition 3.1 and hence is sub-harmonic.

*As a particular case, when $\mathcal{V} = \{v, w\}$, **the maximum of two sub-harmonic functions is sub-harmonic.***

THEOREM 3.4. *Let Ω be an open and bounded set and $g \in L^\infty(\partial\Omega)$. Then the function*

$$H_g(x) := \sup_{v \in S_g} v(x)$$

is harmonic in Ω .

ALTERNATIVE SIMPLER PROOF: Due to Remark 3.6 (with $\mathcal{V} = S_g$), we have that $H_g(x) \in S_g$ and in particular is sub-harmonic. Let us conclude it is harmonic.

Take $B \subset\subset \Omega$ and consider L to be the harmonic lifting of H_g in B . On the one hand, since it is an harmonic lifting, $L(x) \geq H_g(x)$. On the other hand, since $L(x)$ is sub-harmonic, we have that $L \in S_g$, and therefore $L(x) \leq H_g(x) = \sup_{v \in S_g} v(x)$. As a consequence, we obtain $L = H_g$, so L agrees with its harmonic lifting, meaning it is harmonic in B . Since we can do this for any ball B , we conclude H_g is harmonic. \square

OLD PROOF DONE IN CLASS. Fix $x \in \Omega$. We want to show that H_g is harmonic in x . Let v_n be a sequence in S_g such that $v_n(x) \rightarrow H_g(x)$.

We fix $r > 0$ such that $B_r(x) \subset\subset \Omega$ and consider for every n the harmonic lifting V_n of v_n in $B_r(x)$. Then $V_n \in S_g$ and $V_n(x) \rightarrow H_g(x)$. By Ascoli–Arzelà’s theorem for harmonic functions (see Problem 1 in Exercise Sheet 7),³ possibly passing to a subsequence (that we still denote with V_n) we get that $V_n \rightarrow V$ uniformly in $B_\rho(x)$ for every $\rho < r$. Moreover, we have that V is harmonic in $B_r(x)$, $V(y) \leq H_g(y)$ for every $y \in B_r(x)$, and $V(x) = H_g(x)$.

We claim now that there exists $\rho < r$ such that $V(y) = H_g(y)$ for every $y \in B_\rho(x)$. If it is true, we are done, since then H_g is harmonic in x .

We assume that the claim is not true, so for every ρ we find $z \in B_\rho(x)$ such that $V(z) < H_g(z)$. We prove that this leads to a contradiction.

We proceed at point z exactly in the same way as we did for x (taking $w_n \in S_g$ such that $w_n(z) \rightarrow H_g(z)$, doing the harmonic lifting and taking limit along a subsequence. However, we notice that we can enforce $w_n \geq V_n$, since otherwise we could replace w_n by $\max\{w_n, V_n\}$ in order for this to hold. Thus, we obtain W harmonic in $B_\rho(x)$, with $V(y) \leq W(y)$ on $B_\rho(x)$ and $W(z) = H_g(z) > V(z)$.

So, $V - W$ is harmonic in $B_\rho(x)$ such that $V - W \leq 0$, and $V(x) - W(x) = 0$. This implies, by the strong maximum principle, that $V \equiv W$ in $B_\rho(x)$, in contradiction with the fact that $W(z) > V(z)$. \square

³ Alternatively, we can define a new sequence $\{\tilde{V}_n := \max_{j \leq n} V_j\}_n$, which is now a non-decreasing sequence; we can then use Harnack’s convergence theorem.

1.3.3. *Study of the boundary behaviour.* In [Theorem 3.4](#), we proved that, for every bounded function g , there exists a harmonic function $H_g \in C^\infty(\Omega)$ which solves

$$\begin{cases} -\Delta H_g = 0, & x \in \Omega \\ H_g \geq g, & x \in \partial\Omega. \end{cases}$$

Now, we assume that $g \in C(\partial\Omega)$ and we wonder under which conditions H_g is the solution of the Dirichlet problem [\(1.10\)](#), in particular under which conditions we can prove that, for all $x_0 \in \partial\Omega$,

$$\lim_{\substack{x \rightarrow x_0, \\ x \in \Omega}} H_g(x) = g(x_0). \quad (1.12)$$

Indeed, if we prove this identity, we get that $H_g \in C^\infty(\Omega) \cap C(\bar{\Omega})$ and coincides with g on the boundary of Ω .

REMARK 3.7. *Observe that in general we cannot expect that [\(1.12\)](#) holds true for every Ω bounded. Let $\Omega := \{x \in \mathbb{R}^2 : 0 < |x| < 1\}$ and $g \in C(\partial\Omega)$ such that $g(x) = 0$ for $|x| = 1$ and $g(0) = 1$. Then $H_g \equiv 0$ (and, in particular, it is not a solution of the Dirichlet problem with boundary data g since $H_g(0) = 0 \neq 1$). In fact, $0 \in S_g$, so $H_g(x) \geq 0$ for every $x \in \Omega$. Let $v \in S_g$. So, by the weak maximum principle, $v(x) < 1$ for every $x \in \Omega$. Fix $\delta > 0$ and $\varepsilon = \varepsilon(\delta) \in (0, 1)$ such that $-\delta \log(\varepsilon) > 1$. Consider now the function $w_\delta(x) = -\delta \log|x|$. This is harmonic in $\varepsilon < |x| < 1$, moreover $w_\delta(x) = 0$ if $|x| = 1$ and $w_\delta(x) = -\delta \log \varepsilon > 1$ if $|x| = \varepsilon$. This implies, by the weak maximum principle, that $v(x) \leq w_\delta(x)$ for every $\varepsilon \leq |x| \leq 1$. Moreover, $w_\delta(x) > 1 \geq v(x)$ also for every $|x| \leq \varepsilon$. Then $w_\delta(x) \geq v(x)$ for every $0 < |x| \leq 1$ and every $v \in S_g$, which implies that $H_g(x) \leq -\delta \log|x|$ for every $0 < |x| \leq 1$, and every $\delta > 0$, which gives the conclusion letting $\delta \rightarrow 0$.*

The continuity assumption [\(1.12\)](#) on the boundary is connected with the geometric properties of the boundary through the concept of barrier.

DEFINITION 3.3 (Local barrier). *Let $x_0 \in \partial\Omega$. Then w is a local barrier at x_0 if there exists a neighborhood U of x_0 such that $w \in C(\bar{\Omega} \cap \bar{U})$ and*

- (1) w is super-harmonic in $\Omega \cap U$;
- (2) $w(x_0) = 0$ and $w(x) > 0$ for every $x \in \bar{\Omega} \cap \bar{U} \setminus \{x_0\}$.

A barrier w in x_0 , is a local barrier with $U = \mathbb{R}^n$.

REMARK 3.8. *Given a local barrier in x_0 , it is always possible to construct a barrier in x_0 as follows: given $r > 0$ such that $B_{2r}(x_0) \subset\subset U$ and $m := \inf_{(U \setminus B_r(x_0)) \cap \Omega} w > 0$, let*

$$w(x) := \begin{cases} \min\{m, w(x)\}, & x \in \overline{B(x_0, 2r)} \cap \bar{\Omega}, \\ m, & \text{elsewhere.} \end{cases}$$

Then w is a barrier in x_0 .

DEFINITION 3.4 (Regular points). *Let $x_0 \in \partial\Omega$. Then x_0 is a regular point (with respect to the Laplacian), if there exists a (local) barrier at x_0 .*

THEOREM 3.5. *Let Ω be an open bounded set, $g \in L^\infty(\partial\Omega)$. Let $x_0 \in \partial\Omega$. If x_0 is regular (with respect to the Laplacian) and g is continuous in x_0 , then [\(1.12\)](#) holds in x_0 .*

PROOF. Fix $\varepsilon > 0$, then there exists $\delta > 0$ such that, for all $y \in \partial\Omega$, with $|y - x_0| \leq \delta$, we have $|g(y) - g(x_0)| \leq \varepsilon$ (since g is continuous). Let $M := \|g\|_{C^0(\partial\Omega)}$. Let w be a barrier in x_0 . We have that $K := \min_{x \in \bar{\Omega} \setminus B_\delta(x_0)} w(x) > 0$. Let us define

$$\underline{w}(x) := g(x_0) - \varepsilon - \frac{2M}{K}w(x), \quad \bar{w}(x) := g(x_0) + \varepsilon + \frac{2M}{K}w(x).$$

We have that \underline{w} is sub-harmonic and, for $x \in \partial\Omega$, $w(x) \leq g(x)$. Indeed,

$$\text{if } |x - \xi| < \delta, \quad \underline{w}(x) = \underbrace{g(x_0) - g(x) - \varepsilon}_{< 0} - \frac{2M}{K}w(x) + g(x) \leq g(x),$$

$$\text{if } |x - \xi| \geq \delta, \quad \underline{w}(x) \leq g(x) - \varepsilon + \underbrace{g(\xi) - g(x) - \frac{2M}{K}w(x)}_{\leq 0 \text{ since } w(x) \geq K} \leq g(x).$$

Similarly, we can prove that \bar{w} is super-harmonic and $\bar{w} \geq g(x)$ on $\partial\Omega$.

In conclusion, $\underline{w}(x) \leq H_g(x) \leq \bar{w}(x)$, for all $x \in \bar{\Omega}$, which implies

$$|H_g(x) - g(x_0)| \leq \epsilon + \frac{2M}{K}w(x).$$

Since $w(x) \rightarrow 0$ as $x \rightarrow x_0$, there exists $\tilde{\delta} > 0$ such that $w(x) < \frac{K}{2M}\epsilon$ for all $|x - x_0| \leq \tilde{\delta}$, and $|u(x) - g(x_0)| \leq 2\epsilon$ for all $x \in \bar{\Omega}$ such that $|x - x_0| \leq \tilde{\delta}$ which proves the continuity of H_g in x_0 . \square

From [Theorem 3.4](#) and [Theorem 3.5](#) we get the following result.

COROLLARY 3.1. *For every $g \in C(\partial\Omega)$, the Dirichlet problem (1.10) admits a unique solution $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ if and only if all the boundary points of Ω are regular.*

PROOF. Step 1. If g is continuous and all the points of the boundary are regular, then H_g is a solution of (1.10), and it is unique by weak maximum principle.

Step 2. If (1.10) admits a solution for every continuous boundary data, take $x_0 \in \partial\Omega$ and the solution u to (1.10) with $g(x) = |x - x_0|$. Then the solution u to (1.10) is a barrier in x_0 . \square

1.3.4. Regular boundary points. It remains open the question: for which domains Ω all the boundary points are regular? Sufficient conditions for this property to hold can be stated in terms of local geometric (for $n > 2$) or topological (for $n = 2$) properties of the boundary.⁴

We mention some of these conditions.

DEFINITION 3.5 (Exterior ball condition). *Let Ω be an open set of \mathbb{R}^n . We say that Ω has the exterior ball condition if, at every point $x \in \partial\Omega$, there exists $y \in \mathbb{R}^n \setminus \Omega$ and $r > 0$ such that $B_r(y) \subset \mathbb{R}^n \setminus \bar{\Omega}$ and $\overline{B_r(y)} \cap \bar{\Omega} = \{x\}$.*

REMARK 3.9. *Observe that if Ω is convex, then the exterior ball condition is satisfied, due to the Hahn Banach separation theorem. Indeed at every point of $\partial\Omega$ it is possible to construct an hyperplane passing through that point and such that Ω is entirely contained in one of the two half spaces in which the space is divided by the hyperplane.*

Observe that Ω of class C^1 is not sufficient to assure that the exterior ball condition is satisfied. For example, consider $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > x_1^2 \log |x_1|\}$. Then Ω is of class C^1 but in $(0, 0)$ the exterior ball condition is not satisfied. Indeed to prove this, let $f(x) = x^2 \log |x|$ and $g(x) = \sqrt{r^2 - x^2} - r$, for $r > 0$ to be fixed. Note that $f(0) = g(0) = 0$. If the exterior ball condition were satisfied in $(0, 0)$, then there would exist $r > 0$ such that $f(x) > g(x)$ for every $x \in (-r, r), x \neq 0$. But this is not the case, since $f'(x) < g'(x)$ for $x \rightarrow 0^+$ and $f'(x) > g'(x)$ for $x \rightarrow 0^-$.

If Ω is of class C^2 , then we can show that it satisfies the exterior ball condition (also the interior ball condition).

PROPOSITION 3.2. *Let Ω be a bounded open set and $x_0 \in \partial\Omega$ such that in x_0 the exterior ball condition is satisfied. Then x_0 is regular (with respect to the Laplacian).*

PROOF. Let Ω be a bounded open set and $x_0 \in \partial\Omega$ such that in x_0 it is satisfied the exterior ball condition: there exists $y_0 \in \mathbb{R}^n \setminus \Omega$ and $r_0 > 0$ such that $\bar{B}(y_0, r_0) \cap \bar{\Omega} = \{x_0\}$. Then x_0 is regular (with respect to the Laplacian).

It can be checked that a barrier is given by

$$w(x) := w(x) = \begin{cases} \frac{1}{r_0^{n-2}} - \frac{1}{|x-y_0|^{n-2}} & n > 2 \\ \log \frac{|x-y_0|}{r_0} & n = 2. \end{cases}$$

\square

⁴ The detailed study of this subject was initiated by Norbert Wiener [Wie24], and then extended to uniformly elliptic divergence-form equations with smooth coefficients by Werner Püschel [Püs32], and then to uniformly elliptic divergence form equations with bounded measurable coefficients by Walter Littman, Guido Stampacchia, and Hans Weinberger [LSW63]. We refer also to the useful book [Lan72] for further information and context. For fully-nonlinear equations (and, in particular, equations not in divergence form), see [LL23] and the references discussed therein.

DEFINITION 3.6 (Exterior cone condition). *Let Ω be a bounded open set. We say that Ω has the exterior cone condition if, at every point $x \in \partial\Omega$, there exists a cone⁵ C with $\text{int } C \neq \emptyset$ and a neighborhood U of x such that $(x + C) \cap U \subset \mathbb{R}^n \setminus \Omega$.*

PROPOSITION 3.3. *Let Ω be a bounded open set and $x_0 \in \partial\Omega$ such that in x_0 the exterior cone condition is satisfied. Then x_0 is regular (with respect to the Laplacian).*

REMARK 3.10. *It is possible to prove that if Ω is of class C^1 , then at every boundary point of Ω the exterior cone condition (and also the interior cone condition) is satisfied.*

Actually, in order for the (exterior and interior) cone condition to be satisfied it is sufficient that the boundary of Ω is Lipschitz.

REMARK 3.11. *In dimension $n = 2$, much more irregular domains can be considered. It can be proved that $x_0 \in \partial\Omega$ is a regular boundary point if it is the endpoint of a single arc lying in the exterior of Ω .*

2. Dirichlet problem for the Poisson equation

We consider now the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases} \quad (2.1)$$

We could again take inspiration from the Green representation formula and construct a candidate solution as

$$u(y) = \int_{\Omega} G(x, y) f(x) dx - \int_{\partial\Omega} \partial_{\nu_x} G(x, y) g(x) dS(x),$$

where G is the Green function of the Laplace equation for the domain Ω . We know already by maximum principle that if a solution exists it is unique. However, for an arbitrary domain, we do not have, in general, the exact expression of the Green function. We proceed therefore in a slightly different way. We consider first the function $w : \mathbb{R}^n \rightarrow \mathbb{R}$, called the *Newtonian potential* of f , given by

$$w(y) = \int_{\Omega} \Phi(x - y) f(x) dx, \quad y \in \mathbb{R}^n, \quad (2.2)$$

where Φ is the fundamental solution of the Laplace equation.

We will argue that, for a sufficiently smooth f , we have

$$w \in C^2(\Omega) \cap C^0(\bar{\Omega}), \quad \text{and} \quad -\Delta w = f \text{ in } \Omega; \quad (2.3)$$

next, we consider the problem

$$\begin{cases} \Delta u_0 = 0, & x \in \Omega, \\ u_0 = g - w, & x \in \partial\Omega, \end{cases}$$

which has a unique solution provided Ω is bounded with all boundary points regular and $g - w \in C^0(\partial\Omega)$, thanks to the results in [Section 1.3](#). It follows that $u = u_0 + w$ is the unique solution of the original problem [\(2.1\)](#).

In summary, the study of the well-posedness of [\(2.1\)](#) boils down to showing that the Newtonian potential of f satisfies [\(2.3\)](#).

THEOREM 3.6. *Let Ω be a bounded domain and $f \in C_c^2(\Omega)$. Then, the Newtonian potential [\(2.2\)](#) of f satisfies $w \in C^2(\mathbb{R}^n)$ and $-\Delta w = f$ in Ω .*

PROOF (DONE IN PROBLEM CLASS PROBLEM SHEET 7). Since f has compact support in Ω , its extension by zero outside Ω , which we denote \tilde{f} , is still a C^2 function. We can then rewrite the Newtonian potential as

$$w(y) = \int_{\Omega} \Phi(x - y) f(x) dx = \int_{\mathbb{R}^n} \Phi(x - y) \tilde{f}(x) dx = \int_{\mathbb{R}^n} \Phi(z) \tilde{f}(z + y) dz.$$

Since $\tilde{f} \in C_c^2(\mathbb{R}^n)$ and Φ is locally integrable we have that $w \in C^2(\mathbb{R}^n)$ and

$$-\Delta w(y) = - \int_{\mathbb{R}^n} \Phi(z) \Delta \tilde{f}(z + y) dz = - \int_{B_R(0)} \Phi(x - y) \Delta \tilde{f}(x) dx,$$

⁵ We recall that $C \subseteq \mathbb{R}^n$ is a (convex) cone if, for every $x, y \in C$, then $x + y \in C$ and $\lambda x \in C$ for every $\lambda > 0$.

where $B_R(0)$ is any sufficiently large ball that contains Ω . Green's representation formula yields

$$\begin{aligned} \tilde{f}(y) &= \int_{B_R(0)} \Phi(x-y)(-\Delta \tilde{f}(x)) \, dx \\ &\quad - \int_{\partial B_R(0)} \partial_\nu \Phi(x-y) \tilde{f}(x) \, dS(x) + \int_{\partial B_R(0)} \Phi(x-y) \partial_\nu \tilde{f}(x) \, dS(x); \end{aligned}$$

since $\tilde{f} \in C_c^2(B_R(0))$, the boundary terms vanish and we conclude that $-\Delta w(y) = \tilde{f}(y) = f(y)$ for all $y \in \Omega$. \square

COROLLARY 3.2. *Let Ω be a bounded domain with all boundary points regular, $g \in C^0(\partial\Omega)$ and $f \in C_0^2(\Omega)$. Then there exists a unique solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of (2.1).*

REMARK 3.12. *One could generalize the theorems above and show that, if $f \in C^{0,\alpha}$, the Newtonian potential satisfies $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $-\Delta w = f$. If Ω is a bounded domain with all boundary points regular and $g \in C^0(\partial\Omega)$, then the Dirichlet problem for the Poisson equation admits a unique solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and, moreover, $u \in C^{2,\alpha}(\Omega)$.*

Here, we recall that a function $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Hölder continuous in Ω with exponent $\alpha \in (0, 1]$ if, in any compact set $K \subset \Omega$,

$$|f|_{\alpha, K} := \sup_{\substack{x, y \in K, \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty;$$

and $C^{0,\alpha}(\Omega)$ denotes the space of locally α -Hölder continuous functions in Ω .

Sobolev spaces and weak solutions to second-order elliptic PDEs

1. Sobolev spaces

1.1. L^p spaces. Let $\Omega \subset \mathbb{R}^n$ be a domain and denote by $L^1(\Omega)$ the space of Lebesgue integrable functions from Ω to \mathbb{R} , where, as usual, two functions are identified if they coincide almost everywhere, i.e., $L^1(\Omega)$ is the space of equivalence classes where two functions are equivalent if the set where they differ has zero Lebesgue measure. Lect. 8, 5.11

We recall the definition of L^p spaces.¹

DEFINITION 4.1 (L^p spaces). For $1 \leq p < \infty$, we define

$$L^p(\Omega) := \{f : \Omega \rightarrow \mathbb{R}, \text{ measurable, s.t. } |f|^p \in L^1(\Omega)\}$$

and, for $p = \infty$,

$$L^\infty(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \text{ measurable, s.t. there exists } C > 0 \text{ s.t. } |f| \leq C \text{ a.e. in } \Omega\}$$

For $p \in [1, \infty]$, we denote define

$$L^p_{loc}(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \text{ s.t. } f \in L^p(K) \text{ for any } K \subset\subset \Omega\}.$$

For a function $f \in L^p(\Omega)$ and $p \in [1, \infty]$, we denote

$$\|f\|_{L^p(\Omega)} := \left(\int_{\Omega} |f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x \in \Omega} |f(x)| := \inf\{C : |f| \leq C \text{ a.e. in } \Omega\}.$$

We recall (without proof) some fundamental properties of L^p spaces.

Completedness: With the definitions above, the normed vector space $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is a Banach space² for any $1 \leq p < \infty$, i.e., every Cauchy sequence $\{f_n\}_{n \in \mathbb{N}} \subset L^p(\Omega)$ has a limit in $L^p(\Omega)$. We recall, moreover, that every Cauchy sequence $\{f_n\}_{n \in \mathbb{N}} \subset L^p(\Omega)$ has a sub-sequence converging pointwise a.e. in Ω . For $p = 2$, the space $L^2(\Omega)$ is a Hilbert space³ with inner product $(f, g)_{L^2(\Omega)} := \int_{\Omega} fg$.

Hölder inequality:⁴ Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, with $1/p + 1/q = 1$. Then $fg \in L^1(\Omega)$ and

$$\int_{\Omega} f(x)g(x) dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$$

More generally, if $f_i \in L^{p_i}(\Omega)$, for $i = 1, \dots, r$ with $\sum_{i=1}^r 1/p_i = 1$, then $\prod_{i=1}^r f_i \in L^1(\Omega)$ and

$$\int_{\Omega} \prod_{i=1}^r f_i(x) dx \leq \prod_{i=1}^r \|f_i\|_{L^{p_i}(\Omega)}$$

Embeddings: Let Ω be bounded. For any $1 \leq p \leq q < \infty$, if $f \in L^q(\Omega)$, then $f \in L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} \leq |\Omega|^{1/p-1/q} \|f\|_{L^q(\Omega)}.$$

If $f \in L^\infty(\Omega)$, then $f \in L^p(\Omega)$ for all $1 \leq p < \infty$ and $\|f\|_{L^p(\Omega)} \leq |\Omega|^{1/p} \|f\|_{L^\infty(\Omega)}$. Moreover, $\lim_{p \rightarrow \infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)}$.

¹ Sometimes also called *Lebesgue spaces*, after Henri Lebesgue, although they were first introduced by Frigyes Riesz [Rie10] (see [Pie07, Section 1.1.4] for further historical discussions).

² Named after Stefan Banach (see [Pie07, Chapter 1] for further historical information).

³ Named after David Hilbert (see [Pie07, Chapter 1, Section 1.5]).

⁴ Named after Otto Hölder [Höl89], but previously proven by Leonard James Rogers [Rog88]. See [Mal98].

Thus, the embedding $L^p(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. This result is not true if the measure of Ω is infinite, but in general we have the following interpolation result: if $1 \leq p \leq q \leq r$, then $L^p(\Omega) \cap L^r(\Omega) \hookrightarrow L^q(\Omega)$ and

$$\|f\|_q \leq \|f\|_p^\theta \|f\|_r^{1-\theta},$$

where $0 \leq \theta \leq 1$ is given by $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$.

Dual of L^p and reflexivity: We recall that the (topological) dual of a Banach space $(V, \|\cdot\|_V)$, denoted V' , is the space of linear and bounded functionals $G : V \rightarrow \mathbb{R}$. It is also a Banach space when endowed with the norm $\|G\|_{V'} := \sup_{\substack{f \in V \\ f \neq 0}} \frac{|G(f)|}{\|f\|_V}$. For $1 \leq p < \infty$, the dual space

of $L^p(\Omega)$ can be identified with $L^q(\Omega)$ with $q = \frac{p}{p-1}$: for any functional $G \in (L^p(\Omega))'$, there exists a unique function $g \in L^q(\Omega)$ such that

$$G(f) = \int_{\Omega} g(x)f(x) dx, \quad \text{for all } f \in L^p(\Omega).$$

Moreover, $\|G\|_{(L^p(\Omega))'} = \|g\|_{L^q(\Omega)}$. Thanks to this identification, the spaces $L^p(\Omega)$ for $p \in (1, \infty)$ are reflexive, i.e., the “double dual” $(L^p(\Omega))''$ can be identified with $L^p(\Omega)$ itself. This property does not hold for $p = 1, \infty$ and, in fact, $(L^\infty(\Omega))' \supset L^1(\Omega)$ with strict inclusion.

Density of continuous functions and separability: $C_c^0(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \leq p < \infty$, i.e., for any $f \in L^p(\Omega)$ and $\epsilon > 0$, there exists $g \in C_c^0(\Omega)$ such that $\|f - g\|_{L^p(\Omega)} \leq \epsilon$. Such density result is not true, however, in $L^\infty(\Omega)$. From this result it also follows that all $L^p(\Omega)$ spaces with $1 \leq p < \infty$ are separable, i.e., they have a countable dense subspace.⁵

Mollification: Let η_ϵ denote the standard mollifier and, for any $f \in L^1_{\text{loc}}(\Omega)$ and $\epsilon > 0$, denote by $f_\epsilon : \Omega_\epsilon \rightarrow \mathbb{R}$ the ϵ -mollification of f :

$$f_\epsilon(x) = (\eta_\epsilon * f)(x) = \int_{\Omega} \eta_\epsilon(x-y)f(y) dy, \quad x \in \Omega_\epsilon,$$

with $\Omega_\epsilon := \{y \in \Omega : \text{dist}(y, \partial\Omega) > \epsilon\}$. Then $f_\epsilon \in C^\infty(\Omega_\epsilon)$ for any $\epsilon > 0$. Moreover, if $f \in L^p_{\text{loc}}(\Omega)$, with $1 \leq p < \infty$, for any $V \subset\subset \Omega$, we have $\|f_\epsilon\|_{L^p(V)} \leq \|f\|_{L^p(V)}$ and $\lim_{\epsilon \rightarrow 0} \|f - f_\epsilon\|_{L^p(V)} = 0$.

Density of $C_c^\infty(\Omega)$: Using ϵ -mollification, we can show that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, for any $1 \leq p < \infty$.

Fundamental lemma of the calculus of variations: Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in L^1_{\text{loc}}(\Omega)$. If $\int_{\Omega} f\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then $f = 0$ a.e. in Ω .

1.2. Weak derivatives and Sobolev spaces. In Section 3, we presented the notions of distribution and of distributional derivative of a distribution (which always exist). Here we present the notion of weak derivative, where we require some additional integrability.

DEFINITION 4.2 (Weak derivative). Let $u \in L^1_{\text{loc}}(\Omega)$ and $\alpha \in \mathbb{N}^n$ be a multi-index. We say that u has α^{th} weak derivative if its distributional derivative satisfies $D^\alpha u \in L^1_{\text{loc}}(\Omega)$, i.e., there exists $v \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_{\Omega} v\phi dx = (-1)^{|\alpha|} \int_{\Omega} uD^\alpha \phi dx, \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

LEMMA 4.1 (Uniqueness of weak derivative). An α^{th} -weak derivative of $u \in L^1_{\text{loc}}(\Omega)$, if it exists, is uniquely defined up to a zero measure set.

PROOF. Suppose that there exist two α^{th} -weak derivatives $v, \tilde{v} \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_{\Omega} v\phi = (-1)^{|\alpha|} \int_{\Omega} uD^\alpha \phi = \int_{\Omega} \tilde{v}\phi, \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

Then $\int_{\Omega} (v - \tilde{v})\phi = 0$ for all $\phi \in \mathcal{D}(\Omega)$, which implies $v = \tilde{v}$ a.e. in Ω by the fundamental lemma of the calculus of variations. \square

⁵ To show this, it is enough to take a sequence of compact subdomains $\Omega_m \subset\subset \Omega$ such that $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$ and the spaces P_m of polynomials in Ω_m , extended by zero on Ω_m^c , having rational coefficients. Then $P = \bigcup_{m=1}^{\infty} P_m$ is countable and dense in $C_c^0(\Omega)$ hence in $L^p(\Omega)$ as well.

REMARK 4.1. If $u \in C^k(\Omega)$, then the classical α -th derivatives $D^\alpha u$, $|\alpha| \leq k$, coincide with the α -th weak derivatives. Indeed, if we denote by $v_\alpha \in L^1_{loc}(\Omega)$ the α -th weak derivative, we have

$$\int_{\Omega} v_\alpha \phi \, dx = (-1)^{|\alpha|} \int_{\Omega} u D^\alpha \phi \, dx = \int_{\Omega} D^\alpha u \phi \, dx, \quad \text{for all } \phi \in \mathcal{D}(\Omega),$$

which implies $D^\alpha u = v_\alpha$ a.e. in Ω and, since $D^\alpha u$ is continuous, v_α admits a continuous version coinciding with $D^\alpha u$. Hence, the notion of weak-derivative extends that of classical derivative and, for this reason, we use the same symbol $D^\alpha u$ to denote either classical or weak (or distributional) derivative depending on the context.

The next lemma extends another familiar result, stating that the weak derivative of a limit coincides with the limit of the weak derivatives.

LEMMA 4.2 (Convergence of weak derivatives). Consider a sequence of functions $f_n \in L^1_{loc}(\Omega)$. For a fixed multi-index α , assume that each f_n admits the weak derivative $g_n = D^\alpha f_n$. If $f_n \rightarrow f$ and $g_n \rightarrow g$ in $L^1_{loc}(\Omega)$, then $g = D^\alpha f$.

PROOF. For every test function $\phi \in C_c^\infty(\Omega)$, a direct computation yields

$$\begin{aligned} \int_{\Omega} g \phi \, dx &= \lim_{n \rightarrow \infty} \int_{\Omega} g_n \phi \, dx = \lim_{n \rightarrow \infty} (-1)^{|\alpha|} \int_{\Omega} f_n D^\alpha \phi \, dx \\ &= (-1)^{|\alpha|} \int_{\Omega} f D^\alpha \phi \, dx \end{aligned}$$

By definition, this means that g is the α -th weak derivative of f . \square

We are now ready to define *Sobolev spaces*.⁶

DEFINITION 4.3 (Sobolev spaces). For $1 \leq p \leq \infty$, we define

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega), \text{ for all } \alpha \in \mathbb{N}^n, |\alpha| \leq k\}.$$

For $p = 2$, the space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$ and, for $k = 0$, $W^{0,p}(\Omega) = L^p(\Omega)$. Finally, we define

$$W^{k,p}_{loc} := \{f : \Omega \rightarrow \mathbb{R} \text{ s.t. } f \in W^{k,p}(K) \text{ for all } K \subset\subset \Omega\}.$$

On the space $W^{k,p}(\Omega)$, we define the semi-norms⁷

$$\begin{aligned} |f|_{W^{k,p}(\Omega)} &:= \left(\sum_{|\alpha|=k} \int_{\Omega} |D^\alpha f|^p \right)^{1/p}, \quad \text{for } 1 \leq p < \infty, \\ |f|_{W^{k,\infty}(\Omega)} &:= \max_{|\alpha|=k} \sup_{\Omega} D^\alpha f, \end{aligned}$$

and the norms

$$\begin{aligned} \|f\|_{W^{k,p}(\Omega)} &:= \left(\sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha f|^p \right)^{1/p} \\ &\simeq \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)} \simeq \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)}, \quad \text{for } 1 \leq p < \infty, \\ \|f\|_{W^{k,\infty}(\Omega)} &:= \max_{|\alpha| \leq k} \sup_{\Omega} D^\alpha f \simeq \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}. \end{aligned}$$

We recall that two norms $\|\cdot\|$ and $|\cdot|$ on a vector space X are equivalent if there exist constants $c_1, c_2 \in (0, \infty)$ such that

$$\|x\| \leq c_1 |x| \leq c_2 \|x\| \quad \text{for all } x \in X$$

In this case, we often write (as above) $\|\cdot\| \simeq |\cdot|$. Note that the property of a set to be open, closed, compact, or complete in a normed space is not affected if the norm is replaced by an equivalent norm.

We will often use the shorthand notation $\|\cdot\|_{k,p,\Omega}$ or simply $\|\cdot\|_{k,p}$ for $\|\cdot\|_{W^{k,p}(\Omega)}$, if no ambiguity arises.

⁶ Named after Sergei Sobolev [Sob91] (see also the discussion in [Nau02]).

⁷ A seminorm q on a vector space has all the properties of a norm except that $q(f) = 0$ need not imply $f = 0$.

LEMMA 4.3. *The application $\|\cdot\|_{k,p} : W^{k,p}(\Omega) \rightarrow \mathbb{R}_+$ is a norm for any $1 \leq p \leq \infty$.*

PROOF. The proof relies on the fact that $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{\ell^p(\mathbb{R}^m)}$ are norms. Indeed, clearly, $\|\lambda f\|_{k,p} = |\lambda| \|f\|_{k,p}$ for any $\lambda \in \mathbb{R}$; moreover, $\|f\|_{k,p} = 0$ implies, in particular, $\|f\|_{L^p} = 0$, hence $f = 0$ a.e. in Ω . Concerning the triangular inequality, for all $f, g \in W^{k,p}(\Omega)$ and $1 \leq p < \infty$, we have

$$\begin{aligned} \|f + g\|_{k,p} &= \left(\sum_{|\alpha| \leq k} \|D^\alpha f + D^\alpha g\|_{L^p}^p \right)^{1/p} \\ &\leq \left(\sum_{|\alpha| \leq k} (\|D^\alpha f\|_{L^p} + \|D^\alpha g\|_{L^p})^p \right)^{1/p} \\ &\leq \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p}^p \right)^{1/p} + \left(\sum_{|\alpha| \leq k} \|D^\alpha g\|_{L^p}^p \right)^{1/p} \\ &= \|f\|_{k,p} + \|g\|_{k,p}. \end{aligned}$$

Similarly, for $p = \infty$, we have

$$\begin{aligned} \|f + g\|_{k,\infty} &= \max_{|\alpha| \leq k} \|D^\alpha f + D^\alpha g\|_{L^\infty(\Omega)} \\ &\leq \max_{|\alpha| \leq k} (\|D^\alpha f\|_{L^\infty(\Omega)} + \|D^\alpha g\|_{L^\infty(\Omega)}) \leq \|f\|_{k,\infty} + \|g\|_{k,\infty} \end{aligned}$$

Hence $\|\cdot\|_{k,p}$ satisfies all the properties of a norm. \square

LEMMA 4.4. *The normed vector space $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ is a Banach space for every $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. In particular, the space $H^k(\Omega) = W^{k,2}(\Omega)$ is a Hilbert space, for every $k \in \mathbb{N}$, with inner product*

$$(f, g)_{H^k} := \sum_{|\alpha| \leq k} \int_{\Omega} D^\alpha f \cdot D^\alpha g \, dx.$$

PROOF. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $W^{k,p}(\Omega)$, i.e., for all $\epsilon > 0$, there exists $N > 0$ such that

$$\|f_n - f_m\|_{k,p} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f_n - D^\alpha f_m\|_{L^p}^p \right)^{1/p} \leq \epsilon, \quad \text{for all } n, m \geq N.$$

As a consequence, for each $\alpha \in \mathbb{N}^n$, $|\alpha| \leq k$, the sequence $\{D^\alpha f_n\}_{n=1}^\infty$ is Cauchy in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete, there exists $f_\alpha \in L^p$ such that $D^\alpha f_n \rightarrow f_\alpha$ in $L^p(\Omega)$. In particular, for $\alpha = (0, \dots, 0)$, $f_n \xrightarrow{L^p} f_{(0,\dots,0)} =: f$. We claim that $f_\alpha = D^\alpha f$, for all $|\alpha| \leq k$. Indeed,

$$\int_{\Omega} f D^\alpha \phi = \lim_{n \rightarrow \infty} \int_{\Omega} f_n D^\alpha \phi = \lim_{n \rightarrow \infty} (-1)^{|\alpha|} \int_{\Omega} D^\alpha f_n \phi = (-1)^{|\alpha|} \int_{\Omega} f_\alpha \phi, \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

The fact that one can exchange the limit and integration is just a consequence of Hölder inequality since $D^\alpha \phi \in L^q(\Omega)$ with $1/q + 1/p = 1$ and $|\int f D^\alpha \phi - \int f_n D^\alpha \phi| \leq \|f - f_n\|_{L^p} \|D^\alpha \phi\|_{L^q} \xrightarrow{n \rightarrow \infty} 0$. The same argument applies to show that $\lim_{n \rightarrow \infty} \int_{\Omega} D^\alpha f_n \phi = \int_{\Omega} f_\alpha \phi$. Therefore, $D^\alpha f_n \xrightarrow{L^p} D^\alpha f$, for all $|\alpha| \leq k$, implying that $f_n \rightarrow f$ in $W^{k,p}$.

Finally, the bilinear form $(\cdot, \cdot)_{H^k}$ induces the norm $\|\cdot\|_{k,2}$ and it is immediate to verify that it satisfies all the assumptions of an inner product (thanks to the fact that $\int_{\Omega} f \cdot g$ is an inner product in $L^2(\Omega)$). \square

REMARK 4.2. *The inner product in Lemma 4.4 induces the norm*

$$\|f\|_{H^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha f|^2 \right)^{1/2},$$

while the equivalent norm $\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^2(\Omega)}$ is not induced by a scalar product.

LEMMA 4.5 (Properties of weak derivatives). *Let $f, g \in W^{k,p}(\Omega)$ and $\alpha \in \mathbb{N}^n, |\alpha| \leq k$. The following properties of the α^{th} -weak derivatives hold.*

Linearity: *For all $\lambda, \mu \in \mathbb{R}$, we have $\lambda f + \mu g \in W^{k,p}(\Omega)$ and*

$$D^\alpha(\lambda f + \mu g) = \lambda D^\alpha f + \mu D^\alpha g.$$

Commutativity: *For all $\alpha, \beta \in \mathbb{N}^n$ such that $|\alpha| + |\beta| \leq k$, we have*

$$D^\beta(D^\alpha f) = D^\alpha(D^\beta f) = D^{\alpha+\beta} f.$$

Leibniz's⁸ rule: *For all $\xi \in C^\infty(\Omega)$, we have $\xi f \in W_{loc}^{k,p}(\Omega)$ and*

$$D^\alpha(\xi f) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \xi D^{\alpha-\beta} f.$$

LEMMA 4.6. *Let $f \in W^{k,p}(\Omega)$, with $1 \leq p < \infty$, and $f_\epsilon := \eta_\epsilon * f : \Omega \rightarrow \mathbb{R}$. Then $f_\epsilon \xrightarrow{\epsilon \rightarrow 0} f$ in $L^p(\Omega)$ and $f_\epsilon \xrightarrow{\epsilon \rightarrow 0} f$ in $W^{k,p}(K)$, for any $K \subset\subset \Omega$.*

PROOF. We know already that $f_\epsilon \rightarrow f$ in $L^p(\Omega)$ when $\epsilon \rightarrow 0$, so we are left to prove that $D^\alpha f_\epsilon \xrightarrow{\epsilon \rightarrow 0} D^\alpha f$ in $L^p(K)$ for any $\alpha \in \mathbb{N}^n, |\alpha| \leq k$ and $K \subset\subset \Omega$. We first show that $D^\alpha f_\epsilon = \eta_\epsilon * D^\alpha f$ in $\Omega_\epsilon := \{y \in \Omega, \text{dist}(y, \partial\Omega) > \epsilon\}$, where $D^\alpha f_\epsilon$ is a classical derivative since $f_\epsilon \in C^\infty(\Omega_\epsilon)$, whereas $D^\alpha f$ is a weak derivative. We have indeed $\forall x \in \Omega_\epsilon$

$$\begin{aligned} D^\alpha f_\epsilon(x) &= D^\alpha \int_\Omega \eta_\epsilon(x-y) f(y) dy = \int_\Omega D_x^\alpha \eta_\epsilon(x-y) f(y) dy \\ &= (-1)^{|\alpha|} \int_\Omega \underbrace{D_y^\alpha \eta_\epsilon(x-y)}_{\in \mathcal{D}(\Omega)} f(y) dy \\ &= \int_\Omega \eta_\epsilon(x-y) D^\alpha f(y) dy = (\eta_\epsilon * D^\alpha f)(x). \end{aligned}$$

It follows that $D^\alpha f_\epsilon \rightarrow D^\alpha f$ in $L^p(K)$, for all $K \subset\subset \Omega$. Hence,

$$\|f - f_\epsilon\|_{W^{k,p}(K)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f - D^\alpha f_\epsilon\|_{L^p(K)}^p \right)^{\frac{1}{p}} \xrightarrow{\epsilon \rightarrow 0} 0.$$

□

EXAMPLE 4.1. *Consider $\Omega = (0,1) \subset \mathbb{R}$ and the constant function $f = 1$ in Ω , whose (classical/weak) derivative is $f' = 0$ in Ω and clearly $f \in W^{1,p}(\Omega)$ for any $1 \leq p \leq \infty$. Let $f_\epsilon(x) = \int_0^1 \eta_\epsilon(x-y) f(y) dy$, whose derivative is*

$$f'_\epsilon(x) = \int_0^1 \partial_x \eta_\epsilon(x-y) f(y) dy = \int_0^1 -\partial_y \eta_\epsilon(x-y) dy = \eta_\epsilon(x) - \eta_\epsilon(x-1).$$

We notice that, for $\epsilon < \frac{1}{2}$, we have $|\eta_\epsilon(x) - \eta_\epsilon(x-1)| = \eta_\epsilon(x) + \eta_\epsilon(x-1)$ for all $x \in [0,1]$.

We claim that $f'_\epsilon \rightarrow 0$ in $L^p(0,1)$ for any $p \geq 1$. Indeed,

$$\|f'_\epsilon\|_{L^p((0,1))} \geq \|f'_\epsilon\|_{L^1((0,1))} = \int_0^1 \eta_\epsilon(x) dx + \int_0^1 \eta_\epsilon(x-1) dx = 1.$$

In conclusion, $f_\epsilon \rightarrow f$ in $W^{1,p}(0,1)$ for any $p \geq 1$. However, we have $f'_\epsilon \rightarrow 0$ in $L^p(K)$ for any $K \subset\subset (0,1)$.

Lect. 9, 12.11

1.3. Approximation of Sobolev functions. To begin with, we consider Sobolev functions defined on all of \mathbb{R}^n . They may be approximated in the Sobolev norm by test functions.

THEOREM 4.1. *For $k \in \mathbb{N}$ and $1 \leq p < \infty$, the space $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$*

PROOF. We will prove the result in two steps.

Step 1. First, we show that $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

Let $\eta^\epsilon \in C_c^\infty(\mathbb{R}^n)$ be the standard mollifier and $f \in W^{k,p}(\mathbb{R}^n)$. Then $\eta^\epsilon * f \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ and, for $|\alpha| \leq k$,

$$D^\alpha(\eta^\epsilon * f) = \eta^\epsilon * (D^\alpha f) \rightarrow D^\alpha f \quad \text{in } L^p(\mathbb{R}^n) \text{ as } \epsilon \rightarrow 0^+.$$

It follows that $\eta^\epsilon * f \rightarrow f$ in $W^{k,p}(\mathbb{R}^n)$ as $\epsilon \rightarrow 0$.

Step 2. Now we prove that $C_c^\infty(\Omega)$ is dense in $W^{k,p}(\mathbb{R}^n)$. Let us suppose that $f \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$, and let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that

$$\phi(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 2. \end{cases}$$

Define $\phi^R(x) = \phi(x/R)$ and $f^R = \phi^R f \in C_c^\infty(\mathbb{R}^n)$. Then, by the Leibnitz rule,

$$D^\alpha f^R = \phi^R D^\alpha f + \frac{1}{R} h^R,$$

where h^R is a function that is bounded in L^p uniformly in R . Hence, by Lebesgue's dominated convergence theorem

$$D^\alpha f^R \rightarrow D^\alpha f \quad \text{in } L^p \text{ as } R \rightarrow \infty,$$

so $f^R \rightarrow f$ in $W^{k,p}(\mathbb{R}^n)$ as $R \rightarrow \infty$. This concludes the proof. \square

If Ω is a proper open subset of \mathbb{R}^n , then $C_c^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$. Instead, its closure is the space of functions $W_0^{k,p}(\Omega)$ that “vanish on the boundary $\partial\Omega$ ”. We discuss this further in [Section 1.5](#). The space $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$ for any open set Ω (cf. [MS64]), so that $W^{k,p}(\Omega)$ may alternatively be defined as the completion of the space of smooth functions in Ω whose derivatives of order less than or equal to k belong to $L^p(\Omega)$. Such functions need not extend to continuous functions on $\bar{\Omega}$ or be bounded on Ω .

1.4. Embedding theorems. Can we estimate the $L^q(\mathbb{R}^n)$ norm of a smooth, compactly supported function in terms of the $L^p(\mathbb{R}^n)$ -norm of its derivative?

1.4.1. $1 \leq p < n$. We will show that, given $1 \leq p < n$, this is possible for a unique value of q , called the Sobolev conjugate of p . We are looking for an estimate of the form

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C \|Df\|_{L^p(\mathbb{R}^n)} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n), \quad (1.1)$$

for some constant $C = C(p, q, n)$. For $\lambda > 0$, let f_λ denote the rescaled function

$$f_\lambda(x) := f\left(\frac{x}{\lambda}\right).$$

Then, changing variables $x \mapsto \lambda x$ in the integrals that define the L^p, L^q norms, with $1 \leq p, q < \infty$, and using the fact that

$$Df_\lambda = \frac{1}{\lambda}(Df)_\lambda$$

we find that

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |Df_\lambda|^p dx \right)^{1/p} &= \lambda^{n/p-1} \left(\int_{\mathbb{R}^n} |Df|^p dx \right)^{1/p}, \\ \left(\int_{\mathbb{R}^n} |f_\lambda|^q dx \right)^{1/q} &= \lambda^{n/q} \left(\int_{\mathbb{R}^n} |f|^q dx \right)^{1/q}. \end{aligned}$$

These norms must scale according to the same exponent if we are to have an inequality of the desired form, otherwise we can violate the inequality by taking $\lambda \rightarrow 0$ or $\lambda \rightarrow \infty$. The equality of exponents implies that $q = p^*$ where p^* satisfies

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Note that we need $1 \leq p < n$ to ensure that $p^* > 0$, in which case $p < p^* < \infty$. We assume that $n \geq 2$. From this, we motivate the following definition.

DEFINITION 4.4 (Sobolev conjugate exponents). *If $1 \leq p < n$, then the Sobolev conjugate p^* of p is*

$$p^* = \frac{np}{n-p}$$

Thus, an estimate of the form (1.1) is possible only if $q = p^*$; we will show that (1.1) is, in fact, true when $q = p^*$.

This result was obtained by Sobolev [Sob38] for $1 < p < n$ and by Emilio Gagliardo and Luis Nirenberg independently up to the endpoint $p = 1$ (see [Nir59; Gag58])

THEOREM 4.2 (Gagliardo–Nirenberg–Sobolev’s inequality). *Let $1 \leq p < n$, where $n \geq 2$, and let p^* be the Sobolev conjugate of p given in Definition 4.4. Then*

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Df\|_{L^p(\mathbb{R}^n)}, \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n),$$

where

$$C(n, p) = \frac{p}{2n} \left(\frac{n-1}{n-p} \right). \quad (1.2)$$

REMARK 4.3 (Optimal Sobolev constant). *The constant stated in Theorem 4.2 is not optimal. Instead, for $p = 1$, the best constant is*

$$C(n, 1) = \frac{1}{n\alpha_n^{1/n}},$$

where α_n is the volume of the unit ball, or

$$C(n, 1) = \frac{1}{n\sqrt{\pi}} \left[\Gamma\left(1 + \frac{n}{2}\right) \right]^{1/n}$$

where Γ is the Euler Gamma function. Equality is obtained in the limit of functions that approach the characteristic function of a ball. This result for the best Sobolev constant is equivalent to the isoperimetric inequality that a sphere has minimal area among all surfaces enclosing a given volume.

For $1 < p < n$, the best constant is

$$C(n, p) = \frac{1}{n^{1/p}\sqrt{\pi}} \left(\frac{p-1}{n-p} \right)^{1-1/p} \left[\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)} \right]^{1/n}$$

and equality holds for functions of the form

$$f(x) = \left(a + b|x|^{p/(p-1)} \right)^{1-n/p},$$

where a, b are positive constants, which are called Aubin–Talenti bubbles.⁹

EXAMPLE 4.2. *The Sobolev inequality in Theorem 4.2 does not hold in the limiting case $p \rightarrow n, p^* \rightarrow \infty$. If $\phi(x)$ is a smooth cut-off function that is equal to one for $|x| \leq 1$ and zero for $|x| \geq 2$, and*

$$f(x) = \phi(x) \log \log \left(1 + \frac{1}{|x|} \right),$$

then $Df \in L^n(\mathbb{R}^n)$, and $f \in W^{1,n}(\mathbb{R})$, but $f \notin L^\infty(\mathbb{R}^n)$.

Before describing the proof, we introduce some notation, explain the main idea, and establish a preliminary inequality.

For $1 \leq i \leq n$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, let

$$x'_i = (x_1, \dots, \hat{x}_i, \dots, x_n) \in \mathbb{R}^{n-1}$$

where the ‘hat’ means that the i -th coordinate is omitted. We write $x = (x_i, x'_i)$ and denote the value of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at x by

$$f(x) = f(x_i, x'_i).$$

If f is smooth with compact support, then the fundamental theorem of calculus implies that

$$f(x) = \int_{-\infty}^{x_i} \partial_{x_i} f(t, x'_i) dt$$

⁹ Named after Thierry Aubin [Aub76] and Giorgio Talenti [Tal76].

Taking absolute values, we get

$$|f(x)| \leq \int_{-\infty}^{\infty} |\partial_{x_i} f(t, x'_i)| dt.$$

We can improve the constant in this estimate by using the fact that

$$\int_{-\infty}^{\infty} \partial_{x_i} f(t, x'_i) dt = 0.$$

LEMMA 4.7. *Let us suppose that $g : \mathbb{R} \rightarrow \mathbb{R}$ is an integrable function with compact support such that $\int g dt = 0$. If*

$$f(x) = \int_{-\infty}^x g(t) dt,$$

then

$$|f(x)| \leq \frac{1}{2} \int |g(t)| dt.$$

PROOF. Let $g = g_+ - g_-$ where the nonnegative functions g_+, g_- are defined by $g_+ = \max(g, 0)$, $g_- = \max(-g, 0)$. Then $|g| = g_+ + g_-$ and

$$\int g_+ dt = \int g_- dt = \frac{1}{2} \int |g| dt.$$

It follows that

$$\begin{aligned} f(x) &\leq \int_{-\infty}^x g_+(t) dt \leq \int_{-\infty}^{\infty} g_+(t) dt = \frac{1}{2} \int |g| dt \\ f(x) &\geq - \int_{-\infty}^x g_-(t) dt \geq - \int_{-\infty}^{\infty} g_-(t) dt = -\frac{1}{2} \int |g| dt, \end{aligned}$$

which proves the result. \square

Thus, for $1 \leq i \leq n$ we have

$$|f(x)| \leq \frac{1}{2} \int_{-\infty}^{\infty} |\partial_{x_i} f(t, x'_i)| dt.$$

The idea of the proof is to average a suitable power of this inequality over the i -directions and integrate the result to estimate f in terms of Df . In order to do this, we use the following inequality, which estimates the L^1 -norm of a function of $x \in \mathbb{R}^n$ in terms of the L^{n-1} -norms of n functions of $x'_i \in \mathbb{R}^{n-1}$ whose product bounds the original function pointwise.

Thus, for $1 \leq i \leq n$ we have

$$|f(x)| \leq \frac{1}{2} \int_{-\infty}^{\infty} |\partial_{x_i} f(t, x'_i)| dt.$$

The idea of the proof is to average a suitable power of this inequality over the i -directions and integrate the result to estimate f in terms of Df . In order to do this, we use the following inequality, which estimates the L^1 -norm of a function of $x \in \mathbb{R}^n$ in terms of the L^{n-1} -norms of n functions of $x'_i \in \mathbb{R}^{n-1}$ whose product bounds the original function pointwise.

LEMMA 4.8 (Gagliardo's product inequality). *Let us suppose that $n \geq 2$ and*

$$\{g_i \in C_c^\infty(\mathbb{R}^{n-1}) : 1 \leq i \leq n\}$$

are non-negative functions. If we define $g \in C_c^\infty(\mathbb{R}^n)$ by

$$g(x) := \prod_{i=1}^n g_i(x'_i),$$

then

$$\int g dx \leq \prod_{i=1}^n \|g_i\|_{n-1} \tag{1.3}$$

REMARK 4.4. If $n = 2$, [Lemma 4.8](#) states that

$$\int g_1(x_2) g_2(x_1) dx_1 dx_2 \leq \left(\int g_1(x_2) dx_2 \right) \left(\int g_2(x_1) dx_1 \right),$$

which follows immediately from Fubini's theorem.

If $n = 3$, [Lemma 4.8](#) states that

$$\begin{aligned} & \int g_1(x_2, x_3) g_2(x_1, x_3) g_3(x_1, x_2) dx_1 dx_2 dx_3 \\ & \leq \left(\int g_1^2(x_2, x_3) dx_2 dx_3 \right)^{1/2} \left(\int g_2^2(x_1, x_3) dx_1 dx_3 \right)^{1/2} \left(\int g_3^2(x_1, x_2) dx_1 dx_2 \right)^{1/2}. \end{aligned}$$

To prove the inequality in this case, we fix x_1 and apply Cauchy–Schwarz' inequality to the x_2x_3 -integral of $g_1 \cdot g_2g_3$. We then use the inequality for $n = 2$ to estimate the x_2x_3 -integral of g_2g_3 , and integrate the result over x_1 . An analogous approach works for higher n .

REMARK 4.5. Note that under the scaling $g_i \mapsto \lambda g_i$, both sides of [\(1.3\)](#) scale in the same way,

$$\int g dx \mapsto \left(\prod_{i=1}^n \lambda_i \right) \int g dx, \quad \prod_{i=1}^n \|g_i\|_{n-1} \mapsto \left(\prod_{i=1}^n \lambda_i \right) \prod_{i=1}^n \|g_i\|_{n-1}$$

as must be true for any inequality involving norms. Also, under the spatial rescaling $x \mapsto \lambda x$, we have

$$\int g dx \mapsto \lambda^{-n} \int g dx$$

while $\|g_i\|_p \mapsto \lambda^{-(n-1)/p} \|g_i\|_p$, so

$$\prod_{i=1}^n \|g_i\|_p \mapsto \lambda^{-n(n-1)/p} \prod_{i=1}^n \|g_i\|_p$$

Thus, if $p = n - 1$ the two terms scale in the same way, which explains the appearance of the L^{n-1} -norms of the g_i 's on the right hand side of [\(1.3\)](#).

PROOF. We argue by induction. We have the claim for $n = 2$ owing to [Remark 4.5](#). Supposing that it is true for $n - 1$ where $n \geq 3$, we want to prove it for n .

For $1 \leq i \leq n$, let $g_i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be the functions given in the theorem. Fix $x_1 \in \mathbb{R}$ and define $g_{x_1} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ by

$$g_{x_1}(x'_1) = g(x_1, x'_1)$$

For $2 \leq i \leq n$, let $x'_i = (x_1, x'_{1,i})$ where

$$x'_{1,i} = (\hat{x}_1, \dots, \hat{x}_i, \dots, x_n) \in \mathbb{R}^{n-2}$$

Define $g_{i,x_1} : \mathbb{R}^{n-2} \rightarrow \mathbb{R}$ and $\tilde{g}_{i,x_1} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ by

$$g_{i,x_1}(x'_{1,i}) = g_i(x_1, x'_{1,i}).$$

Then

$$g_{x_1}(x'_1) = g_1(x'_1) \prod_{i=2}^n g_{i,x_1}(x'_{1,i}).$$

Using Hölder's inequality with $q = n - 1$ and $q' = (n - 1)/(n - 2)$, we get

$$\begin{aligned} \int g_{x_1} dx'_1 &= \int g_1 \left(\prod_{i=2}^n g_{i,x_1}(x'_{1,i}) \right) dx'_1 \\ &\leq \|g_1\|_{n-1} \left[\int \left(\prod_{i=2}^n g_{i,x_1}(x'_{1,i}) \right)^{(n-1)/(n-2)} dx'_1 \right]^{(n-2)/(n-1)} \end{aligned}$$

The induction hypothesis implies that

$$\begin{aligned} \int \left(\prod_{i=2}^n g_{i,x_1}(x'_{1,i}) \right)^{(n-1)/(n-2)} dx'_1 &\leq \prod_{i=2}^n \|g_{i,x_1}^{(n-1)/(n-2)}\|_{n-2} \\ &\leq \prod_{i=2}^n \|g_{i,x_1}\|_{n-1}^{(n-1)/(n-2)} \end{aligned}$$

Hence,

$$\int g_{x_1} dx'_1 \leq \|g_1\|_{n-1} \prod_{i=2}^n \|g_{i,x_1}\|_{n-1}$$

Integrating this equation over x_1 and using the generalized Hölder inequality with $p_2 = p_3 = \dots = p_n = n-1$, we get

$$\begin{aligned} \int g dx &\leq \|g_1\|_{n-1} \int \left(\prod_{i=2}^n \|g_{i,x_1}\|_{n-1} \right) dx_1 \\ &\leq \|g_1\|_{n-1} \left(\prod_{i=2}^n \int \|g_{i,x_1}\|_{n-1}^{n-1} dx_1 \right)^{1/(n-1)} \end{aligned}$$

Thus, since

$$\begin{aligned} \int \|g_{i,x_1}\|_{n-1}^{n-1} dx_1 &= \int \left(\int |g_{i,x_1}(x'_{1,i})|^{n-1} dx'_{1,i} \right) dx_1 \\ &= \int |g_i(x'_i)|^{n-1} dx'_i \\ &= \|g_i\|_{n-1}^{n-1} \end{aligned}$$

we find that

$$\int g dx \leq \prod_{i=1}^n \|g_i\|_{n-1}$$

□

We are now ready to prove [Theorem 4.2](#).

PROOF OF THEOREM 4.2. First, we prove the result for $p = 1$ and then for $1 < p < n$.

Case 1: $p = 1$. For $1 \leq i \leq n$, we have, by [Lemma 4.7](#),

$$|f(x)| \leq \frac{1}{2} \int |\partial_{x_i} f(t, x'_i)| dt$$

Multiplying these inequalities and taking the $(n-1)$ -th root, we get

$$|f|^{n/(n-1)} \leq \frac{1}{2^{n/(n-1)}} g, \quad \text{with } g := \prod_{i=1}^n \tilde{g}_i,$$

where $\tilde{g}_i(x) = g_i(x'_i)$ with

$$g_i(x'_i) = \left(\int |\partial_{x_i} f(t, x'_i)| dt \right)^{1/(n-1)}.$$

[Lemma 4.8](#) implies that

$$\int g dx \leq \prod_{i=1}^n \|g_i\|_{n-1}$$

So, since

$$\|g_i\|_{n-1} = \left(\int |\partial_{x_i} f| dx \right)^{1/(n-1)},$$

it follows that

$$\int |f|^{n/(n-1)} dx \leq \frac{1}{2^{n/(n-1)}} \left(\prod_{i=1}^n \int |\partial_{x_i} f| dx \right)^{1/(n-1)}.$$

Note that $n/(n-1) = 1^*$ is the Sobolev conjugate of 1.

Using the arithmetic–geometric mean inequality,

$$\left(\prod_{i=1}^n a_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^n a_i,$$

we get

$$\int |f|^{n/(n-1)} dx \leq \left(\frac{1}{2n} \sum_{i=1}^n \int |\partial_{x_i} f| dx \right)^{n/(n-1)},$$

i.e.,

$$\|f\|_{1^*} \leq \frac{1}{2n} \|Df\|_1.$$

Case 2: $1 < p < n$ For any $s > 1$, we have

$$\frac{d}{dx} |x|^s = s \operatorname{sign} x |x|^{s-1}.$$

Thus,

$$\begin{aligned} |f(x)|^s &= \int_{-\infty}^{x_i} \partial_{x_i} |f(t, x'_i)|^s dt \\ &= s \int_{-\infty}^{x_i} |f(t, x'_i)|^{s-1} \operatorname{sgn} [f(t, x'_i)] \partial_{x_i} f(t, x'_i) dt \end{aligned}$$

Using [Lemma 4.7](#), it follows that

$$|f(x)|^s \leq \frac{s}{2} \int_{-\infty}^{\infty} |f^{s-1}(t, x'_i) \partial_{x_i} f(t, x'_i)| dt$$

and multiplication of these inequalities gives

$$|f(x)|^{sn} \leq \left(\frac{s}{2} \right)^n \prod_{i=1}^n \int_{-\infty}^{\infty} |f^{s-1}(t, x'_i) \partial_{x_i} f(t, x'_i)| dt.$$

Applying [Lemma 4.8](#) with the functions

$$g_i(x'_i) = \left[\int_{-\infty}^{\infty} |f^{s-1}(t, x'_i) \partial_{x_i} f(t, x'_i)| dt \right]^{1/(n-1)},$$

we obtain

$$\|f\|_{sn/(n-1)}^{sn} \leq \frac{s}{2} \prod_{i=1}^n \|f^{s-1} \partial_{x_i} f\|_1.$$

By Hölder's inequality, we have

$$\|f^{s-1} \partial_{x_i} f\|_1 \leq \|f^{s-1}\|_{p'} \|\partial_{x_i} f\|_p$$

On the other hand,

$$\|f^{s-1}\|_{p'} = \|f\|_{p'(s-1)}^{s-1}$$

Choosing $s > 1$ so that

$$s = p \left(\frac{n-1}{n-p} \right) \iff \frac{sn}{n-1} = p^*,$$

then

$$\|f\|_{p^*} \leq \frac{s}{2} \left(\prod_{i=1}^n \|\partial_{x_i} f\|_p \right)^{1/n}.$$

Using the arithmetic–geometric mean inequality, we get

$$\|f\|_{p^*} \leq \frac{s}{2n} \left(\sum_{i=1}^n \|\partial_{x_i} f\|_p^p \right)^{1/p},$$

which proves the result. \square

We can use the Sobolev inequality to prove various embedding theorems.

DEFINITION 4.5. We say that a Banach space X is continuously embedded (or embedded for short) in a Banach space Y if there is a one-to-one, bounded linear map $\iota : X \rightarrow Y$.

We often think of ι as identifying elements of the smaller space X with elements of the larger space Y ; if X is a subset of Y , then ι is the inclusion map. The boundedness of ι means that there is a constant C such that $\|\iota x\|_Y \leq C\|x\|_X$ for all $x \in X$, so the weaker Y -norm of ιx is controlled by the stronger X -norm of x . We write an embedding as $X \hookrightarrow Y$ or as $X \subset Y$ when the boundedness is understood.

THEOREM 4.2 does not, of course, imply that $f \in L^p(\mathbb{R}^n)$ whenever $Df \in L^p(\mathbb{R}^n)$, since constant functions have zero derivative.

To ensure that $f \in L^{p^*}(\mathbb{R}^n)$, we also need to impose a decay condition on f that eliminates the constant functions. In the following theorem, this is provided by the assumption that $f \in L^p(\mathbb{R}^n)$

THEOREM 4.3. Suppose that $1 \leq p < n$ and $p \leq q \leq p^*$ where p^* is the Sobolev conjugate of p . Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ and $\|f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{W^{1,p}(\mathbb{R}^n)}$ for all $f \in W^{1,p}(\mathbb{R}^n)$ for some constant $C = C(n, p, q)$.

REMARK 4.6. Let us present an heuristic argument for the lower bound $q \geq p$. Let us consider the function $f(x) = (1 + |x|)^{-\alpha}$, with $\alpha > 0$. Then $f \in L^r(\mathbb{R}^n)$ if $\alpha r > n$. So, if $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, then $\alpha p > n$ should imply $\alpha q > n$ for every $\alpha > 0$, but this yields $p \leq q$ (otherwise there would exist $\alpha > 0$ such that $q < n/\alpha < p$).

PROOF. Step 1. If $f \in W^{1,p}(\mathbb{R}^n)$, then, by THEOREM 4.1, there is a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^n)$ that converges to f in $W^{1,p}(\mathbb{R}^n)$. THEOREM 4.2 implies that $f_n \rightarrow f$ in $L^{p^*}(\mathbb{R}^n)$. In detail: $\{Df_n\}$ converges to Df in L^p so it is Cauchy in L^p ; since

$$\|f_n - f_m\|_{p^*} \leq C\|Df_n - Df_m\|_p,$$

then $\{f_n\}$ is Cauchy in L^{p^*} ; therefore $f_n \rightarrow \tilde{f}$ for some $\tilde{f} \in L^{p^*}$ since L^{p^*} is complete; and \tilde{f} is equivalent to f since a subsequence of $\{f_n\}$ converges pointwise a.e. to \tilde{f} , because of the L^{p^*} convergence, and to f , because of the L^p -convergence. Thus, $f \in L^{p^*}(\mathbb{R}^n)$ and

$$\|f\|_{p^*} \leq C\|Df\|_p$$

Step 2. Since $f \in L^p(\mathbb{R}^n)$, using the interpolation between L^p spaces, we have, for $p < q < p^*$,

$$\|f\|_q \leq \|f\|_p^\theta \|f\|_{p^*}^{1-\theta}$$

where $0 < \theta < 1$ is defined by

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}.$$

Therefore, using THEOREM 4.2 and the inequality

$$a^\theta b^{1-\theta} \leq [\theta^\theta (1-\theta)^{1-\theta}]^{1/p} (a^p + b^p)^{1/p},$$

we get

$$\begin{aligned} \|f\|_q &\leq C^{1-\theta} \|f\|_p^\theta \|Df\|_p^{1-\theta} \\ &\leq C^{1-\theta} [\theta^\theta (1-\theta)^{1-\theta}]^{1/p} (\|f\|_p^p + \|Df\|_p^p)^{1/p} \\ &\leq C^{1-\theta} [\theta^\theta (1-\theta)^{1-\theta}]^{1/p} \|f\|_{W^{1,p}}. \end{aligned}$$

□

Instead of the assumption that $f \in L^p(\mathbb{R}^n)$, we can impose the following weaker decay condition.

DEFINITION 4.6. A Lebesgue measurable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ vanishes at infinity if for every $\epsilon > 0$ the set $\{x \in \mathbb{R}^n : |f(x)| > \epsilon\}$ has finite Lebesgue measure.

The Sobolev embedding theorem remains true for functions that vanish at infinity.

THEOREM 4.4. Suppose that $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ is weakly differentiable with $Df \in L^p(\mathbb{R}^n)$ where $1 \leq p < n$ and f vanishes at infinity. Then $f \in L^{p^*}(\mathbb{R}^n)$ and

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Df\|_{L^p(\mathbb{R}^n)}$$

where C is given in (1.2).

REMARK 4.7. If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then f vanishes at infinity. Note that this does not imply that $\lim_{|x| \rightarrow \infty} f(x) = 0$.

EXAMPLE 4.3. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f = \sum_{n \in \mathbb{N}} \chi_{I_n}, \quad I_n = \left[n, n + \frac{1}{n^2} \right]$$

where χ_I is the characteristic function of the interval I . Then

$$\int_{\mathbb{R}} f \, dx = \sum_{n \in \mathbb{N}} \frac{1}{n^2} < \infty$$

so $f \in L^1(\mathbb{R})$. The limit of $f(x)$ as $|x| \rightarrow \infty$ does not exist since $f(x)$ takes on the values 0 and 1 for arbitrarily large values of x . Nevertheless, f vanishes at infinity since for any $\epsilon < 1$,

$$|\{x \in \mathbb{R} : |f(x)| > \epsilon\}| = \sum_{n \in \mathbb{N}} \frac{1}{n^2},$$

which is finite.

EXAMPLE 4.4. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/\log x, & \text{if } x \geq 2, \\ 0, & \text{if } x < 2, \end{cases}$$

vanishes at infinity, but $f \notin L^p(\mathbb{R})$ for any $1 \leq p < \infty$.

1.4.2. $p > n$. If the weak derivative of a function that vanishes at infinity belongs to $L^p(\mathbb{R}^n)$ with $p < n$, then the function has improved integrability properties and belongs to $L^{p^*}(\mathbb{R}^n)$. Even though the function is weakly differentiable, it need not be continuous. In this section, we show that if the derivative belongs to $L^p(\mathbb{R}^n)$ with $p > n$ then the function (or a pointwise a.e. equivalent version of it) is continuous, and in fact Hölder continuous.

The following result is due to Charles Bradford Morrey jun. (see [Mor40; Mor66]). The main idea is to estimate the difference $|f(x) - f(y)|$ in terms of Df by the mean value theorem, average the result over a ball $B_r(x)$ and estimate the result in terms of $\|Df\|_p$ by Hölder's inequality.

THEOREM 4.5 (Morrey's inequality). Let $n < p < \infty$ and

$$\alpha = 1 - \frac{n}{p},$$

with $\alpha = 1$ if $p = \infty$. Then there are constants $C = C(n, p)$ such that

$$[f]_{\alpha} \leq C \|Df\|_p \quad \text{for all } f \in C_c^{\infty}(\mathbb{R}^n) \quad (1.4)$$

$$\sup_{\mathbb{R}^n} |f| \leq C \|f\|_{W^{1,p}} \quad \text{for all } f \in C_c^{\infty}(\mathbb{R}^n), \quad (1.5)$$

where $[\cdot]_{\alpha}$ denotes the Hölder seminorm $[\cdot]_{\alpha, \mathbb{R}^n}$, i.e.,

$$|f|_{\alpha, \mathbb{R}^n} := \sup_{\substack{x, y \in \mathbb{R}^n, \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < \infty.$$

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PROOF. Step 1. First we prove that there exists a constant C , depending only on n , such that, for any ball $\overline{B_r(x)}$,

$$\int_{B_r(x)} |f(x) - f(y)| \, dy \leq C \int_{B_r(x)} \frac{|Df(y)|}{|x - y|^{n-1}} \, dy. \quad (1.6)$$

Let $w \in \partial B_1(0)$ be a unit vector. For $s > 0$,

$$f(x + sw) - f(x) = \int_0^s \frac{d}{dt} f(x + tw) \, dt = \int_0^s Df(x + tw) \cdot w \, dt,$$

and, therefore, since $|w| = 1$,

$$|f(x + sw) - f(x)| \leq \int_0^s |Df(x + tw)| \, dt.$$

Integrating this inequality with respect to w over the unit sphere, we get

$$\int_{\partial B_1(0)} |f(x) - f(x + sw)| \, dS(w) \leq \int_{\partial B_1(0)} \left(\int_0^s |Df(x + tw)| \, dt \right) \, dS(w).$$

Changing the order of integration yields

$$\begin{aligned} \int_{\partial B_1(0)} \left(\int_0^s |Df(x + tw)| \, dt \right) \, dS(w) &= \int_{\partial B_1(0)} \int_0^s \frac{|Df(x + tw)|}{t^{n-1}} t^{n-1} \, dt \, dS(w) \\ &= \int_{B_x(x)} \frac{|Df(y)|}{|x - y|^{n-1}} \, dy. \end{aligned}$$

Thus,

$$\int_{\partial B_1(0)} |f(x) - f(x + sw)| \, dS(w) \leq \int_{B_x(x)} \frac{|Df(y)|}{|x - y|^{n-1}} \, dy.$$

Using this inequality, we can also estimate

$$\begin{aligned} \int_{B_r(x)} |f(x) - f(y)| \, dy &= \int_0^r \left(\int_{\partial B_s(0)} |f(x) - f(x + sw)| \, dS(w) \right) s^{n-1} \, ds \\ &\leq \int_0^r \left(\int_{B_s(x)} \frac{|Df(y)|}{|x - y|^{n-1}} \, dy \right) s^{n-1} \, ds \\ &\leq \left(\int_0^r s^{n-1} \, ds \right) \left(\int_{B_r(x)} \frac{|Df(y)|}{|x - y|^{n-1}} \, dy \right) \\ &\leq \frac{r^n}{n} \int_{B_r(x)} \frac{|Df(y)|}{|x - y|^{n-1}} \, dy, \end{aligned}$$

where we estimated the integral over $B_s(x)$ by the integral over $B_r(x)$ for $s \leq r$. This gives (1.6) with $C = (n\alpha_n)^{-1}$.

Step 2. To prove (1.4), let us suppose that $x, y \in \mathbb{R}^n$. Let $r = |x - y|$ and $\Omega = B_r(x) \cap B_r(y)$. Then averaging the inequality

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(y) - f(z)|$$

with respect to z over Ω , we get

$$|f(x) - f(y)| \leq \int_{\Omega} |f(x) - f(z)| \, dz + \int_{\Omega} |f(y) - f(z)| \, dz. \quad (1.7)$$

From (1.6) and Hölder's inequality,

$$\begin{aligned} \int_{\Omega} |f(x) - f(z)| \, dz &\leq \int_{B_r(x)} |f(x) - f(z)| \, dz \\ &\leq C \int_{B_r(x)} \frac{|Df(y)|}{|x - y|^{n-1}} \, dy \\ &\leq C \left(\int_{B_r(x)} |Df|^p \, dz \right)^{1/p} \left(\int_{B_r(x)} \frac{dz}{|x - z|^{p'(n-1)}} \right)^{1/p'} \end{aligned}$$

We have

$$\left(\int_{B_r(x)} \frac{dz}{|x - z|^{p'(n-1)}} \right)^{1/p'} = C \left(\int_0^r \frac{r^{n-1} dr}{r^{p'(n-1)}} \right)^{1/p'} = Cr^{1-n/p},$$

where C denotes a generic constant depending on n and p . Thus,

$$\int_{\Omega} |f(x) - f(z)| \, dz \leq Cr^{1-n/p} \|Df\|_{L^p(\mathbb{R}^n)}$$

with a similar estimate for the integral in which x is replaced by y . Using these estimates in (1.7) and setting $r = |x - y|$, we get

$$|f(x) - f(y)| \leq C|x - y|^{1-n/p} \|Df\|_{L^p(\mathbb{R}^n)}, \quad (1.8)$$

which proves (1.4)

Step 3. Finally, we prove (1.5). For any $x \in \mathbb{R}^n$, using (1.8), we find that

$$\begin{aligned} |f(x)| &\leq \int_{B_1(x)} |f(x) - f(y)| \, dy + \int_{B_1(x)} |f(y)| \, dy \\ &\leq C \|Df\|_{L^p(\mathbb{R}^n)} + C \|f\|_{L^p(B_1(x))} \\ &\leq C \|f\|_{W^{1,p}(\mathbb{R}^n)} \end{aligned}$$

and taking the supremum with respect to x , we get (1.5). \square

Combining these estimates for

$$\|f\|_{C^{0,\alpha}} = \sup |f| + [f]_\alpha$$

and using a density argument, we can prove the following theorem. We denote by $C_0^{0,\alpha}(\mathbb{R}^n)$ the space of Hölder continuous functions f whose limit as $x \rightarrow \infty$ is zero, meaning that for every $\epsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that $|f(x)| < \epsilon$ if $x \in \mathbb{R}^n \setminus K$.

THEOREM 4.6. *Let $n < p < \infty$ and $\alpha = 1 - n/p$. Then*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow C_0^{0,\alpha}(\mathbb{R}^n)$$

and there exists a constant $C = C(n, p)$ such that

$$\|f\|_{C^{0,\alpha}} \leq C \|f\|_{W^{1,p}} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n)$$

For $p = \infty$, we have that $f \in W^{1,\infty}(\mathbb{R}^n)$ is globally Lipschitz continuous, with

$$[f]_1 \leq C \|Df\|_{L^\infty(\mathbb{R}^n)}$$

A function in $W^{1,\infty}(\mathbb{R}^n)$ need not approach zero at infinity. We have in this case the following characterization of Lipschitz functions.

THEOREM 4.7. *A function $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ is globally Lipschitz continuous if and only if it is weakly differentiable and $Df \in L^\infty(\mathbb{R}^n)$.*

When $n < p \leq \infty$, the above estimates can be used to prove that the pointwise derivative of a Sobolev function exists almost everywhere and agrees with the weak derivative.

THEOREM 4.8. *If $f \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ for some $n < p \leq \infty$, then f is differentiable pointwise a.e. and the pointwise derivative coincides with the weak derivative.*

1.4.3. General embedding theorem. More generally, we state the following result.

THEOREM 4.9. *Let $m \in \mathbb{N}^*$, $1 \leq p \leq \infty$. We have:*

- If $kp < n$: $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, for $p \leq q \leq \frac{np}{n-kp}$;
- If $kp = n$: $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, for $p \leq q < \infty$;
- If $kp > n$: $W^{k,p}(\mathbb{R}^n) \hookrightarrow C^{m,\alpha}(\mathbb{R}^n)$, with $\begin{cases} m = \left\lfloor k - \frac{n}{p} \right\rfloor, & \alpha = \left\{ k - \frac{n}{p} \right\}, & \text{if } \frac{n}{p} \notin \mathbb{N}, \\ m = k - \frac{n}{p} - 1, & \alpha = 1, & \text{if } \frac{n}{p} \in \mathbb{N}, \end{cases}$
where $\lfloor \cdot \rfloor$ denotes the integer part function and $\{ \cdot \}$ the fractional part function.

EXAMPLE 4.5. *There exists a function $u \in H^1(\mathbb{R}^2)$ but $u \notin L^\infty(\mathbb{R}^2)$. Let $\psi \in \mathcal{D}(\mathbb{R}^2)$ such that $0 \leq \psi \leq 1$ and*

$$\psi(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2}, \\ 0 & \text{if } |x| > \frac{3}{4}. \end{cases}$$

The desired counterexample is given by

$$u(x) = |\ln |x||^\alpha \psi(x), \quad 0 < \alpha < \frac{1}{2}.$$

Since there is a logarithmic pole at $x = 0$, the function is not in $L^\infty(\mathbb{R}^2)$. We show that $u \in H^1(\mathbb{R}^2)$. It follows that $u \in L^2(\mathbb{R}^2)$, since

$$\int_{\mathbb{R}^2} |u(x)|^2 \, dx \leq \int_{|x| \leq \frac{3}{4}} |\ln |x||^{2\alpha} \, dx = 2\pi \int_0^{\frac{3}{4}} |\ln \rho|^{2\alpha} \rho \, d\rho.$$

There are no issues with the integrability of the given function. Observe that u is C^∞ for $x \neq 0$. Therefore, for $x \neq 0$, the function admits classical derivatives. We show that these derivatives belong to L^2 .

$$\begin{aligned} \int_{\mathbb{R}^2} |D_j u(x)|^2 dx &\leq \int_{\mathbb{R}^2} \left(\alpha |\ln|x||^{\alpha-2} \ln|x| \frac{1}{|x|} \frac{x_j}{|x|} \psi + |\ln|x||^\alpha \partial_{x_j} \psi(x) \right)^2 dx \\ &\leq 2\alpha^2 \int_{|x| \leq \frac{3}{4}} |\ln|x||^{2(\alpha-1)} \frac{1}{|x|^2} \frac{x_j^2}{|x|^2} dx + 2\|\nabla\psi\|_\infty^2 \int_{\frac{1}{2} < |x| < \frac{3}{4}} |\ln|x||^{2\alpha} dx. \end{aligned}$$

The second integral is evidently finite. For the first integral, we switch to polar coordinates:

$$\int_{|x| \leq \frac{3}{4}} \frac{|\ln|x||^{2(\alpha-1)}}{|x|^2} dx = 2\pi \int_0^{\frac{3}{4}} \frac{|\ln\rho|^{2(\alpha-1)}}{\rho} d\rho = 2\pi \left[\frac{(\ln\rho)^{2\alpha-1}}{2\alpha-1} \right]_0^{\frac{3}{4}}.$$

Since $\alpha < \frac{1}{2}$, the integral converges as $\rho \rightarrow 0$. To conclude that $u \in H^1(\mathbb{R}^2)$, we must show that these classical derivatives coincide with the distributional derivatives, which is non-trivial because $x = 0$ is a singular point of the derivative. Let $\phi \in \mathcal{D}(\mathbb{R}^2)$. We want to prove that

$$\int_{\mathbb{R}^2} D_j u \phi dx = - \int_{\mathbb{R}^2} u \frac{\partial \phi}{\partial x_j} dx.$$

Starting from the second term:

$$\int_{\mathbb{R}^2} u \frac{\partial \phi}{\partial x_j} dx = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon(0)} u \frac{\partial \phi}{\partial x_j} dx.$$

This identity holds as a consequence of the dominated convergence theorem. Outside the origin, we can integrate by parts using classical derivatives:

$$\int_{\mathbb{R}^2} u \frac{\partial \phi}{\partial x_j} dx = \lim_{\varepsilon \rightarrow 0} \left(- \int_{\mathbb{R}^2 \setminus B_\varepsilon(0)} \frac{\partial u}{\partial x_j} \phi dx + \int_{\partial B_\varepsilon(0)} u \phi \nu_j d\sigma \right).$$

Here, ν_j is the j -th component of the outward normal vector. Using Lebesgue's theorem again for the first term, we have

$$\int_{\mathbb{R}^2} u \frac{\partial \phi}{\partial x_j} dx = - \int_{\mathbb{R}^2} \frac{\partial u}{\partial x_j} \phi dx + \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(0)} u \phi \nu_j d\sigma.$$

To conclude, we must show that

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(0)} u \phi \nu_j d\sigma = 0.$$

For $\varepsilon < 1$, we have $u|_{B_\varepsilon(0)} = |\ln\varepsilon|^\alpha \psi(\varepsilon) = |\ln\varepsilon|^\alpha$. Thus,

$$\int_{\partial B_\varepsilon(0)} u \phi \nu_j d\sigma = |\ln\varepsilon|^\alpha \left| \int_{\partial B_\varepsilon(0)} \phi \nu_j d\sigma \right| \leq |\ln\varepsilon|^\alpha \|\phi\|_\infty 2\pi\varepsilon.$$

EXAMPLE 4.6. There exists a function in $H^1(\mathbb{R}^2)$ but not in $L_{loc}^\infty(\mathbb{R}^2)$, meaning it is unbounded in any open subset of \mathbb{R}^2 , not merely at a single point as in the previous proof.

Let u be a positive function in $H^1(\mathbb{R}^2)$ such that $\lim_{x \rightarrow 0} u(x) = +\infty$. For instance, one can choose the function defined in the previous proof. Let $\{x_n\}$ be a dense sequence in \mathbb{R}^2 , and consider the series

$$\sum_{k=0}^{+\infty} 2^{-k} u(x - x_k).$$

We show that this series converges in $H^1(\mathbb{R}^2)$. Indeed, since the $H^1(\mathbb{R}^2)$ -norm is invariant under translations, we have

$$\sum_{k=0}^{+\infty} 2^{-k} \|u(\cdot - x_k)\|_{H^1(\mathbb{R}^2)} = \sum_{k=0}^{+\infty} 2^{-k} \|u\|_{H^1(\mathbb{R}^2)} = 2\|u\|_{H^1(\mathbb{R}^2)}.$$

Since $H^1(\mathbb{R}^2)$ is a Banach space, any normally convergent series also converges. Denote by w the sum of the series in the sense of $H^1(\mathbb{R}^2)$.

On the other hand, by the construction of u , the series has positive terms, so at each point, the series either converges or diverges positively. Observe that w is also the pointwise limit. Indeed, convergence in $H^1(\mathbb{R}^2)$ to w implies convergence in $L^2(\mathbb{R}^2)$ to w , which in turn implies the pointwise convergence almost everywhere of a subsequence of the partial sums to w .

Fix $h \in \mathbb{N}$, and consider $\lim_{x \rightarrow x_h} u(x - x_h) = +\infty$. Therefore, at every point x_h , we have $\lim_{x \rightarrow x_h} w(x) = +\infty$.

However, since the given sequence is dense in \mathbb{R}^2 , the function $w \in H^1(\mathbb{R}^2)$, but it is not bounded on any open subset of \mathbb{R}^2 .

1.5. Boundary values of Sobolev functions. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $k \in \mathbb{N}$ and $1 \leq p < \infty$. After noticing that $\mathcal{D}(\Omega) \subset W^{k,p}(\Omega)$, we can define

$$W_0^{k,p}(\Omega) := \overline{\mathcal{D}}^{\|\cdot\|_{k,p}}$$

(the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_{k,p}$). We also use the notation $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

PROPOSITION 4.1. *Let Ω be an open set with C^1 boundary and $1 \leq p < \infty$. If $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, then*

$$u|_{\partial\Omega} = 0 \iff u \in W_0^{1,p}(\Omega).$$

REMARK 4.8. *To generalize this result to the case $k \geq 1$, we need to ask Ω to be of class C^k and one proves that all the derivatives up to $m - 1$ order are null at the boundary. In other words, one should not confuse $W_0^{1,p} \cap W^{k,p}$ with $W_0^{k,p}$.*

We have already discussed that, if $\Omega = \mathbb{R}^n$, then $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$. On the other hand, the following results hold.

THEOREM 4.10 (Meyers–Serrin’s theorem). *Assume Ω is bounded and let $u \in W^{k,p}(\Omega)$, with $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that*

$$u_m \rightarrow u \quad \text{in } W^{k,p}(\Omega) \text{ as } m \rightarrow \infty.$$

In other words, $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

THEOREM 4.11. *Assume Ω is bounded and $\partial\Omega \in C^1$. Let $u \in W^{k,p}(\Omega)$, with $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(\bar{\Omega})$ such that*

$$u_m \rightarrow u \quad \text{in } W^{k,p}(\Omega) \text{ as } m \rightarrow \infty.$$

In other words, $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$.

1.5.1. Extension operator. We consider now the question whether, given a function $f \in W^{k,p}(\Omega)$, it is possible to construct an extension $\tilde{f} \in W^{k,p}(\mathbb{R}^n)$ such that $\tilde{f}|_\Omega = f$ and $\|\tilde{f}\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{k,p}(\Omega)}$ for some $C > 0$ independent of f .

For $f \in L^p(\Omega)$, we can take the simple extension by zero outside the domain: $\tilde{f} = f$ in Ω and $\tilde{f} = 0$ in Ω^c . Such extension clearly satisfies $\tilde{f} \in L^p(\mathbb{R}^n)$ and $\|\tilde{f}\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\Omega)}$ for any $p \in [1, \infty]$.

When it comes to functions in $W^{k,p}(\Omega)$, $k \geq 1$, however, the extension by zero outside Ω does not lead, in general, to a function in $W^{k,p}(\mathbb{R}^n)$ so the procedure to construct an extension operator is more delicate.

DEFINITION 4.7 (Extension operator). *We say that $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$ is an extension operator if*

- (1) E is linear, i.e., $E(\alpha f + \beta g) = \alpha E f + \beta E g$ for all $f, g \in W^{k,p}(\Omega)$ and for all $\alpha, \beta \in \mathbb{R}$;
- (2) E is bounded, i.e., there exists $C > 0$ such that $\|E f\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{k,p}(\Omega)}$ for all $f \in W^{k,p}(\Omega)$;
- (3) $E f = f$ a.e. in Ω for all $f \in W^{k,p}(\Omega)$;
- (4) if Ω is bounded, $E(f)$ is compactly supported for all $f \in W^{k,p}(\Omega)$.

THEOREM 4.12 (Extension theorem). *Let Ω be a domain in \mathbb{R}^n with bounded boundary $\partial\Omega$ of class C^k . Then there exists an extension operator $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$ for any $1 \leq p < \infty$.*

1.5.2. *Trace theorems.* Let $\Omega \in \mathbb{R}^n$ be a domain with nonempty bounded and smooth boundary $\partial\Omega$, say of class C^1 . We recall that a function $f \in L^p(\Omega)$ is defined only up to zero measure sets (i.e., it is defined only almost everywhere). Since $|\partial\Omega| = 0$, it is meaningless to talk about the “boundary value” of f since we can modify the value of f on $\partial\Omega$ without changing the equivalence class to which f belongs. We ask now whether for a function $f \in W^{k,p}(\Omega)$, with $k \geq 1$, the picture changes, i.e., whether it makes sense to talk about the value of f (also called the *trace*) on $\partial\Omega$.

It turns out that the answer is yes. The way to proceed is the following: since $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$, given $f \in W^{k,p}(\Omega)$ we can find a sequence $f_k \in C^\infty(\bar{\Omega})$ such that $f_k \xrightarrow{k \rightarrow \infty} f$ in $W^{k,p}(\Omega)$. For each f_k , the trace $f_k|_{\partial\Omega}$ is uniquely defined, so that we define the *trace of f on $\partial\Omega$* as $\lim_{k \rightarrow \infty} f_k|_{\partial\Omega}$. The crucial question is whether such limit exists and with respect to which topology. The following theorem answers this question.

THEOREM 4.13 (Trace theorem). *Let $\Omega \subset \mathbb{R}^n$ be a domain with bounded boundary $\partial\Omega$ of class C^1 . Then there exists a linear and bounded operator $\tau : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$, $1 \leq p < \infty$ such that $\tau f = f|_{\partial\Omega}$ for any $f \in C^0(\bar{\Omega}) \cap W^{1,p}(\Omega)$.*

From this, we may deduce the following result.

THEOREM 4.14 (Trace-zero functions in $W^{1,p}$). *Let us assume that Ω is bounded and $\partial\Omega$ is C^1 . Let us suppose, furthermore, that $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $\tau u = 0$ on $\partial\Omega$.*

1.6. Compactness results. A Banach space X is *compactly embedded* in a Banach space Y , written $X \hookrightarrow\hookrightarrow Y$ or $X \Subset Y$, if the embedding $\iota : X \rightarrow Y$ is compact. That is, ι maps bounded sets in X to precompact sets in Y ; or, equivalently, if $\{x_n\}$ is a bounded sequence in X , then $\{x_n\}$ has a convergent subsequence in Y .

An important property of the Sobolev embeddings is that they are compact on domains with finite measure. This corresponds to the rough principle that uniform bounds on higher derivatives imply compactness with respect to lower derivatives. The compactness of the Sobolev embeddings, due to Franz Rellich [Rel30] and Vladimir Iosifovich Kondrashov [Kon45].

THEOREM 4.15 (Rellich–Kondrashov’s compact embedding theorem). *Suppose that Ω is a bounded open set in \mathbb{R}^n with C^1 boundary, $k, m \in \mathbb{N}$ with $k \geq m$, and $1 \leq p < \infty$.*

(1) *If $kp < n$, then*

$$\begin{aligned} W^{k,p}(\Omega) &\Subset L^q(\Omega) && \text{for } 1 \leq q < np/(n - kp); \\ W^{k,p}(\Omega) &\subset L^q(\Omega) && \text{for } q = np/(n - kp) \end{aligned}$$

More generally, if $(k - m)p < n$, then

$$\begin{aligned} W^{k,p}(\Omega) &\Subset W^{m,q}(\Omega) && \text{for } 1 \leq q < np/(n - (k - m)p) \\ W^{k,p}(\Omega) &\subset W^{m,q}(\Omega) && \text{for } q = np/(n - (k - m)p) \end{aligned}$$

(2) *If $kp = n$, then*

$$W^{k,p}(\Omega) \Subset L^q(\Omega) \quad \text{for } 1 \leq q < \infty$$

(3) *If $kp > n$, then*

$$W^{k,p}(\Omega) \Subset C^{0,\mu}(\bar{\Omega})$$

for $0 < \mu < k - n/p$ if $k - n/p < 1$, for $0 < \mu < 1$ if $k - n/p = 1$, and for $\mu = 1$ if $k - n/p > 1$; and

$$W^{k,p}(\Omega) \subset C^{0,\mu}(\bar{\Omega})$$

for $\mu = k - n/p$ if $k - n/p < 1$.

More generally, if $(k - m)p > n$, then

$$W^{k,p}(\Omega) \Subset C^{m,\mu}(\bar{\Omega})$$

for $0 < \mu < k - m - n/p$ if $k - m - n/p < 1$, for $0 < \mu < 1$ if $k - m - n/p = 1$, and for $\mu = 1$ if $k - m - n/p > 1$; and

$$W^{k,p}(\Omega) \subset C^{m,\mu}(\bar{\Omega})$$

for $\mu = k - m - n/p$ if $k - m - n/p = 0$. These results hold for arbitrary bounded open sets Ω if $W^{k,p}(\Omega)$ is replaced by $W_0^{k,p}(\Omega)$.

EXAMPLE 4.7. If $u \in W^{n,1}(\mathbb{R}^n)$, then $u \in C_0(\mathbb{R}^n)$. This can be seen from the equality

$$u(x) = \int_0^{x_1} \cdots \int_0^{x_n} \partial_1 \cdots \partial_n u(x') \, dx'_1 \cdots dx'_n$$

which holds for all $u \in C_c^\infty(\mathbb{R}^n)$ and a density argument. In general, however, it is not true that $u \in L^\infty$ in the critical case $kp = n$.

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1.7. Poincaré's inequalities. In some cases, the norm of $W_0^{1,p}(\Omega)$ may be simplified using Poincaré inequality [Poi90], which allows one to obtain bounds on a function using bounds on its derivatives and the geometry of its domain of definition.

THEOREM 4.16 (Poincaré's inequality). Suppose that Ω is an open subset of \mathbb{R}^N bounded in one direction.¹⁰ Then there is a constant $C_{Poi} > 0$ (depending only on p and Ω) such that, for any $u \in W_0^{1,p}(\Omega)$, with $p \in [1, \infty)$,

$$\|u\|_{L^p(\Omega)} \leq C_{Poi} \|\nabla u\|_{L^p(\Omega)}.$$

REMARK 4.9. Finding the optimal constant in Poincaré's inequality, sometimes called the Poincaré constant for the domain Ω , is, in general, a difficult that depends on the value of p and the geometry of the domain Ω . For example (and without pretense of completeness), we refer to [AD04; PW60] for the cases $p = 1$ and $p = 2$ in bounded, convex, Lipschitz domains.

PROOF. Without loss of generality, we write $x = (x_1, x')$ with $x' = (x_2, \dots, x_n)$ and suppose that Ω is bounded in the direction of x_1 , i.e., there exists $a, b > 0$ such that, for all $x \in \Omega$, we have $x_1 \in]a, b[$.

We assume that $u \in C_c^\infty(\Omega)$. This will be sufficient since, for any $u \in W_0^{1,p}(\Omega)$, there exists a sequence $\{u_k\}_k \subset C_c^\infty(\Omega)$ such that $u_k \xrightarrow{k \rightarrow \infty} u$ in $W^{1,p}(\Omega)$. Therefore, if we prove Poincaré's inequality for C_c^∞ functions, we can then pass to the limit and recover the statement for $W_0^{1,p}$ functions. We extend u to the whole space \mathbb{R}^n by setting $u(x) = 0$ for $x \notin \Omega$.

Step 1. We start with the case $p = 2$ (just because it is somewhat simpler). Using the variables $x = (x_1, x')$ with $x' = (x_2, \dots, x_n)$, we compute

$$u^2(x_1, x') = \int_a^{x_1} 2u \partial_{x_1} u(t, x') \, dt.$$

An integration by parts yields

$$\begin{aligned} \|u\|_{L^2}^2 &= \int_{\mathbb{R}^n} u^2(x) \, dx = \int_{\mathbb{R}^{n-1}} \int_a^b 1 \cdot \left(\int_a^{x_1} 2u \partial_{x_1} u(t, x') \, dt \right) dx_1 dx' \\ &= \int_{\mathbb{R}^{n-1}} \int_a^b (b - x_1) 2u \partial_{x_1} u(x_1, x') \, dx_1 dx' \leq 2(b - a) \int_{\mathbb{R}^n} |u| |\partial_{x_1} u| \, dx \\ &\leq 2(b - a) \|u\|_{L^2} \|\partial_{x_1} u\|_{L^2}. \end{aligned}$$

Dividing both sides by $\|u\|_{L^2}$, we obtain the result with $C_{Poi} := 2(b - a)$.

Step 2. For $p \in [1, \infty)$, with $p \neq 2$, we the argument is similar and we leave it as an exercise. \square

A generalization of this inequality to $W_0^{k,p}$ is available, due to Kurt Otto Friedrichs [Fri27].

In particular, Poincaré's inequality holds true if Ω is supposed to be bounded. This result has the following important consequence.

COROLLARY 4.1. If Ω is bounded in one direction or has a finite measure, then in $W_0^{1,p}(\Omega)$ the norm $\|f\|_{W_0^{1,p}(\Omega)} := \|Df\|_{L^p(\Omega)}$ is equivalent to the norm $\|f\|_{W^{1,p}(\Omega)}$, i.e., there exist two constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{W_0^{1,p}(\Omega)} \leq \|f\|_{W^{1,p}(\Omega)} \leq C_2 \|f\|_{W_0^{1,p}(\Omega)}.$$

¹⁰ An open set $\Omega \subset \mathbb{R}^N$ is said to be *bounded in one direction* if there is $e \in \mathbb{R}^N$, $\|e\| = 1$, and two real numbers $a, b \in \mathbb{R}$ such that

$$x \cdot e \in]a, b[, \quad \text{for any } x \in \Omega.$$

PROOF. For every $f \in W_0^{1,p}(\Omega)$, we have $\|f\|_{L^p(\Omega)}^p \leq C_{\text{Poi}}^p \|Df\|_{L^p(\Omega)}^p$ which implies

$$\|f\|_{W_0^{1,p}(\Omega)}^p = \|f\|_{L^p(\Omega)}^p + \|Df\|_{L^p(\Omega)}^p \leq (1 + C_{\text{Poi}}^p) \|f\|_{W_0^{1,p}(\Omega)}^p$$

and this proves the second inequality with $C_2 = (1 + C_{\text{Poi}}^p)^{1/p}$. The first inequality trivially holds with $C_1 = 1$. \square

In particular, Poincaré's inequality also implies that we may use as an equivalent inner-product on H_0^1 an expression that involves only the derivatives of the functions and not the functions themselves.

COROLLARY 4.2. *If Ω is an open set that is bounded in some direction, then $H_0^1(\Omega)$ equipped with the inner product*

$$(u, v)_0 = \int_{\Omega} Du \cdot Dv \, dx, \quad (1.9)$$

is a Hilbert space, and the corresponding norm is equivalent to the standard norm on $H_0^1(\Omega)$.

PROOF. We denote the norm associated with the inner-product (1.9) by

$$\|u\|_{H_0^1(\Omega)} = \left(\int_{\Omega} |Du|^2 \, dx \right)^{1/2}$$

and the standard norm and inner product by

$$\begin{aligned} \|u\|_{H^1(\Omega)} &= \left(\int_{\Omega} [u^2 + |Du|^2] \, dx \right)^{1/2} \\ (u, v)_1 &= \int_{\Omega} (uv + Du \cdot Dv) \, dx. \end{aligned}$$

Then, using Poincaré's inequality, we have

$$\|u\|_{H_0^1(\Omega)} \leq \|u\|_{H^1(\Omega)} \leq (C + 1)^{1/2} \|u\|_0.$$

Thus, the two norms are equivalent; in particular, $(H_0^1, (\cdot, \cdot)_0)$ is complete since $(H_0^1, (\cdot, \cdot)_1)$ is complete, so it is a Hilbert space with respect to the inner product (1.9). \square

For functions that do not vanish at the boundary, we can still prove an inequality of this type, attributed to Henri Poincaré and Wilhelm Wirtinger: as an application of the compact embedding theorem, we can prove an estimate on the difference between a function u and its average value on a domain Ω .

THEOREM 4.17 (Poincaré–Wirtinger's inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set with C^1 boundary, and let $p \in [1, \infty]$. Then there exists a constant $C > 0$ (depending only on p and Ω), such that*

$$\left\| u - \int_{\Omega} u \, dx \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \quad \text{for every } u \in W^{1,p}(\Omega).$$

PROOF. Let us suppose, for the sake of finding a contradiction, that the conclusion is false. Then we could find a sequence of functions $u_k \in W^{1,p}(\Omega)$ with

$$\left\| u_k - \int_{\Omega} u_k \, dx \right\|_{L^p(\Omega)} > k \|\nabla u_k\|_{L^p(\Omega)} \quad \text{for every } k \in \{1, 2, \dots\}.$$

Then the renormalized functions

$$v_k := \frac{u_k - \int_{\Omega} u_k \, dx}{\left\| u_k - \int_{\Omega} u_k \, dx \right\|_{L^p(\Omega)}}$$

satisfy

$$\int_{\Omega} v_k \, dx = 0, \quad \|v_k\|_{L^p(\Omega)} = 1, \quad \|Dv_k\|_{L^p(\Omega)} < \frac{1}{k}, \quad \text{for } k \in \{1, 2, \dots\}. \quad (1.10)$$

Since the sequence $(v_k)_{k \geq 1}$ is bounded in $W^{1,p}(\Omega)$, if $p < \infty$, we can use the Rellich–Kondrachov's compactness theorem and find a subsequence that converges in $L^p(\Omega)$ to some function v . If $p > n$, then the functions v_k are uniformly bounded and Hölder continuous. Using Ascoli–Arzela's

compactness theorem, we can thus find a subsequence that converges in $L^\infty(\Omega)$ to some function v .

By (1.10), the sequence of weak gradients also converges, namely $\nabla v_k \rightarrow 0$ in $L^p(\Omega)$. By Lemma 4.2, the zero function is the weak gradient of the limit function v .

We now have

$$\int_{\Omega} v \, dx = \lim_{k \rightarrow \infty} \int_{\Omega} v_k \, dx = 0.$$

Moreover, since $\nabla v = 0 \in L^p(\Omega)$, then the function v must be constant on the connected set Ω ; hence $v(x) = 0$ for a.e. $x \in \Omega$. But this is in contradiction with

$$\|v\|_{L^p(\Omega)} = \lim_{k \rightarrow \infty} \|v_k\|_{L^p(\Omega)} = 1.$$

□

1.8. Dual Sobolev spaces. By duality, we can now define Sobolev spaces of negative order.

DEFINITION 4.8. *Let $k > 0$ and $p \in [1, \infty)$. Let Ω be an open set in \mathbb{R}^n . We define the space $W^{-k,p}(\Omega) := \left(W_0^{k,p'}(\Omega)\right)'$ with $\frac{1}{p} + \frac{1}{p'} = 1$ (for $p = 2$, we write $H^{-k}(\Omega) := W^{-k,2}(\Omega)$), equipped with the norm*

$$\|T\|_{W^{-k,p}(\Omega)} := \sup_{w \in W_0^{k,p'}(\Omega)} \frac{|\langle T, w \rangle|}{\|w\|_{W^{k,p'}(\Omega)}}.$$

Identifying $L^p(\Omega)$ with the dual space of $L^{p'}(\Omega)$, we infer that $L^p(\Omega) \hookrightarrow W^{-k,p}(\Omega)$. Moreover, one can check that any element $T \in W^{-k,p}(\Omega)$ is a distribution.

EXAMPLE 4.8. *The Dirac mass at a point $x \in \Omega$ belongs to $W^{-k,p}(\Omega)$ if $kp' > n$.*

THEOREM 4.18. *Let $p \in [1, \infty)$. Let Ω be an open bounded set in \mathbb{R}^n . For all $f \in W^{-1,p}(\Omega)$, there exist functions $\{g_i\}_{i=1,\dots,n}$, belonging to $L^p(\Omega)$, such that $\|f\|_{W^{-1,p}(\Omega)} = \max_{i \in \{0,\dots,n\}} \|g_i\|_{L^p(\Omega)}$ and*

$$\langle f, v \rangle = \int_{\Omega} g_0 v \, dx + \sum_{i=1}^n \int_{\Omega} g_i \partial_{x_i} v \, dx, \quad \text{for all } v \in W_0^{1,p'}(\Omega).$$

More generally, for all $m \in \mathbb{N}$, we have $f \in W^{-m,p}(\Omega)$ if and only if $f = \sum_{|\alpha| \leq m} \partial^\alpha g_\alpha$ for some $g_\alpha \in L^p(\Omega)$.

For future use, we focus on the particular case $k = 1$ and $p = 2$. The space of bounded linear maps $f : H_0^1(\Omega) \rightarrow \mathbb{R}$ is denoted by $H^{-1}(\Omega) = (H_0^1(\Omega))'$, and the action of $f \in H^{-1}(\Omega)$ on $\phi \in H_0^1(\Omega)$ by $\langle f, \phi \rangle$. The norm of $f \in H^{-1}(\Omega)$ is given by

$$\|f\|_{H^{-1}} = \sup \left\{ \frac{|\langle f, \phi \rangle|}{\|\phi\|_{H_0^1}} : \phi \in H_0^1, \phi \neq 0 \right\}.$$

A function $f \in L^2(\Omega)$ defines a linear functional $F_f \in H^{-1}(\Omega)$ by

$$\langle F_f, v \rangle = \int_{\Omega} f v \, dx = (f, v)_{L^2} \quad \text{for all } v \in H_0^1(\Omega)$$

Here, $(\cdot, \cdot)_{L^2}$ denotes the standard inner product on $L^2(\Omega)$. The functional F_f is bounded on $H_0^1(\Omega)$ with $\|F_f\|_{H^{-1}} \leq \|f\|_{L^2}$ since, by Cauchy-Schwarz' inequality,¹¹

$$|\langle F_f, v \rangle| \leq \|f\|_{L^2} \|v\|_{L^2} \leq \|f\|_{L^2} \|v\|_{H_0^1}$$

We identify F_f with f , and write both simply as f . Such linear functionals are, however, not the only elements of $H^{-1}(\Omega)$. As shown in Theorem 4.18, $H^{-1}(\Omega)$ may be identified with the space of distributions on Ω that are sums of first-order distributional derivatives of functions in $L^2(\Omega)$. Thus, after identifying functions with regular distributions, we have the following triple of Hilbert spaces

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega), \quad H^{-1}(\Omega) = (H_0^1(\Omega))'$$

¹¹ Also called Cauchy–Bunyakovsky–Schwarz' inequality, after Augustin-Louis Cauchy [Cau21], Viktor Bunyakovsky [Bun59], and Hermann Schwarz [Sch88].

Moreover, if $f \in L^2(\Omega) \subset H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$, then

$$\langle f, u \rangle = (f, u)_{L^2},$$

so the duality pairing coincides with the L^2 -inner product when both are defined.

2. Weak solutions of elliptic PDEs

Let us consider the Dirichlet problem for the Laplacian with homogeneous boundary conditions on a bounded domain Ω in \mathbb{R}^n ,

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \quad (2.1)$$

First, suppose that the boundary of Ω is smooth and $u, f : \bar{\Omega} \rightarrow \mathbb{R}$ are smooth functions. Multiplying the PDE $-\Delta u = f$ by a test function $\phi \in \mathcal{D}(\Omega)$, integrating the result over Ω , and using the divergence theorem, we get

$$\int_{\Omega} Du \cdot D\phi \, dx = \int_{\Omega} f\phi \, dx \quad \text{for all } \phi \in C_c^\infty(\Omega). \quad (2.2)$$

The boundary terms vanish because $\phi = 0$ on the boundary. Conversely, if f and Ω are smooth, then any smooth function u that satisfies (2.2) is a solution of the Dirichlet problem (2.1).

Next, we formulate weaker assumptions under which (2.2) makes sense. By the Cauchy–Schwartz inequality, the integral on the left-hand side of (2.2) is finite if Du belongs to $L^2(\Omega)$, so we suppose that $u \in H^1(\Omega)$. Moreover, we impose the boundary condition $u = 0$ on $\partial\Omega$ of (2.1) in a weak sense by requiring that $u \in H_0^1(\Omega)$. The left hand side of (2.2) then extends by continuity to $\phi \in H_0^1(\Omega) = C_c^\infty(\Omega)$. The right hand side of (2.2) is well-defined for all $\phi \in H_0^1(\Omega)$ if $f \in L^2(\Omega)$, but this is not the most general f for which it makes sense; we can define the right-hand for any f in the dual space of $H_0^1(\Omega)$.

DEFINITION 4.9 (Weak solutions of the Dirichlet problem (2.1)). *Let Ω be an open set in \mathbb{R}^n and $f \in H^{-1}(\Omega)$. A function $u : \Omega \rightarrow \mathbb{R}$ is a weak solution of (2.1) if $u \in H_0^1(\Omega)$ and*

$$\int_{\Omega} Du \cdot D\phi \, dx = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega).$$

REMARK 4.10. *If Ω is smooth and $g : \partial\Omega \rightarrow \mathbb{R}$ is a function on the boundary that is in the range of the trace map $\tau : H^1(\Omega) \rightarrow L^2(\partial\Omega)$, say $g = \tau w$, then we obtain a weak formulation of the inhomogeneous Dirichlet problem*

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega, \end{cases} \quad (2.3)$$

by requiring $u - w \in H_0^1(\Omega)$ instead of $u \in H_0^1(\Omega)$. The definition is otherwise the same. One can prove that the range of the trace map on $H^1(\Omega)$ for a smooth domain Ω is the fractional-order Sobolev space $H^{1/2}(\partial\Omega)$.

REMARK 4.11 (Distributional solutions of the Poisson equation). *Let us comment on some other ways to define weak solutions of Poisson’s equation. If we integrate by parts again in (2.2), we find that every smooth solution u of (2.1) satisfies*

$$-\int_{\Omega} u\Delta\phi \, dx = \int_{\Omega} f\phi \, dx \quad \text{for all } \phi \in C_c^\infty(\Omega). \quad (2.4)$$

This condition makes sense without any differentiability assumptions on u , and we can define a locally integrable function $u \in L_{loc}^1(\Omega)$ to be a weak solution of $-\Delta u = f$ for $f \in L_{loc}^1(\Omega)$ if it satisfies (2.4). One problem with using this definition is that general functions $u \in L^p(\Omega)$ do not have enough regularity to make sense of their boundary values on $\partial\Omega$.

More generally, we can define distributional solutions $T \in \mathcal{D}'(\Omega)$ of Poisson’s equation $-\Delta T = f$ with $f \in \mathcal{D}'(\Omega)$ by

$$-\langle T, \Delta\phi \rangle = \langle f, \phi \rangle \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

While these definitions appear more general, owing to some elliptic regularity results they turn out not to extend the class of weak solutions we consider in Definition 4.9 if $f \in H^{-1}(\Omega)$.

3. Existence of weak solutions for elliptic PDEs via Riesz' representation theorem

In this section, we establish the existence and uniqueness of weak solutions to (2.1).

THEOREM 4.19 (Well-posedness of weak solutions to the Dirichlet problem (2.1)). *Suppose that Ω is an open set in \mathbb{R}^n that is bounded in some direction and $f \in H^{-1}(\Omega)$. Then there is a unique weak solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ in the sense of Definition 4.9.*

The main tool is Riesz' representation theorem, which states that a Hilbert space can be identified with its dual.¹²

THEOREM 4.20 (Riesz representation of linear functionals). *Let $(H, (\cdot, \cdot))$ be a Hilbert space.*

- (1) *For every $x \in H$, the map $y \mapsto (y, x)$ is a continuous linear functional on H .*
- (2) *Let $y \mapsto Ay$ be a continuous linear functional on H . Then there exists a unique element $a \in H$ such that $Ay = (y, a)$ for every $y \in H$.*

PROOF OF THEOREM 4.19. We equip $H_0^1(\Omega)$ with the inner product (1.9). Then, since Ω is bounded in some direction, the resulting norm is equivalent to the standard norm, and f is a bounded linear functional on $(H_0^1(\Omega), (\cdot, \cdot)_0)$. By Riesz' representation theorem, there exists a unique $u \in H_0^1(\Omega)$ such that

$$(u, \phi)_0 = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega),$$

which is equivalent to the condition that u is a weak solution. □

The same approach works for other *symmetric* linear elliptic PDEs, as we will see below.

3.1. Inhomogeneous Dirichlet problem. Let $f \in H^{-1}(\Omega)$ and $g \in H^1(\Omega)$. For the inhomogeneous Dirichlet problem (2.3), we consider the closed convex set

$$K(\Omega) = H_0^1(\Omega) + g := \{w \in H^1(\Omega) : w - g \in H_0^1(\Omega)\}.$$

A function $u \in K(\Omega)$ is a weak solution of (2.3) if

$$\int_{\Omega} \nabla u \cdot \nabla \phi \, dx = \int_{\Omega} f \phi \, dx, \quad \text{for all } \phi \in H_0^1(\Omega).$$

REMARK 4.12. *Let $f \in C(\Omega)$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Then u is a weak solution of (2.3) if and only if u is a classical solution of (2.3).*

To show the existence and uniqueness of weak solutions to (2.3), we apply Riesz' representation theorem to the linear functional $G : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$G(\phi) := \int_{\Omega} f \phi - \int_{\Omega} \nabla g \cdot \nabla \phi.$$

Then there exists a unique $w \in H_0^1(\Omega)$ such that $G(\phi) = (\nabla w, \nabla \phi)_0$. Then $u := w + g$ is the unique function in $K(\Omega)$ such that

$$\int_{\Omega} \nabla v \cdot \nabla \phi = \int_{\Omega} f \phi,$$

i.e., the unique weak solution of (2.3).

REMARK 4.13. *Uniqueness can be established a priori via energy methods. Let $u_1, u_2 \in K(\Omega)$. Then, for all $\phi \in H_0^1(\Omega)$, we have*

$$\int_{\Omega} \nabla(u_1 - u_2) \cdot \nabla \phi = 0.$$

In particular, letting $\phi = u_1 - u_2$, we conclude $u_1 = u_2$ a.e.

¹² This result is also known as *Riesz–Fréchet representation theorem*, after Frigyes Riesz [Rie07] and Maurice René Fréchet [Fré07].

3.2. Homogeneous Dirichlet problem for $-\Delta + I$. Consider the Dirichlet problem

$$\begin{cases} -\Delta u + u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Then $u \in H_0^1(\Omega)$ is a weak solution if

$$\int_{\Omega} (Du \cdot D\phi + u\phi) \, dx = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega).$$

This is equivalent to the condition that

$$(u, \phi)_1 = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega),$$

where $(\cdot, \cdot)_1$ is the standard inner product on $H_0^1(\Omega)$. Thus, Riesz' representation theorem implies the existence of a unique weak solution.

REMARK 4.14. *Note that in this example we do not use the Poincaré inequality, so the result applies to arbitrary open sets, including $\Omega = \mathbb{R}^n$. In that case, $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, and we get a unique solution $u \in H^1(\mathbb{R}^n)$ of $-\Delta u + u = f$ for every $f \in H^{-1}(\mathbb{R}^n)$. Moreover, using the standard norms, we have $\|u\|_{H^1} = \|f\|_{H^{-1}}$. Thus the operator $-\Delta + I$ is an isometry of $H^1(\mathbb{R}^n)$ onto $H^{-1}(\mathbb{R}^n)$.*

3.3. Homogeneous Dirichlet problem for $-\Delta + \mu I$. Let $\mu > 0$. A function $u \in H_0^1(\Omega)$ is a weak solution of

$$\begin{cases} -\Delta u + \mu u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

if

$$(u, \phi)_{\mu} = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega),$$

where

$$(u, v)_{\mu} = \int_{\Omega} (\mu uv + Du \cdot Dv) \, dx$$

The norm $\|\cdot\|_{\mu}$ associated with this inner product is equivalent to the standard one, since

$$\frac{1}{C} \|u\|_{\mu}^2 \leq \|u\|_1^2 \leq C \|u\|_{\mu}^2$$

where $C := \max\{\mu, 1/\mu\}$. We therefore again get the existence of a unique weak solution from Riesz' representation theorem.

EXAMPLE 4.9. *Consider the last example for $\mu < 0$. If we have a Poincaré inequality $\|u\|_{L^2} \leq C \|Du\|_{L^2}$ for Ω , which is the case if Ω is bounded in some direction, then*

$$(u, u)_{\mu} = \int_{\Omega} (\mu u^2 + |Du|^2) \, dx \geq (1 - C|\mu|) \int_{\Omega} |Du|^2 \, dx$$

Thus $\|u\|_{\mu}$ defines a norm on $H_0^1(\Omega)$ that is equivalent to the standard norm if $-1/C < \mu < 0$, and we get a unique weak solution in this case also, provided that $|\mu|$ is sufficiently small.

For bounded domains, the Dirichlet Laplacian has an infinite sequence of real eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ such that there exists a nonzero solution $u \in H_0^1(\Omega)$ of $-\Delta u = \lambda_n u$. The best constant in Poincaré's inequality can be shown to be the minimum eigenvalue λ_1 , and this method does not work if $\mu \leq -\lambda_1$. For $\mu = -\lambda_n$, a weak solution does not exist for every $f \in H^{-1}(\Omega)$, and if one does exist it is not unique since we can add to it an arbitrary eigenfunction. Thus, not only does the method fail, but the conclusion of [Theorem 4.19](#) may be false.

3.4. Homogeneous Dirichlet problem for symmetric elliptic operators. Let us consider the operator

$$Lu = - \sum_{i,j=1}^n \partial_{x_i} (a_{ij} \partial_{x_j} u),$$

where the coefficients are assumed to be bounded, symmetric ($a_{ij} = a_{ji}$), and satisfy the uniform ellipticity condition. That is, for some $\theta > 0$,

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad \text{for all } x \in \Omega, \text{ and all } \xi \in \mathbb{R}^n.$$

The function $u \in H_0^1(\Omega)$ will be a weak solution of the Dirichlet problem for this operator,

$$\begin{cases} Lu = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

if

$$a(u, \phi) = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega),$$

where $a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ is the symmetric bilinear form associated with the operator, and is given by

$$a(u, v) := \sum_{i,j=1}^n \int_{\Omega} a_{ij} \partial_{x_j} u \partial_{x_i} v \, dx$$

If Ω is bounded in some direction, then boundedness of a_{ij} , uniform ellipticity, and Poincaré's inequality will imply that the symmetric bilinear form a defines an inner product on $H_0^1(\Omega)$, with the induced norm being equivalent to the standard norm of $H_0^1(\Omega)$. This will again imply that $f \in H^{-1}$ is a bounded linear functional on the Hilbert space $(H_0^1(\Omega), a)$, and hence Riesz' representation theorem will imply the existence of a unique weak solution of the Dirichlet problem for this operator.

REMARK 4.15. *The bilinear form a of course arises from integration by parts of the left hand side of the equation after multiplying by the function v . Thus, having the derivative in front of the entire term $a_{ij} \partial_{x_j} u$ is crucial, since we are not assuming that the coefficients a_{ij} are weakly differentiable. In such cases, we will say that the elliptic operator is in the divergence form.*

3.5. Homogeneous Neumann problem for $-\Delta + I$. Let Ω be an open, connected, bounded set with C^1 boundary. If $f \in L^2(\Omega)$, u is a weak solution of

$$\begin{cases} -\Delta u + u = f, & x \in \Omega, \\ \partial_{\nu} u = 0, & x \in \partial\Omega, \end{cases} \quad (3.1)$$

if

$$\int_{\Omega} \nabla u \cdot \nabla \phi + \int_{\Omega} u \phi = \int_{\Omega} f \phi, \quad \text{for all } \phi \in H^1(\Omega).$$

This is equivalent to the condition that

$$(u, \phi)_1 = \langle f, \phi \rangle \quad \text{for all } \phi \in H^1(\Omega),$$

where $(\cdot, \cdot)_1$ is the standard inner product on $H^1(\Omega)$. Riesz' representation theorem yields the existence of a unique weak solution.

REMARK 4.16. *Let $f \in C(\Omega)$. If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a classical solution, then it is a weak solution of (3.1).*

Viceversa, if $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a weak solution, then it is a classical solution of (3.1). To check this, let us consider a test function $v \in \mathcal{D}(\Omega) \subset H^1(\Omega)$. Since u is a weak solution, using Gauss-Green's formulas, we have

$$\int_{\Omega} f v = \int_{\Omega} \nabla v \cdot \nabla u + \int_{\Omega} u v = \int_{\Omega} (-\Delta u) v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, dS + \int_{\Omega} u v = \int_{\Omega} (-\Delta u) v + \int_{\Omega} u v.$$

In the last equality, we used the assumption on the support of v . Therefore, for every $v \in \mathcal{D}(\Omega)$, we have

$$\int_{\Omega} (-\Delta u + u - f) v = 0.$$

By the fundamental lemma of the calculus of variations, $(-\Delta u + u - f) = 0$ almost everywhere in Ω , but since u and f are continuous, this equality holds for every $x \in \Omega$.

It remains to prove that

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Let $v \in C^\infty(\bar{\Omega}) \subset H^1(\Omega)$. We know that u satisfies $(-\Delta u + u - f) = 0$ and is also a weak solution of the Neumann problem (3.1). From this, we deduce that

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} (-\Delta u) v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, dS = \int_{\Omega} (-u + f) v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, dS.$$

That is,

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, dS = \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} uv - \int_{\Omega} fv = 0.$$

It can be shown that the set $\{v|_{\partial\Omega} : v \in C^\infty(\bar{\Omega})\}$ is dense in $L^2(\partial\Omega)$. Therefore, by continuity,

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, dS = 0 \quad \text{for all } v \in L^2(\partial\Omega),$$

since $\frac{\partial u}{\partial \nu} \in L^2(\partial\Omega)$ is orthogonal to a dense subspace. Ultimately, $\partial_\nu u = 0$ on $\partial\Omega$.

It is said that the Neumann boundary value problem has natural boundary conditions, because the space in which the weak solution is defined does not explicitly contain information about $\partial_\nu u$, but this condition is automatically satisfied.

3.6. Homogeneous Neumann problem for the Poisson equation. Let Ω be an open, connected, bounded set with C^1 boundary. Let $f \in L^2(\Omega)$ and consider

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \partial_\nu u = 0, & x \in \partial\Omega. \end{cases} \quad (3.2)$$

Let $f \in L^2(\Omega)$. We say that $u \in H^1(\Omega)$ is a weak solution of (3.2) if

$$\int_{\Omega} \nabla u \cdot \nabla \phi = \int_{\Omega} f \phi, \quad \text{for all } \phi \in H^1(\Omega).$$

THEOREM 4.21. *If a weak solution of (3.2) exists, then it is unique up to additive constants.*

PROOF. Let $u_1, u_2 \in H^1(\Omega)$ be two weak solutions of (3.2). Then, for every $\phi \in H^1(\Omega)$, we have

$$\int_{\Omega} \nabla(u_1 - u_2) \cdot \nabla \phi = 0.$$

Choosing, in particular, $\phi = u_1 - u_2$, yields $\nabla(u_1 - u_2) = 0$ a.e., and so $u_1 = u_2 + c$ (for some constant $c \in \mathbb{R}$), since Ω is connected. \square

THEOREM 4.22. *There exists a weak solution of (3.2) if and only if $\int_{\Omega} f(x) \, dx = 0$.*

PROOF. Step 1. If a weak solution exists, then we pick $\phi \equiv \text{const.}$ in the weak formulation and deduce $\int_{\Omega} f(x) \, dx = 0$.

Step 2. Let us suppose $\int_{\Omega} f(x) \, dx = 0$ and define

$$V = \left\{ u \in H^1(\Omega) : \int_{\Omega} u \, dx = 0 \right\}.$$

This is a closed subspace of $H^1(\Omega)$ (for example, one can see this as V is the kernel of the linear continuous functional $\langle 1, \cdot \rangle \in L^2(\Omega)$). By Poincaré–Wirtinger’s inequality, V is a Hilbert space with norm $\|\nabla u\|_{L^2(\Omega)}$. Since weak solutions are unique up to an additive constant, we may assume that they lie all in V (otherwise, we subtract the average). It suffices to test weak solutions with $\phi \in V$. Indeed, since $\int_{\Omega} f = 0$, we have

$$\int \phi f = \int \left(\phi - \int \phi \right) f.$$

Applying Riesz’ representation theorem on the Hilbert space V , we get existence of a unique weak solution of (3.2) in V . \square

4. Existence of weak solutions for elliptic PDEs via variational methods

We have established the existence of a weak solution by use of Riesz’ representation theorem. In this section, we use a different approach, via *variational methods*. The Riesz representation theorem is, however, typically proved by a similar argument to the one used in the *direct method of the calculus of variations*, so in essence the proofs are equivalent.

DEFINITION 4.10. A functional $J : X \rightarrow \mathbb{R}$ on a Banach space X is differentiable at $x \in X$ if there is a bounded linear functional $A : X \rightarrow \mathbb{R}$ such that

$$\lim_{h \rightarrow 0} \frac{|J(x+h) - J(x) - Ah|}{\|h\|_X} = 0$$

If A exists, then it is unique, and it is called the differential of J at x , denoted $DJ(x) = A$.

This definition expresses the basic idea of a differentiable function as one which can be approximated locally by a linear map. If J is differentiable at every point of X , then $DJ : X \rightarrow X^*$ maps $x \in X$ to the linear functional $DJ(x) \in X^*$ that approximates J near x .

A weaker notion of differentiability is the existence of *directional derivatives*

$$\delta J(x; h) = \lim_{\epsilon \rightarrow 0} \left[\frac{J(x + \epsilon h) - J(x)}{\epsilon} \right] = \left. \frac{d}{d\epsilon} J(x + \epsilon h) \right|_{\epsilon=0}.$$

If the directional derivative at x exists for every $h \in X$ and is a bounded linear functional on h , then $\delta J(x; h) = \delta J(x)h$ where $\delta J(x) \in X^*$. We call $\delta J(x)$ the *Gâteaux derivative* of J at x . The derivative DJ is then called the *Fréchet derivative* to distinguish it from the directional or Gâteaux derivative. If J is differentiable at x , then it is Gâteaux-differentiable at x and $DJ(x) = \delta J(x)$, but the converse is not true.

EXAMPLE 4.10. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(0, 0) = 0$ and

$$f(x, y) = \left(\frac{xy^2}{x^2 + y^4} \right)^2 \quad \text{if } (x, y) \neq (0, 0).$$

Then f is Gâteaux-differentiable at 0, with $\delta f(0) = 0$, but f is not Fréchet differentiable at 0.

If $J : X \rightarrow \mathbb{R}$ attains a local minimum at $x \in X$ and J is differentiable at x , then for every $h \in X$ the function $J_{x;h} : \mathbb{R} \rightarrow \mathbb{R}$ defined by $J_{x;h}(t) = J(x + th)$ is differentiable at $t = 0$ and attains a minimum at $t = 0$. It follows that

$$\frac{dJ_{x;h}}{dt}(0) = \delta J(x; h) = 0 \quad \text{for every } h \in X.$$

Hence $DJ(x) = 0$.

We say that an elliptic problem is in *variational form* if its weak solutions are critical points of a suitable functional.

If we have a variational problem, in the *direct method of the calculus of variations*, we prove the existence of a minimizer of J by showing that a minimizing sequence $\{u_n\}$ converges in a suitable sense to a minimizer u . The direct method of the calculus of variations is encoded in the proof of the (generalized) Weierstrass theorem.¹³

THEOREM 4.23 ((Generalized) Weierstrass' theorem). *Let X be a reflexive space, $F : X \rightarrow \mathbb{R}$ be a coercive and weakly lower-semicontinuous functional. Then $\inf_X F > -\infty$ and it is a minimum., i.e., there exists $\bar{x} \in X$ such that $F(\bar{x}) = \inf_X F$. Moreover, if F is strictly convex, then the minimum point is unique.*

REMARK 4.17. Let X be a Banach space, we recall that $F : X \rightarrow]-\infty, +\infty]$ is called

- *weakly lower-semicontinuous* if, for every $x \in X$ and every $x_n \rightarrow x$, we have $\liminf_{n \rightarrow \infty} F(x_n) \geq F(x)$;
- *weakly lower-semicontinuous* if, for every $x \in X$ and every $x_n \rightarrow x$, we have $\liminf_{n \rightarrow \infty} F(x_n) \geq F(x)$;
- *coercive* if $\lim_{\|x\|_X \rightarrow \infty} F(x) = +\infty$;
- *convex* if, for every $x, y \in X$ and every $t \in [0, 1]$, we have $F(tx + (1-t)x) \leq tF(x) + (1-t)F(y)$.

We also recall that, if F is convex and lower-semicontinuous, then it is weakly lower-semicontinuous.

¹³ Named after Karl Theodor Wilhelm Weierstrass.

4.1. Homogeneous Dirichlet problem for the Poisson equation. Given $f \in H^{-1}(\Omega)$, we define a quadratic functional $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \langle f, u \rangle. \quad (4.1)$$

PROPOSITION 4.2. *The functional $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ in (4.1) is well-defined and differentiable. Its derivative $DJ(u) : H_0^1(\Omega) \rightarrow \mathbb{R}$ at $u \in H_0^1(\Omega)$ is given by*

$$DJ(u)h = \int_{\Omega} Du \cdot Dh dx - \langle f, h \rangle \quad \text{for } h \in H_0^1(\Omega).$$

PROOF. Given $u \in H_0^1(\Omega)$, define the linear map $A : H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$Ah = \int_{\Omega} Du \cdot Dh dx - \langle f, h \rangle.$$

Then A is bounded, with $\|A\| \leq \|Du\|_{L^2} + \|f\|_{H^{-1}}$, since

$$|Ah| \leq \|Du\|_{L^2} \|Dh\|_{L^2} + \|f\|_{H^{-1}} \|h\|_{H_0^1} \leq (\|Du\|_{L^2} + \|f\|_{H^{-1}}) \|h\|_{H_0^1}$$

For $h \in H_0^1(\Omega)$, we have

$$J(u+h) - J(u) - Ah = \frac{1}{2} \int_{\Omega} |Dh|^2 dx.$$

It follows that

$$|J(u+h) - J(u) - Ah| \leq \frac{1}{2} \|h\|_{H_0^1}^2,$$

and therefore

$$\lim_{h \rightarrow 0} \frac{|J(u+h) - J(u) - Ah|}{\|h\|_{H_0^1}} = 0$$

which proves that J is differentiable on $H_0^1(\Omega)$ with $DJ(u) = A$. \square

Note that $DJ(u) = 0$ if and only if u is a weak solution of the Dirichlet problem Poisson's equation

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (4.2)$$

in the sense of Definition 4.9. Thus, we have the following result.

COROLLARY 4.3. *If $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined in (4.1) attains a minimum at $u \in H_0^1(\Omega)$, then u is a weak solution of $-\Delta u = f$.*

PROOF. Let us define $g(t) = J(u + tv)$ for $v \in H_0^1(\Omega)$. We know that g has a minimum point at $t = 0$. In particular,

$$\begin{aligned} g(t) &= J(u + tv) = \frac{1}{2} \int_{\Omega} |\nabla(u + tv)|^2 dx - \int_{\Omega} f(u + tv) dx \\ &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{t^2}{2} \int_{\Omega} |\nabla v|^2 dx + t \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Omega} fu dx - t \int_{\Omega} fv dx. \end{aligned}$$

Then, the condition $g'(0) = 0$ ensures

$$\int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Omega} fv dx = 0.$$

\square

Finally, we can prove that J has one and only one minimum. Indeed, we can show that J is coercive, strictly convex, and lower-semicontinuous (which proves the claim owing to Theorem 4.23).

Coercivity: We aim to estimate $J(v)$ from below:

$$J(v) = \frac{1}{2} \|v\|_{H_0^1(\Omega)}^2 - \int_{\Omega} fv \geq \frac{1}{2} \|v\|_{H_0^1(\Omega)}^2 - \|f\|_2 \|v\|_2 \geq \frac{1}{2} \|v\|_{H_0^1(\Omega)}^2 - C_P \|f\|_2 \|v\|_{H_0^1(\Omega)},$$

where C_P is the constant from Poincaré's inequality. Thus, $J(v) \rightarrow +\infty$ as $\|v\|_{H_0^1(\Omega)} \rightarrow +\infty$ (since $J(v)$ contains a quadratic polynomial in $\|v\|_{H_0^1(\Omega)}$).

Strict Convexity: Using the strict convexity of the single-variable function $f(x) = x^2$ (which follows from $f'' = 2 > 0$), we have:

$$\begin{aligned} & J(tv_1 + (1-t)v_2) - (tJ(v_1) + (1-t)J(v_2)) \\ &= \frac{1}{2} \int_{\Omega} |t\nabla v_1 + (1-t)\nabla v_2|^2 dx - \int_{\Omega} f(tv_1 + (1-t)v_2) dx \\ &\quad - \frac{t}{2} \int_{\Omega} |\nabla v_1|^2 dx - \frac{1-t}{2} \int_{\Omega} |\nabla v_2|^2 dx + t \int_{\Omega} f v_1 dx + (1-t) \int_{\Omega} f v_2 dx \\ &< \frac{1}{2} \int_{\Omega} t|\nabla v_1|^2 + (1-t)|\nabla v_2|^2 dx - \frac{t}{2} \int_{\Omega} |\nabla v_1|^2 dx - \frac{1-t}{2} \int_{\Omega} |\nabla v_2|^2 dx = 0. \end{aligned}$$

Lower Semicontinuity: Since the functional J is continuous, it is lower semicontinuous with respect to the strong topology. Furthermore, since it is convex, it is also weakly lower semicontinuous.

Thus, by the Weierstrass theorem, there exists a unique $u \in H_0^1(\Omega)$ that minimizes J .

4.2. Inhomogeneous Dirichlet problem for the Poisson equation. We now turn to the inhomogeneous Dirichlet problem for the Poisson equation:

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega, \end{cases} \quad (4.3)$$

We will use the notation of Section 3.1.

THEOREM 4.24. *Let $f \in L^2(\Omega)$ and $g \in H^1(\Omega) \cap C(\bar{\Omega})$. Then there exists one and only one weak solution of (4.3).*

PROOF. Let $J : K \rightarrow \mathbb{R}$ be defined as

$$J(u) = \frac{1}{2} \int |\nabla u|^2 dx - \int f u dx.$$

To this functional, we associate the functional $\tilde{J} : H_0^1(\Omega) \rightarrow \mathbb{R}$, defined by

$$\tilde{J}(w) := J(w + g).$$

We aim to prove that \tilde{J} has a unique minimum point. Explicitly, we have

$$\tilde{J}(w) = J(w + \varphi) = \frac{1}{2} \int_{\Omega} |\nabla w|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 dx + \int_{\Omega} \nabla w \cdot \nabla \varphi dx - \int_{\Omega} f w dx - \int_{\Omega} f \varphi dx.$$

Since \tilde{J} is defined on $H_0^1(\Omega)$, which is a reflexive space, we apply Weierstrass' theorem to find the minimum points by proving that \tilde{J} is coercive and weakly lower semi-continuous. In fact, we will find a unique minimum of \tilde{J} , which, when shifted back, will correspond to the unique minimum of J .

To prove the coercivity of \tilde{J} , we use Hölder's inequality:

$$- \int_{\Omega} \nabla w \cdot \nabla \varphi dx \leq \left| \int_{\Omega} \nabla w \cdot \nabla \varphi dx \right| \leq \|\nabla w\|_2 \|\nabla \varphi\|_2.$$

From Poincaré's inequality, it follows that

$$\int_{\Omega} f w dx \leq \|f\|_2 \|w\|_2 \leq C \|f\|_2 \|\nabla w\|_2.$$

Changing the signs in the previous inequalities, we get

$$|\tilde{J}(w)| \geq \frac{1}{2} \|\nabla w\|_2^2 + \frac{1}{2} \|\nabla \varphi\|_2^2 - \|\nabla w\|_2 \|\nabla \varphi\|_2 - C \|f\|_2 \|\nabla w\|_2 - \int_{\Omega} f \varphi dx.$$

Thus, as $\|\nabla w\|_2 \rightarrow +\infty$, we have $|\tilde{J}(w)| \rightarrow +\infty$, since $\tilde{J}(w)$ contains a quadratic polynomial in $\|\nabla w\|_2$.

Next, we show that \tilde{J} is weakly lower semicontinuous. Indeed:

- The term $\frac{1}{2} \int_{\Omega} |\nabla w|^2 dx$ is weakly lower semicontinuous because it represents the norm squared of the space in question (proof of this fact omitted).
- The terms $\int_{\Omega} \nabla w \cdot \nabla \varphi dx - \int_{\Omega} f w dx$ are weakly continuous because they are linear and continuous.

- The terms $\frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 dx - \int_{\Omega} f \varphi dx$ are constant.

Hence, by Weierstrass' theorem, \tilde{J} has a minimum point. We now prove the strict convexity of \tilde{J} to ensure the uniqueness of this minimum. Since \tilde{J} contains constant terms and linear terms, its strict convexity reduces to the strict convexity of the squared H_0^1 -norm. Specifically:

$$\begin{aligned} & \tilde{J}(tw_1 + (1-t)w_2) - \left(t\tilde{J}(w_1) + (1-t)\tilde{J}(w_2) \right) \\ &= \frac{1}{2} \left[\|tw_1 + (1-t)w_2\|_{H_0^1(\Omega)}^2 - t\|w_1\|_{H_0^1(\Omega)}^2 - (1-t)\|w_2\|_{H_0^1(\Omega)}^2 \right]. \end{aligned}$$

Let w_0 be the minimum obtained from Weierstrass' theorem for \tilde{J} on $H_0^1(\Omega)$. Then $u_0 = w_0 + \varphi$ is the minimum of the functional J on K . Clearly, $u_0 \in K$. We find that u_0 is a weak solution, meaning that for every $v \in H_0^1(\Omega)$,

$$\int_{\Omega} \nabla u_0 \cdot \nabla v dx = \int_{\Omega} f v dx.$$

To see this, we define

$$g(t) := \tilde{J}(w_0 + tv).$$

We know that g has a unique minimum at $t = 0$. In particular:

$$\begin{aligned} g(t) &= \tilde{J}(w_0 + tv) \\ &= \frac{1}{2} \int_{\Omega} |\nabla w_0|^2 dx + t^2 \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx + t \int_{\Omega} \nabla w_0 \cdot \nabla v dx \\ &\quad + \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 dx + \int_{\Omega} \nabla w_0 \cdot \nabla \varphi dx + t \int_{\Omega} \nabla v \cdot \nabla \varphi dx \\ &\quad - \int_{\Omega} f u_0 dx - t \int_{\Omega} f v dx - \int_{\Omega} f \varphi dx. \end{aligned}$$

This is again a quadratic polynomial in t . The condition $g'(0) = 0$ ensures

$$\int_{\Omega} \nabla w_0 \cdot \nabla v dx + \int_{\Omega} \nabla v \cdot \nabla \varphi dx - \int_{\Omega} f v dx = 0.$$

□

4.3. Homogeneous Neumann problem for $-\Delta + I$. We aim to prove that the weak solution of (3.1) is the unique minimum of a functional, which can be obtained using the Weierstrass Theorem. Consider the functional

$$J : H^1(\Omega) \rightarrow \mathbb{R}, \quad J(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx + \frac{1}{2} \int_{\Omega} |v|^2 dx - \int_{\Omega} f v dx.$$

We verify the hypotheses of the Weierstrass Theorem:

Coercivity: We estimate $J(v)$ from below by a quadratic polynomial in $\|v\|_{H^1(\Omega)}$:

$$J(v) = \frac{1}{2} \|v\|_{H^1(\Omega)}^2 - \int_{\Omega} f v dx \geq \frac{1}{2} \|v\|_{H^1(\Omega)}^2 - \|f\|_2 \|v\|_2 \geq \frac{1}{2} \|v\|_{H^1(\Omega)}^2 - \|f\|_2 \|v\|_{H^1(\Omega)}.$$

Thus, $J(v) \rightarrow +\infty$ as $\|v\|_{H^1(\Omega)} \rightarrow +\infty$.

Strict Convexity: Due to the linearity of the term J_2 , we have

$$J(tv_1 + (1-t)v_2) - (tJ(v_1) + (1-t)J(v_2)) = J_1(tv_1 + (1-t)v_2) - (tJ_1(v_1) + (1-t)J_1(v_2)),$$

where J_1 is a squared norm, ensuring strict convexity.

Lower Semicontinuity: We rewrite the functional as

$$J(v) = \frac{1}{2} \|v\|_{H^1(\Omega)}^2 - \int_{\Omega} f v dx = J_1(v) + J_2(v).$$

The term J_1 is continuous and convex, hence weakly lower semicontinuous. The term J_2 is linear and continuous, thus weakly continuous. Therefore, J is weakly lower semicontinuous.

By the Weierstrass Theorem, there exists a unique $u \in H^1(\Omega)$ that minimizes J . In particular, for any $t \in \mathbb{R}$ and any $v \in H^1(\Omega)$, we have $J(u) \leq J(u + tv)$.

We now show that u is a weak solution of (3.1). Define $g(t) = J(u + tv)$. We know that g has a unique minimum at $t = 0$. Specifically:

$$\begin{aligned} g(t) &= J(u + tv) = \frac{1}{2} \int_{\Omega} |\nabla(u + tv)|^2 dx + \frac{1}{2} \int_{\Omega} |u + tv|^2 dx - \int_{\Omega} f(u + tv) dx \\ &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{t^2}{2} \int_{\Omega} |\nabla v|^2 dx + t \int_{\Omega} \nabla u \cdot \nabla v dx \\ &\quad + \frac{1}{2} \int_{\Omega} |u|^2 dx + \frac{t^2}{2} \int_{\Omega} |v|^2 dx + t \int_{\Omega} uv dx - \int_{\Omega} fu dx - t \int_{\Omega} fv dx. \end{aligned}$$

The condition $g'(0) = 0$ ensures

$$\int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} uv dx - \int_{\Omega} fv dx = 0.$$

4.4. The Dirichlet eigenvalues of the Laplacian. We consider the *Dirichlet eigenvalue problem* for the Laplace operator: Given a bounded open set Ω , find $\lambda \in \mathbb{R}$ such that there exists a (non-trivial) weak solution u to the problem

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.4}$$

i.e., there exists $u \in H_0^1(\Omega)$ with $u \neq 0$, such that

$$\int_{\Omega} \nabla \varphi \cdot \nabla u = \lambda \int_{\Omega} u \varphi, \quad \text{for all } \varphi \in H_0^1(\Omega).$$

The function u is called *eigenfunction* associated with the eigenvalue λ .

REMARK 4.18. *The eigenvalue problem for the Laplacian arises, for example, when searching for solutions of the Schrödinger equation*

$$i\partial_t u + \Delta u = 0$$

that do not decay. Specifically, stationary solutions are sought, i.e., solutions of the form

$$u(x, t) = e^{i\lambda t} w(x),$$

with $\lambda > 0$. Substituting this ansatz into the equation yields a new equation:

$$-\lambda w + \Delta w = 0.$$

THEOREM 4.25. *Every Dirichlet eigenvalue of $-\Delta$ in Ω is strictly positive and the associated eigenfunctions satisfy $\|u\|_{L^2} > 0$. Moreover, eigenfunctions associated with distinct eigenvalues are orthogonal with respect to the scalar product of $H_0^1(\Omega)$.*

PROOF. Let λ be an eigenvalue, and let u be an eigenfunction associated with λ . Taking $\varphi = u$ in the definition of a weak solution, we obtain

$$\int_{\Omega} |\nabla u|^2 = \lambda \int_{\Omega} |u|^2.$$

Since $u \neq 0$, the Poincaré inequality ensures that the first term is strictly positive, and thus the second term is also strictly positive. Since $\|u\|_2^2 \geq 0$, the claim follows.

Let $\lambda_1 \neq \lambda_2$ be distinct eigenvalues with corresponding eigenfunctions u_1 and u_2 . By the definition of eigenfunctions, we have

$$\lambda_2 \int_{\Omega} u_2 u_1 = \int_{\Omega} \nabla u_1 \cdot \nabla u_2 = \lambda_1 \int_{\Omega} u_1 u_2.$$

Subtracting side by side, we obtain

$$(\lambda_1 - \lambda_2) \int_{\Omega} u_2 u_1 = 0,$$

which implies

$$\int_{\Omega} u_2 u_1 = 0.$$

From this, we conclude the result, namely $\int_{\Omega} \nabla u_1 \cdot \nabla u_2 = 0$. \square

We are now interested in estimating the smallest eigenvalue from below. Let $C_P(n, \Omega, 2)$ be the Poincaré constant for $H_0^1(\Omega)$, namely,

$$\|u\|_2 \leq C_P(n, \Omega, 2) \|\nabla u\|_2 \quad \text{for all } u \in H_0^1(\Omega).$$

Define $C(\Omega) = [C_P(n, \Omega, 2)]^{-2}$. It follows that

$$\int_{\Omega} |\nabla u|^2 \geq C(\Omega) \int_{\Omega} |u|^2 \quad \text{for all } u \in H_0^1(\Omega).$$

This is equivalent to stating that

$$\frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2} \geq C(\Omega) \quad \text{for all } u \in H_0^1(\Omega), u \neq 0.$$

Let λ be an eigenvalue with corresponding eigenfunction u . We have seen that $\|u\|_2^2 > 0$, and it satisfies

$$\lambda = \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2} \geq C(\Omega) > 0.$$

Furthermore,

$$C(\Omega) \leq \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2} = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_2^2 = 1}} \int_{\Omega} |\nabla u|^2.$$

To prove the last equality, it suffices to substitute u with the function $v = \frac{u}{\|u\|_2}$, which gives the relation \leq . The \geq inequality is straightforward.

Our goal is to show that this infimum is attained by the smallest eigenvalue.

THEOREM 4.26. *Let*

$$\lambda_1 := \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2} = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^2(\Omega)} = 1}} \int_{\Omega} |\nabla u|^2.$$

Then there exists $\bar{u} \in H_0^1(\Omega)$, with $\|\bar{u}\|_{L^2(\Omega)} = 1$ such that $\lambda_1 = \int_{\Omega} |\nabla \bar{u}|^2$ (i.e., the infimum is actually achieved), λ_1 is the smallest Dirichlet eigenvalue of $-\Delta$ in Ω .

PROOF. Let $\{u_n\}$ be a minimizing sequence for λ_1 , with $u_n \in H_0^1(\Omega)$, $\|u_n\|_2 = 1$, and

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^2 = \lambda_1.$$

In particular, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is reflexive (as a Hilbert space), its bounded subsets are weakly relatively compact. Therefore, there exists a subsequence $\{u_{n_k}\}$ that converges weakly in $H_0^1(\Omega)$:

$$u_{n_k} \rightharpoonup \bar{u} \quad \text{with } \bar{u} \in H_0^1(\Omega).$$

Moreover, $u_{n_k} \rightharpoonup \bar{u}$ in $L^2(\Omega)$ and

$$\|\bar{u}\|_{H_0^1(\Omega)}^2 \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_{n_k}|^2 = \lambda_1.$$

We now apply Rellich–Kondrachov’s theorem for $p = q = 2$:

- (i) If $n > 2$, then $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in \left[1, \frac{np}{n-p}\right)$;
- (ii) If $n = 2$, then $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, \infty)$;
- (iii) If $n = 1$, then $H_0^1(\Omega) \hookrightarrow \mathcal{C}_0(\bar{\Omega})$.

For $n > 2$, $2 < \frac{2n}{n-2}$. For $n = 1$, since Ω is bounded, $\mathcal{C}_0(\bar{\Omega}) \hookrightarrow L^2(\Omega)$. In all cases,

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega).$$

Thus, from $\{u_{n_k}\}$, we can extract a subsequence that converges strongly in $L^2(\Omega)$. This new subsequence also converges to \bar{u} .

Using the continuity of the norm, we have

$$\|\bar{u}\|_{L^2(\Omega)} = 1.$$

Recalling that

$$\lambda_1 = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_2 = 1}} \int_{\Omega} |\nabla u|^2,$$

it follows that $\lambda_1 \leq \|\bar{u}\|_{H_0^1(\Omega)}^2$. From the weak convergence properties, we deduce $\lambda_1 = \|\bar{u}\|_{H_0^1(\Omega)}^2$, meaning λ_1 is a minimum, not just an infimum.

Step 2. We now show that λ_1 is an eigenvalue. Let $w \in H_0^1(\Omega)$ such that $\|w\|_2 = 1$ and $\int_{\Omega} |\nabla w|^2 = \lambda_1$. We prove that w is the eigenfunction associated with λ_1 .

Let $v \in H_0^1(\Omega)$ and $t \in \mathbb{R}$. By the definition of λ_1 , we have

$$\lambda_1 \leq \frac{\int_{\Omega} |\nabla(w + tv)|^2}{\int_{\Omega} |w + tv|^2}.$$

Define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ as

$$g(t) = \int_{\Omega} |\nabla(w + tv)|^2 - \lambda_1 \int_{\Omega} |w + tv|^2.$$

Then $g(t) \geq 0$ and $g(0) = 0$. Thus, $t = 0$ is the minimum of this function. It follows that $g'(0) = 0$. Calculating the derivative, we have

$$\begin{aligned} g(t) &= \int_{\Omega} |\nabla w|^2 + 2t \int_{\Omega} \nabla w \cdot \nabla v + t^2 \int_{\Omega} |\nabla v|^2 \\ &\quad - \lambda_1 \int_{\Omega} |w|^2 - 2\lambda_1 t \int_{\Omega} wv - \lambda_1 t^2 \int_{\Omega} |v|^2, \\ g'(t) &= 2 \int_{\Omega} \nabla w \cdot \nabla v + 2t \int_{\Omega} |\nabla v|^2 - 2\lambda_1 \int_{\Omega} wv - 2\lambda_1 t \int_{\Omega} |v|^2. \end{aligned}$$

The condition $g'(0) = 0$ is equivalent to

$$\int_{\Omega} \nabla w \cdot \nabla v - \lambda_1 \int_{\Omega} wv = 0,$$

showing that w is the eigenfunction associated with λ_1 .

It remains to prove that λ_1 is the smallest eigenvalue. Let λ be another eigenvalue with associated eigenfunction u_{λ} . Using u_{λ} in the weak solution formulation for the Dirichlet problem, we have

$$\int_{\Omega} |\nabla u_{\lambda}|^2 = \lambda \int_{\Omega} |u_{\lambda}|^2.$$

It follows that

$$\frac{\int_{\Omega} |\nabla u_{\lambda}|^2}{\int_{\Omega} |u_{\lambda}|^2} = \lambda \geq \lambda_1.$$

This concludes the proof. □

PROPOSITION 4.3. *There exists a strictly increasing sequence of eigenvalues $\{\lambda_n\}$ with finite multiplicity such that $\lambda_n \rightarrow \infty$. Associated with this sequence is a corresponding sequence of eigenfunctions $\{u_n\}$, which forms a maximal orthonormal system in $H_0^1(\Omega)$.*

We do not present the proof, but we comment on the main idea. Given $\lambda_1, \dots, \lambda_n$, one constructs

$$\lambda_{n+1} := \min \{ \|\nabla u\|_2^2 : u \in E_n^{\perp}, \|u\|_2 = 1 \},$$

where E_n^{\perp} is the orthogonal complement of E_n in $H_0^1(\Omega)$, and E_n is generated by the eigenfunctions associated with $\lambda_1, \dots, \lambda_n$.

EXAMPLE 4.11. *Let $\Omega = (0, 1)$. The Dirichlet problem is given by*

$$\begin{cases} u'' + \lambda u = 0, \\ u(0) = u(1) = 0, \end{cases} \quad \text{in } (0, 1).$$

Since $u(0) = 0$, the solution takes the form $u(x) = c \sin(\sqrt{\lambda}x)$. By choosing c appropriately depending on λ , we ensure $\|u\|_2 = 1$.

Imposing $u(1) = 0$, and since $u \neq 0$, we find $\sin(\sqrt{\lambda}) = 0$, which implies $\lambda = \pi^2 n^2$ for $n \in \mathbb{N} \setminus \{0\}$.

The eigenvalues are all simple, and the eigenfunctions form a maximal orthonormal system in $H_0^1(0, 1)$ (as follows from the theory of Fourier series).

4.4.1. *A nonlinear eigenvalue problem.* Let Ω be a bounded open subset of \mathbb{R}^n , with $n \geq 3$, and let $2 \leq p < 2^* = \frac{2n}{n-2}$. We aim to show that

$$M_1 := \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^2}{\left(\int_{\Omega} |u|^p\right)^{2/p}}$$

is a minimum and to determine its value. For $p > 2$, the problem is nonlinear, whereas for $p = 2$, the problem becomes linear and will be revisited in the last section.

Recall that since Ω is bounded, the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ is continuous for every $1 \leq p \leq 2^*$. In particular, $u \in H_0^1(\Omega)$ has a finite $L^p(\Omega)$ -norm. By homogeneity, we can reduce the problem to finding

$$M := \inf_{u \in E} \int_{\Omega} |\nabla u(x)|^2 dx,$$

where

$$E := H_0^1(\Omega) \cap \{u \in L^p(\Omega) : \|u\|_p = 1\}.$$

To see the role of homogeneity, observe that if $u \in E$, then

$$\int_{\Omega} |\nabla u|^2 dx = \frac{\int_{\Omega} |\nabla u|^2 dx}{\|u\|_p^2} \geq M_1.$$

Taking the infimum, we obtain $M \geq M_1$. Conversely, if $u \in H_0^1(\Omega)$, $u \neq 0$, then $\frac{u}{\|u\|_p} \in E$, and thus

$$\frac{\int_{\Omega} |\nabla u|^2 dx}{\|u\|_p^2} = \int_{\Omega} \left| \nabla \frac{u}{\|u\|_p} \right|^2 dx \geq M,$$

which implies $M_1 \geq M$. Therefore, $M_1 = M$.

PROPOSITION 4.4. *M is finite and is a minimum.*

PROOF. Since the functional $u \mapsto \|\nabla u\|_2^2$ is positive, M is finite. Let $\{u_n\}$ be a minimizing sequence for M , with $u_n \in H_0^1(\Omega)$, $\|u_n\|_p = 1$, and

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^2 = M.$$

In particular, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is reflexive (being a Hilbert space), its bounded subsets are weakly relatively compact. Thus, there exists a subsequence $\{u_{n_k}\}$ that converges weakly in $H_0^1(\Omega)$:

$$u_{n_k} \rightharpoonup \bar{u} \quad \text{with } \bar{u} \in H_0^1(\Omega).$$

This implies $\nabla u_{n_k} \rightharpoonup \nabla \bar{u}$ in $L^2(\Omega)$ and $u_{n_k} \rightharpoonup \bar{u}$ in $L^p(\Omega)$. By the norm convergence property,

$$\|\bar{u}\|_{H_0^1(\Omega)}^2 \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_{n_k}|^2 = M.$$

We now apply Rellich—Kondrachov's theorem (here, the assumption $p < 2^*$ is crucial):

$$H_0^1(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega).$$

Thus, from $\{u_{n_k}\}$, we extract a subsequence that converges strongly in $L^p(\Omega)$. Since Ω is bounded and $p \geq 2$, this subsequence also converges weakly in $L^2(\Omega)$, and by the uniqueness of the weak limit, its limit is \bar{u} . Using the continuity of the norm, we have $\|\bar{u}\|_{L^p(\Omega)} = 1$, and hence $\bar{u} \in E$. By the definition of M , we have $M \leq \|\bar{u}\|_{H_0^1(\Omega)}^2$, but from the weak convergence properties, we deduce $M = \|\bar{u}\|_{H_0^1(\Omega)}^2$. Thus, M is a minimum, not just an infimum. \square

PROPOSITION 4.5. *Consider the nonlinear problem for $2 < p < 2^*$ on Ω bounded:*

$$\begin{cases} -\Delta u = M|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It admits a weak solution $\bar{u} \in H_0^1(\Omega)$ such that $\|\bar{u}\|_p = 1$ and

$$\int_{\Omega} \nabla \bar{u} \cdot \nabla v - M \int_{\Omega} |\bar{u}|^{p-2} \bar{u} v = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

Moreover,

$$M = \int_{\Omega} |\nabla \bar{u}|^2 = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^2}{\left(\int_{\Omega} |u|^p\right)^{2/p}}.$$

PROOF. Let $\bar{u} \in H_0^1(\Omega) \cap \{u \in L^p(\Omega) : \|u\|_p = 1\}$ be such that

$$M := \int_{\Omega} |\nabla \bar{u}(x)|^2 dx.$$

We now show that \bar{u} satisfies the following Euler–Lagrange equation (in weak form):

$$\int_{\Omega} \nabla \bar{u} \cdot \nabla v - M \int_{\Omega} |\bar{u}|^{p-2} \bar{u} v = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

Let $v \in H_0^1(\Omega)$ and $t \in \mathbb{R}$. Consider the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g(t) = \int_{\Omega} |\nabla(\bar{u} + tv)|^2 - M \left(\int_{\Omega} |\bar{u} + tv|^p \right)^{2/p}.$$

By the definition of M and \bar{u} , we have $g(t) \geq 0$, $g(0) = 0$, and g has a minimum at $t = 0$. Therefore, $g'(0) = 0$. Differentiating and using the parameter-dependent integral theory, we find

$$\begin{aligned} g'(t) &= 2 \int_{\Omega} \nabla(\bar{u} + tv) \cdot \nabla v - M \frac{2}{p} \left(\int_{\Omega} |\bar{u} + tv|^p \right)^{2/p-1} \int_{\Omega} p |\bar{u} + tv|^{p-2} (\bar{u} + tv) v, \\ g'(0) &= 2 \int_{\Omega} \nabla \bar{u} \cdot \nabla v - 2M \int_{\Omega} |\bar{u}|^{p-2} \bar{u} v. \end{aligned}$$

The condition $g'(0) = 0$ is then equivalent to the weak form of the equation. \square

REMARK 4.19. *Note that the previous proof does not guarantee uniqueness of the solution.*

REMARK 4.20. *Is it necessary for the constant in front of the nonlinear term to be M , the minimum we started from? Clearly not. Let $\mu > 0$, and substitute $u(x)$ with $\mu u(x)$ in the previous problem. We have*

$$\begin{cases} -\mu \Delta v = M \mu^{p-1} |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

which implies

$$\begin{cases} -\Delta v = M \mu^{p-2} |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

For $\lambda > 0$, setting $\mu = \lambda^{1/(p-2)} M^{-1/(p-2)}$, v solves

$$\begin{cases} -\Delta v = \lambda |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

REMARK 4.21. *We cannot consider $\lambda < 0$, because choosing $u = v$ in the weak solution definition would yield*

$$\int |\nabla u|^2 = \lambda \int |u|^p,$$

which forces $u = 0$, violating the condition $\|u\|_p = 1$.

4.5. A semilinear problem. We consider the semilinear problem

$$\begin{cases} -\Delta u = |u|^{p-1}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (4.5)$$

THEOREM 4.27. *If $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$, then there exists a non-trivial weak solution $u \in H_0^1(\Omega)$ of (4.5).*

PROOF. We define $F, G : H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$F(u) = \int_{\Omega} |Du|^2 dx, \quad G(u) = \int_{\Omega} |u|^{p+1} dx - 1$$

and the admissible set

$$\mathcal{A} := \{u \in H_0^1(\Omega) : G(u) = 0\}.$$

Step 1. We show that G is well-defined on $H_0^1(\Omega)$. By Sobolev's embedding theorem,

$$H_0^1(\Omega) \subset L^{p+1}(\Omega) \quad \text{for all } 1 \leq p \leq \frac{2n}{n-2} - 1 = \frac{n+2}{n-2},$$

so G is well defined on $H_0^1(\Omega)$ for all $1 < p < \frac{n+2}{n-2}$.

Step 2. Now that we know the problem is well-defined, we note that F is Gâteaux differentiable on $H_0^1(\Omega)$ with

$$dF[u]v = 2 \int_{\Omega} Du \cdot Dv dx \quad \text{for all } u, v \in H_0^1(\Omega)$$

and that G is C^1 on $H_0^1(\Omega)$ with¹⁴

$$DG(u)v = (p+1) \int_{\Omega} |u|^{p-1}uv dx, \quad \text{for all } u, v \in H_0^1(\Omega).$$

Step 3. The Lagrange multiplier theorem implies that if $\phi \in \mathcal{A}$ is a minimizer of $F|_{\mathcal{A}}$, then there exists a $\lambda \in \mathbb{R}$ such that

$$2 \int_{\Omega} D\phi \cdot Dv dx = \lambda(p+1) \int_{\Omega} |\phi|^{p-1}\phi v dx.$$

for all $v \in H_0^1(\Omega)$, that is, ϕ is a non-trivial weak solution of

$$\begin{cases} -\Delta u = \mu|u|^{p-1}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\mu = \frac{\lambda(p+1)}{2}$. Note that, taking $v = \phi$, gives, in particular

$$2 \int_{\Omega} |D\phi|^2 dx = \lambda(p+1) \int_{\Omega} |\phi|^{p+1} dx$$

so that $\lambda \geq 0$. However, note that if $\lambda = 0$ then $\int_{\Omega} |D\phi|^2 dx = 0$ so that ϕ is a constant in $H_0^1(\Omega)$, i.e., $\phi = 0$ a.e. in Ω , which is a contradiction. Now rescaling ϕ as

$$w = \mu^{1/(p-1)}\phi$$

it follows that w is a non-trivial weak solution of (4.5), as desired.

Step 4. In summary, the existence of non-trivial weak solutions of (4.5) can be established by showing that the problem

$$\lambda = \inf_{u \in \mathcal{A}} F(u)$$

has a minimizer in \mathcal{A} . To apply [Theorem 4.23](#) to show that $F|_{\mathcal{A}}$ has a minimizer, we observe that

(1) The functional F is clearly bounded below on $H_0^1(\Omega)$.

¹⁴ By Taylor's theorem,

$$|u + \epsilon v|^{p+1} = |u|^{p+1} + (p+1)\epsilon|u|^{p-1}uv + \frac{p(p+1)}{2}\epsilon^2|u + \theta v|^{p-1}v^2,$$

for some $0 < \theta(x) < \epsilon$.

(2) F is coercive on $H_0^1(\Omega)$. Indeed, by Poincaré we have

$$\int_{\Omega} u^2 dx \leq C \int_{\Omega} |Du|^2 dx = CF(u)$$

for some constant $C > 0$, and hence $F(u) \rightarrow \infty$ as $\|u\|_{H^1(\Omega)} \rightarrow \infty$.

(3) F is weakly lower semicontinuous on $H_0^1(\Omega)$.

It remains to verify that $G : H_0^1(\Omega) \rightarrow \mathbb{R}$ is continuous with respect to weak convergence. This follows because Rellich–Kondrachov’s theorem with $p = 2$ and $n \geq 3$ implies that

$$H_0^1(\Omega) \subseteq L^{p+1}(\Omega) \text{ for all } 0 \leq p < \frac{2n}{n-2} - 1 = \frac{n+2}{n-2}.$$

Since $G(u) = \int_{\Omega} |u|^{p+1} dx - 1$, we have that G is continuous with respect to weak convergence in $H_0^1(\Omega)$. In summary, by [Theorem 4.23](#), if $1 < p < \frac{n+2}{n-2}$ then there exists a $\phi \in \mathcal{A}$ such that

$$F(\phi) = \min_{u \in \mathcal{A}} F(u)$$

and hence ϕ induces a non-trivial weak solution of [\(4.5\)](#). □

4.6. Pohozaev’s identity and a non-existence result for a supercritical semilinear problem. We conclude this chapter by proving a non-existence result for a semilinear elliptic problem, with supercritical growth, in star-shaped domains.¹⁵

THEOREM 4.28. *Let $\Omega \subset \mathbb{R}^n$ be an open star-shaped set with C^1 boundary, with $n \geq 3$. If $u \in C^2(\bar{\Omega})$ is a classical solution of*

$$\begin{cases} -\Delta u = |u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.6}$$

for some $p > 2^* = 2n/(n-2)$, then $u \equiv 0$ in Ω .

This result is a consequence of Derrick–Pohozaev’s identity.¹⁶ An often used more general form is due to Henri Berestycki and Pierre-Louis Lions [[BL83](#)]. The proof of Derrick–Pohozaev’s identity is a remarkable calculation initiated by multiplying the $-\Delta u = |u|^{p-2}u$ by $x \cdot Du$ and integrating by parts.

LEMMA 4.9 (Derrick–Pohozaev’s identity). *Let u be a solution of [\(4.6\)](#). Then*

$$\frac{n-2}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \nu(x) \cdot x dS(x) = \frac{n}{p} \int_{\Omega} |u|^p dx,$$

where $\nu(x)$ denotes the outward-pointing normal vector to x .

PROOF. We multiply the PDE by $x \cdot Du$ and integrate over Ω , to find

$$\underbrace{\int_{\Omega} (-\Delta u)(x \cdot Du) dx}_{=:A} = \underbrace{\int_{\Omega} |u|^{p-2}u(x \cdot Du) dx}_{=:B}$$

Step 1. The term on the left is

$$\begin{aligned} A &= - \sum_{i,j=1}^n \int_{\Omega} u_{x_i x_i} x_j u_{x_j} dx \\ &= \sum_{i,j=1}^n \int_{\Omega} u_{x_i} (x_j u_{x_j})_{x_i} dx - \sum_{i,j=1}^n \int_{\partial\Omega} u_{x_i} \nu^i x_j u_{x_j} dS \\ &=: A_1 + A_2. \end{aligned}$$

¹⁵ An open set Ω is called *star-shaped with respect to 0* if, for each $x \in \bar{\Omega}$, the line segment $\{\lambda x : 0 \leq \lambda \leq 1\}$ lies in $\bar{\Omega}$. If Ω is convex and $0 \in \Omega$, then Ω is star-shaped with respect to 0, but a general star-shaped region need not be convex.

¹⁶ Named after S. I. Pohožaev [[Poh65](#)] and G. H. Derrick [[Der64](#)].

For A_1 , we compute

$$\begin{aligned} A_1 &= \sum_{i,j=1}^n \int_{\Omega} u_{x_i} \delta_{ij} u_{x_j} + u_{x_i} x_j u_{x_j x_i} dx \\ &= \int_{\Omega} |Du|^2 + \sum_{j=1}^n \left(\frac{|Du|^2}{2} \right)_{x_j} x_j dx \\ &= \left(1 - \frac{n}{2}\right) \int_{\Omega} |Du|^2 dx + \int_{\partial\Omega} \frac{|Du|^2}{2} (\nu \cdot x) dS \end{aligned}$$

Here we really used

$$\Delta u = \nabla \cdot (\nabla u) = \nabla \cdot f$$

and

$$\nabla(x \cdot \nabla u) = \nabla u + x \cdot \nabla^2 u.$$

On the other hand, since $u = 0$ on $\partial\Omega$, $Du(x)$ is parallel to the normal $\nu(x)$ at each point $x \in \partial\Omega$. Thus, $Du(x) = \pm |Du(x)|\nu(x)$. Using this equality, we calculate

$$A_2 = - \int_{\partial\Omega} |Du|^2 (\nu \cdot x) dS.$$

In summary, we deduce

$$A = \frac{2-n}{2} \int_{\Omega} |Du|^2 dx - \frac{1}{2} \int_{\partial\Omega} |Du|^2 (\nu \cdot x) dS.$$

Step 2. For B , we compute

$$\begin{aligned} B &:= \sum_{j=1}^n \int_{\Omega} |u|^{p-2} u x_j u_{x_j} dx \\ &= \sum_{j=1}^n \int_{\Omega} \left(\frac{|u|^p}{p} \right)_{x_j} x_j dx = -\frac{n}{p+1} \int_{\Omega} |u|^p dx. \end{aligned}$$

Step 3. Putting Step 1 and Step 2 together, we conclude

$$\left(\frac{n-2}{2} \right) \int_{\Omega} |Du|^2 dx + \frac{1}{2} \int_{\partial\Omega} |Du|^2 (\nu \cdot x) dS = \frac{n}{p} \int_{\Omega} |u|^p dx.$$

□

We also need a lemma on star-shaped sets.

LEMMA 4.10 (Normals to a star-shaped region). *Let Ω be an open star-shaped set in \mathbb{R}^n with C^1 boundary. Then $x \cdot \nu(x) \geq 0$ for all $x \in \partial\Omega$ (where $\nu(x)$ denotes the unit outward normal to $\partial\Omega$ at x).*

PROOF. Since $\partial\Omega$ is C^1 , if $x \in \partial\Omega$ then for each $\epsilon > 0$ there exists $\delta > 0$ such that $|y - x| < \delta$ and $y \in \bar{\Omega}$ imply $\nu(x) \cdot \frac{(y-x)}{|y-x|} \leq \epsilon$.¹⁷ In particular

$$\limsup_{\substack{y \rightarrow x \\ y \in \bar{\Omega}}} \nu(x) \cdot \frac{(y-x)}{|y-x|} \leq 0.$$

Let $y = \lambda x$ for $0 < \lambda < 1$. Then $y \in \bar{\Omega}$, since Ω is star-shaped. Thus, noticing that

$$\frac{\lambda x - x}{|\lambda x - x|} = \frac{(\lambda - 1)x}{|\lambda - 1||x|} = -\frac{x}{|x|}$$

as $\lambda \in (0, 1)$, we deduce

$$\nu(x) \cdot \frac{x}{|x|} = - \lim_{\lambda \rightarrow 1^-} \nu(x) \cdot \frac{(\lambda x - x)}{|\lambda x - x|} \geq 0.$$

¹⁷We write $\Omega = F^{-1}(-\infty, 0)$ and $\partial\Omega = F^{-1}(0)$ and $\nu = \nabla F$ for a C^1 function F . We are then claiming that, for any $\epsilon > 0$, there is a $\delta > 0$ such that if $|y - x| < \delta$ and $F(y) \leq F(x)$, then

$$\nabla F(x) \cdot \frac{y - x}{|y - x|} < \epsilon.$$

□

PROOF OF THEOREM 4.28. Using Lemma 4.9 and Lemma 4.10, we have

$$\frac{n-2}{2} \int_{\Omega} |\nabla u|^2 \, dx \leq \frac{n}{p} \int_{\Omega} |u|^p \, dx.$$

But, since we assumed that u is a (classical) solution, we have

$$\int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} |u|^p \, dx.$$

Combining these two observations implies

$$\left(\frac{n-2}{2} - \frac{n}{p} \right) \int_{\Omega} |u|^p \, dx \leq 0.$$

Hence, if $u \not\equiv 0$, it follows that $\frac{n-2}{2} - \frac{n}{p} \leq 0$, that is, $p \leq \frac{2n}{n-2}$. This yields $u \equiv 0$ if $p > 2^*$. □

General second-order elliptic PDEs

1. Maximum principles for uniformly elliptic operators

Lect. 11, 26.11

In this section, we revisit the weak and strong maximum principles established earlier for harmonic functions. Specifically, we will apply the methods outlined in [Section 2.3.3](#) and [Section 2.3.6](#) to extend these results to uniformly elliptic operators of the form

$$Lu(x) := - \sum_{i,j=1}^n a_{ij}(x) \partial_{x_i x_j}^2 u(x) + \sum_{i=1}^n b_i(x) \partial_{x_i} u(x) + c(x)u(x), \quad (1.1)$$

acting on functions $u : \Omega \rightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is an open set. Let us introduce the matrix valued function $x \mapsto A(x) = (a_{ij}(x))_{i,j=1}^n$ and the vector valued function $x \mapsto b(x) = (b_i(x))_{i=1}^n$ so that L can be written in more compact form as

$$Lu(x) = -A(x) : D^2u(x) + b(x) \cdot Du(x) + c(x)u(x),$$

where D^2u denotes the Hessian of u , Du its gradient and, for matrices $A, B \in \mathbb{R}^{n \times k}$, we have used the notation $A : B := \sum_{i=1}^n \sum_{j=1}^k A_{ij} B_{ij}$.

We say that the operator L is *elliptic* if the matrix (a_{ij}) is positive definite (this is consistent with [Definition 1.6](#)).

DEFINITION 5.1 (Uniformly elliptic operator). *The operator L in (1.1) is said to be uniformly elliptic in Ω if there exist $\lambda, \Lambda \in \mathbb{R}_+$ such that*

$$0 < \lambda |\xi|^2 \leq \xi^\top A(x) \xi \leq \Lambda |\xi|^2, \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n. \quad (1.2)$$

In what follows, will also assume that there exist constants $B, C > 0$ such that

$$|b(x)| \leq B, \quad 0 \leq c(x) \leq C, \quad \text{for all } x \in \Omega. \quad (1.3)$$

EXAMPLE 5.1. *The Laplacian operator $L = -\Delta$ is uniformly elliptic on any open set, with $\theta = 1$.*

EXAMPLE 5.2. *The Tricomi operator*

$$L = y \partial_x^2 + \partial_y^2$$

is elliptic in $y > 0$ and hyperbolic in $y < 0$. For any $0 < \epsilon < 1$, L is uniformly elliptic in the strip $\{(x, y) : \epsilon < y < 1\}$, with $\theta = \epsilon$, but it is not uniformly elliptic in $\{(x, y) : 0 < y < 1\}$.

REMARK 5.1. *We recall a linear algebra result that will be helpful going forward. Let $A \in \mathbb{R}^{n \times n}$ be a positive definite matrix (i.e., $\xi^\top A \xi > 0$ for all $\xi \in \mathbb{R}^n$) and $D \in \mathbb{R}^{n \times n}$ be a positive semi-definite symmetric matrix. Then*

$$A : D = \sum_{i,j} A_{ij} D_{ij} \geq 0.$$

Indeed, D can be diagonalized as $D = \sum_{l=1}^n \lambda_l v^{(l)} (v^{(l)})^\top$ where $\lambda_l \geq 0$ are the (real) eigenvalues of D and $v^{(l)} \in \mathbb{R}^n$ the corresponding eigenvectors. Then

$$A : D = \sum_{l=1}^n \lambda_l \sum_{i,j=1}^n v_i^{(l)} A_{ij} v_j^{(l)} = \sum_{l=1}^n \lambda_l (v^{(l)})^\top A v^{(l)} \geq 0.$$

1.1. Weak maximum principle. As a first result, we generalize the weak maximum principle to the case of an elliptic operator on a bounded domain. We start from the easier case $c = 0$.

THEOREM 5.1 (Weak maximum principle for uniformly elliptic operators (case $c = 0$)). *Let Ω be a bounded domain, L a uniformly elliptic operator in Ω and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.*

(1) *If $Lu \leq 0$ and $c = 0$ in Ω , then $\max_{x \in \bar{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x)$.*

(2) *If $Lu \geq 0$ and $c = 0$ in Ω , then $\min_{x \in \bar{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x)$.*

PROOF. We will prove only (1) (while the proof of (2) is analogous).

Step 1: $Lu < 0$. We consider first the case $Lu < 0$ and prove the statement by contradiction. Suppose that $\max_{x \in \bar{\Omega}} u(x) > \max_{x \in \partial\Omega} u(x)$. Then, there exists $x_0 \in \Omega$ such that $u(x_0) = \max_{x \in \bar{\Omega}} u(x)$ and, since $u \in C^2(\Omega)$, we have that $-D^2u(x_0)$ is a symmetric and positive semi-definite matrix and $Du(x_0) = 0$. This implies (recalling Remark 5.1)

$$Lu(x_0) = \underbrace{-A(x_0) : D^2u(x_0)}_{\geq 0} + \underbrace{b(x_0) \cdot Du(x_0)}_{=0} \geq 0,$$

which contradicts the assumption.

Step 2: $Lu \leq 0$. Consider now the case $Lu \leq 0$ and the auxiliary (comparison) function $\varphi(x) := e^{\gamma x_1}$ with $\gamma > \frac{B}{\lambda}$ (where B is the constant in (1.3)). Since $a_{11}(x) = e_1^T A(x) e_1 \geq \lambda$, we have

$$L\varphi(x) = -\gamma^2 a_{11}(x) e^{\gamma x_1} + \gamma b_1(x) e^{\gamma x_1} \leq \gamma(-\gamma\lambda + B) e^{\gamma x_1} < 0.$$

Then, for any $\epsilon > 0$, we define $v_\epsilon(x) := u(x) + \epsilon\varphi(x)$, which satisfies $Lv_\epsilon < 0$ in Ω . Hence, by Step 1, $\max_{\bar{\Omega}} v_\epsilon = \max_{\partial\Omega} v_\epsilon$. Letting $\epsilon \rightarrow 0$, this yields $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$. \square

In the case $c \geq 0$, the result has to be slightly weakened.

THEOREM 5.2 (Weak maximum principle for uniformly elliptic operators (case $c \geq 0$)). *Let Ω be a bounded domain, L a uniformly elliptic operator in Ω and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.*

(1) *If $c \geq 0$ and $Lu \leq 0$ in Ω then*

$$\max_{x \in \bar{\Omega}} u(x) \leq \max_{x \in \partial\Omega} u^+(x),$$

where $u^+ := \max\{u, 0\}$.

(2) *If $c \geq 0$ and $Lu \geq 0$ in Ω then*

$$\min_{x \in \bar{\Omega}} u(x) \geq \min_{x \in \partial\Omega} (-u^-(x)),$$

where $u^- := -\min\{u, 0\}$.

(3) *In particular, if $Lu = 0$ in Ω , then $\max_{x \in \bar{\Omega}} |u(x)| = \max_{x \in \partial\Omega} |u(x)|$.*

PROOF. We prove (1) (as the proof of (2) is analogous).

Let $\Omega_+ := \{x \in \bar{\Omega} : u(x) > 0\}$ and $L_0u = -A : D^2u + b \cdot Du$. If $\Omega_+ = \emptyset$, the result is true. Otherwise, in Ω_+ , we have

$$L_0u = Lu - \underbrace{cu}_{\geq 0} \leq 0,$$

so, by Theorem 5.1, $M := \max_{x \in \bar{\Omega}_+} u(x) = \max_{x \in \partial\Omega_+} u(x)$ and $M \geq 0$.

On the one hand, $\max_{x \in \bar{\Omega}} u(x) = \max_{x \in \bar{\Omega}_+} u(x)$. On the other hand, let $x_0 \in \partial\Omega_+$ be such that $u(x_0) = M$. If $x_0 \in \Omega$, then $M = 0$ (by the continuity of u) and if $x_0 \in \partial\Omega$, then $M = u(x_0) = \max_{\partial\Omega} u$. Hence,

$$\max_{x \in \bar{\Omega}} u(x) = \max_{x \in \partial\Omega} u^+(x).$$

Finally, to prove (3), we note that, if $Lu = 0$ in Ω then, in particular,

$$\begin{aligned} Lu \leq 0 & \quad \text{and} \quad u(y) \leq \max_{x \in \partial\Omega} u^+(x) \leq \max_{x \in \partial\Omega} |u(x)| \\ Lu \geq 0 & \quad \text{and} \quad u(y) \geq \min_{x \in \partial\Omega} (-u^-(x)) \geq -\max_{x \in \partial\Omega} |u(x)|. \end{aligned}$$

Hence, $|u(y)| \leq \max_{x \in \partial\Omega} |u(x)|$ for all $y \in \Omega$, which yields (3). \square

From Theorem 5.2, we can deduce a comparison/positivity principle.

COROLLARY 5.1 (Comparison principle). *Let L be uniformly elliptic and $c \geq 0$ in Ω .*

(1) *If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lu \leq 0$ in Ω , $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in $\bar{\Omega}$.*

(2) *If $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $Lu \leq Lv$ in Ω , $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\bar{\Omega}$.*

1.1.1. *A priori bound on the solutions of the Dirichlet problem.* The maximum principle allows us also to derive a simple a priori bound on the solutions of the Dirichlet problem

$$\begin{cases} Lu = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases} \quad (1.4)$$

In turn, this bound immediately gives uniqueness of solutions to the Dirichlet problem (1.4).

PROPOSITION 5.1 (A priori bounds for the Dirichlet problem). *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a solution of the Dirichlet problem (1.4), with Ω bounded, L uniformly elliptic in Ω with $c \geq 0$, $f \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$ and bounded, $g \in C^0(\partial\Omega)$. Then*

$$\|u\|_{C^0(\bar{\Omega})} \leq \|g\|_{C^0(\partial\Omega)} + C \sup_{x \in \Omega} |f(x)|,$$

with $C = \frac{2}{\lambda} e^{2(\frac{B}{\lambda} + 1)D}$, where λ, B are as in (1.2)–(1.3), and $D := \sup_{x \in \Omega} |x|$.

PROOF. Let $L_0 u := -A : D^2 u + b \cdot Du$ and take a comparison function $\tilde{\varphi}$ that satisfies

$$L_0 \tilde{\varphi} \leq -\alpha, \quad |\tilde{\varphi}| \leq \beta \quad \text{in } \Omega,$$

for some $\alpha, \beta > 0$. For instance, we can take the comparison function $\tilde{\varphi}(x) := e^{\gamma x_1}$ considered in the proof of Theorem 5.1, for which we have already established the bound $L_0 \tilde{\varphi}(x) \leq \gamma(-\gamma\lambda + B)e^{\gamma x_1}$. Therefore, if we take $\gamma = 1 + \frac{B}{\lambda}$, we have

$$L_0 \tilde{\varphi}(x) \leq -\gamma\lambda e^{\gamma x_1} \leq -\lambda e^{-\gamma D} =: -\alpha, \quad |\tilde{\varphi}(x)| \leq e^{\gamma D} =: \beta, \quad \text{for all } x \in \Omega$$

Let us now set $G := \|g\|_{C^0(\partial\Omega)}$, $F = \sup_{\Omega} |f|$, and consider another comparison function

$$\varphi(x) := G + (\beta - \tilde{\varphi}(x)) \frac{F}{\alpha},$$

which satisfies $\varphi \geq 0$ in Ω , $\varphi \geq g$ on $\partial\Omega$, and

$$L\varphi = L_0\varphi + c\varphi = -L_0\tilde{\varphi} \frac{F}{\alpha} + \underbrace{c\varphi}_{\geq 0} \geq F.$$

Hence,

$$L(\varphi - u) \geq F - f(x) \geq 0, \quad \varphi - u \geq 0 \quad \text{on } \partial\Omega$$

and, by the weak maximum principle, $\varphi - u \geq 0$ in $\bar{\Omega}$. This yields

$$u(x) \leq G + \frac{2\beta}{\alpha} F.$$

The lower bound can be obtained by taking $-\varphi$ instead of φ . \square

1.2. Strong maximum principle. The strong maximum principle holds as well for a uniformly elliptic operator on a domain Ω .

THEOREM 5.3 (Strong maximum principle). *Let L be uniformly elliptic in Ω (with Ω not necessarily bounded) and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.*

- (1) *If $c = 0$ and $Lu \leq 0$ (resp., $Lu \geq 0$) in Ω and there exists $y \in \Omega$ such that $u(y) = \max_{x \in \bar{\Omega}} u(x)$ (resp., $u(y) = \min_{x \in \bar{\Omega}} u(x)$), then u is constant in $\bar{\Omega}$.*
- (2) *If $c \geq 0$ and $Lu \leq 0$ (resp., $Lu \geq 0$) in Ω and there exists $y \in \Omega$ such that $u(y) = \max_{x \in \bar{\Omega}} u(x)$ and $u(y) \geq 0$ (resp., $u(y) = \min_{x \in \bar{\Omega}} u(x)$ and $u(y) \leq 0$), then u is constant in $\bar{\Omega}$.*

For the proof of Theorem 5.3, we need first to establish, as an auxiliary result, a generalization of Lemma 2.1.

LEMMA 5.1 (Zaremba–Hopf–Oleĭnik’s boundary point lemma). *Let L be uniformly elliptic in Ω , $c \geq 0$, and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that $Lu \leq 0$ and $\max_{\partial\Omega} u \geq 0$ if $c \neq 0$. If there exists $x_0 \in \partial\Omega$ maximizing u in $\bar{\Omega}$ such that*

- (1) $u(x_0) > u(x)$ for all $x \in \Omega$,
- (2) $\partial\Omega$ satisfies an interior sphere condition at x_0 , i.e., there exists $y \in \Omega$ and $r > 0$ such that $B_r(y) \subset \Omega$ and $\bar{B}_r(y) \cap \partial\Omega = \{x_0\}$,

then $\partial_\nu u(x_0) > 0$ (with strict inequality).

PROOF. Let $\rho \in (0, r)$ and consider the annular region $\Sigma := \{x \in B_r(y) : 0 < \rho < |x - y| < r\}$, and $\bar{\Sigma} \cap \partial\Omega = \{x_0\}$.

We define the auxiliary function $\varphi(x) := e^{-\alpha|x-y|^2} - e^{-\alpha r^2}$, which satisfies $\varphi \geq 0$ in Σ and

$$\begin{aligned} L\varphi(x) &= e^{-\alpha|x-y|^2} \left[-\sum_{i,j} a_{ij} 4\alpha^2 (x_i - y_i)(x_j - y_j) + \sum_i 2\alpha (a_{ii} - b_i(x_i - y_i)) + c \right] - ce^{-\alpha r^2} \\ &\leq e^{-\alpha|x-y|^2} [-4\alpha^2 \lambda |x - y|^2 + 2\alpha(n\Lambda + |b||x - y|) + c] \\ &\leq e^{-\alpha|x-y|^2} [-4\alpha^2 \rho^2 \lambda + 2\alpha(n\Lambda + Br) + C]. \end{aligned}$$

Taking α large enough we can make $L\varphi \leq 0$ in Σ . Since $u(x) - u(x_0) < 0$ in Ω , hence on $\partial B_\rho(y)$ which is compact, we have $\max_{x \in \partial B_\rho(y)} (u(x) - u(x_0)) < 0$ and there exists $\epsilon > 0$ such that $v(x) = u(x) - u(x_0) + \epsilon\varphi(x) \leq 0$ on $\partial B_\rho(y)$.

On the other hand, $\varphi = 0$ on $\partial B_r(y)$, therefore $v \leq 0$ on $\partial B_\rho(y) \cup \partial B_r(y) = \partial\Sigma$ and $Lv \leq -cu(x_0) \leq 0$.

So, by the weak maximum principle, $v \leq 0$ in $\bar{\Sigma}$. Being $v(x_0) = 0$ it follows that $\partial_\nu v(x_0) \geq 0$ and

$$\partial_\nu u(x_0) \geq -\epsilon \partial_\nu \varphi(x_0) = -\epsilon e^{\alpha|x_0-y|^2} |x_0 - y| > 0.$$

□

PROOF OF THEOREM 5.3. Let $Lu \leq 0$ in Ω and assume by contradiction that u is non-constant and achieves its maximum M in the interior of Ω (with $M \geq 0$ if $c \neq 0$). Let $\Omega_M := \{y \in \Omega : u(y) = M\}$ and $\Omega^- := \{z \in \bar{\Omega} : u(z) < M\}$.

Let us consider $\bar{z} \in \Omega^-$ such that $r = \text{dist}(\bar{z}, \Omega_M) < \text{dist}(\bar{z}, \partial\Omega)$, so that $B_r(\bar{z}) \subset \Omega^-$ and there exists $x_0 \in \partial\Omega_M$ such that $x_0 \in \partial B_r(\bar{z})$.

Then we can apply Lemma 5.1 in $B_r(\bar{z})$. Indeed, all the hypotheses are verified: namely, $Lu \leq 0$ in $B_r(\bar{z})$, $u(x_0) > u(x)$, for all $x \in B_r(\bar{z})$ (with $u(x_0) = M \geq 0$ if $c \neq 0$), and $B_r(\bar{z})$ satisfies an interior sphere condition at x_0 . We conclude therefore that $\partial_\nu u(x_0) > 0$ but this leads to a contradiction since x_0 is an interior maximum and $Du(x_0) = 0$. □

Beside needed to prove the strong maximum principle, Lemma 5.1 is also useful to establish uniqueness of solutions for Neumann problems.

COROLLARY 5.2 (Uniqueness for the Neumann problem). *Let $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be two solutions of the Neumann problem*

$$\begin{cases} Lu = f, & x \in \Omega, \\ \partial_\nu u = h, & x \in \partial\Omega, \end{cases}$$

with L uniformly elliptic in Ω and Ω satisfying an interior sphere condition at each point of $\partial\Omega$. If $c = 0$ in Ω , then $u - v$ is constant in Ω . If $c > 0$ at some point in Ω , then $u = v$.

PROOF. Suppose that $w = u - v$ is not constant in $\bar{\Omega}$. Since $Lw = Lu - Lv = 0$ in Ω , then, by the strong maximum principle, either w or $-w$ achieves a non-negative maximum M at a point $x_0 \in \partial\Omega$ and is strictly less than M in Ω .

By Lemma 5.1, $\partial_\nu w(x_0) \neq 0$ (either strictly positive or strictly negative), contradicting the boundary condition $\partial_\nu w = 0$ on $\partial\Omega$. Hence, w is constant in $\bar{\Omega}$ and, if $c \neq 0$ in Ω , we have $w = 0$. □

2. Well-posedness of weak solutions of linear second-order elliptic PDEs

For $\mu \in \mathbb{R}$, we consider the Dirichlet problem for $L + \mu I$,

$$\begin{cases} Lu + \mu u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \quad (2.1)$$

Here, I denotes the identity and we consider a uniformly elliptic operator L in the (divergence) form

$$Lu := - \sum_{i,j=1}^n \partial_{x_i} (a_{ij} \partial_{x_j} u) + \sum_{i=1}^n \partial_{x_i} (b_i u) + cu, \quad (2.2)$$

where

$$a_{ij}, b_i, c \in L^\infty(\Omega), \quad a_{ij} = a_{ji} \quad (2.3)$$

and we assume that there exists some constant $\theta > 0$

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2, \quad (2.4)$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$.

We say that $u \in H_0^1(\Omega)$ is a *weak solution* of (2.1)

$$\int_{\Omega} \left\{ \sum_{i,j=1}^n a_{ij} \partial_{x_i} u \partial_{x_j} \phi - \sum_{i=1}^n b_i u \partial_{x_i} \phi + cu \phi \right\} dx + \mu \int_{\Omega} u \phi dx = \langle f, \phi \rangle$$

for all $\phi \in H_0^1(\Omega)$. To write this condition more concisely, we define a bilinear form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$$

by

$$a(u, v) := \int_{\Omega} \left\{ \sum_{i,j=1}^n a_{ij} \partial_{x_i} u \partial_{x_j} v - \sum_i b_i u \partial_{x_i} v + cuv \right\} dx. \quad (2.5)$$

DEFINITION 5.2 (Weak solution of (2.1)). *Let that $\Omega \subset \mathbb{R}^n$ be an open set, $f \in H^{-1}(\Omega)$, and L be the differential operator (2.2). Then $u : \Omega \rightarrow \mathbb{R}$ is a weak solution of (2.1) if $u \in H_0^1(\Omega)$ and*

$$a(u, \phi) + \mu(u, \phi)_{L^2} = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega).$$

REMARK 5.2. *The form a is well-defined and bounded on $H_0^1(\Omega)$. However, it is not symmetric unless $b_i = 0$. We have*

$$a(v, u) = a^*(u, v),$$

where

$$a^*(u, v) = \int_{\Omega} \left\{ \sum_{i,j=1}^n a_{ij} \partial_{x_i} u \partial_{x_j} v + \sum_i b_i (\partial_{x_i} u) v + cuv \right\} dx \quad (2.6)$$

is the bilinear form associated with the formal adjoint L^* of L ,

$$L^*u = - \sum_{i,j=1}^n \partial_{x_i} (a_{ij} \partial_{x_j} u) - \sum_{i=1}^n b_i \partial_{x_i} u + cu. \quad (2.7)$$

REMARK 5.3. *Notice that L^* is the operator we considered in Section 1. We stress that, although L^* is not of exactly the same form as L , since it first derivative term is not in divergence form, the same proof (up to minor changes) of the existence of weak solutions for L , which we will give below, applies to L^* with a in (2.5) replaced by a^* in (2.6).*

In the proof of the maximum principles in Section 1, we used L^ for the sake of convenience (using L instead would have required somewhat more attention).*

The proof of the existence of a weak solution of (2.1) is similar to the proof for the Dirichlet Laplacian, with one exception. If L is not symmetric, we cannot use a to define an equivalent inner product on $H_0^1(\Omega)$ and appeal to the Riesz representation theorem. Instead we use a result due to Peter Lax and Arthur Milgram (see [LM54] and also [Bab71]) which applies to non-symmetric bilinear forms.

We begin by stating the Lax–Milgram theorem for a bilinear form on a Hilbert space. Afterwards, we verify its hypotheses for the bilinear form associated with a general second-order uniformly elliptic PDE and use it to prove the existence of weak solutions.

THEOREM 5.4 (Lax–Milgram’s theorem). *Let \mathcal{H} be a Hilbert space with inner-product $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$, and let $a : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ be a bilinear form on \mathcal{H} . Assume that there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \|u\|^2 \leq a(u, u), \quad |a(u, v)| \leq C_2 \|u\| \|v\| \quad \text{for all } u, v \in \mathcal{H}.$$

Then, for every bounded linear functional $f : \mathcal{H} \rightarrow \mathbb{R}$, there exists a unique $u \in \mathcal{H}$ such that

$$\langle f, v \rangle = a(u, v) \quad \text{for all } v \in \mathcal{H}.$$

We omit the proof of this result, and instead we see how to apply it to the bilinear form in (2.5).

THEOREM 5.5. *Let a be the bilinear form on $H_0^1(\Omega)$ defined in (2.5), where the coefficients satisfy (2.3) and the uniform ellipticity condition (2.4) with constant θ . Then, there exist constants $C_1, C_2 > 0$ and $\gamma \in \mathbb{R}$ such that, for all $u, v \in H_0^1(\Omega)$,*

$$C_1 \|u\|_{H_0^1}^2 \leq a(u, u) + \gamma \|u\|_{L^2}^2, \quad (2.8)$$

$$|a(u, v)| \leq C_2 \|u\|_{H_0^1} \|v\|_{H_0^1}. \quad (2.9)$$

If $b = 0$, we may take $\gamma := \theta - c_0$, with $c_0 := \inf_{\Omega} c$ and, if $b \neq 0$, we may take $\gamma := \frac{1}{2\theta} \sum_{i=1}^n \|b_i\|_{L^\infty}^2 + \frac{\theta}{2} - c_0$.

REMARK 5.4. *Equation (2.8) is a crucial estimate of the H_0^1 -norm of u in terms of $a(u, u)$, using the uniform ellipticity of L , and is called Gårding’s inequality.¹ We notice that the expression for γ given in Theorem 5.5 is not necessarily sharp. For example, as in the case of the Laplacian, the use of Poincaré’s inequality gives smaller values of γ for bounded domains.*

Equation (2.9) states that the bilinear form a is bounded on H_0^1 .

PROOF OF THEOREM 5.5. Step 1. For any $u, v \in H_0^1(\Omega)$, we have

$$\begin{aligned} |a(u, v)| &\leq \sum_{i,j=1}^n \int_{\Omega} |a_{ij} \partial_{x_i} u \partial_{x_j} v| \, dx + \sum_{i=1}^n \int_{\Omega} |b_i u \partial_{x_i} v| \, dx + \int_{\Omega} |cuv| \, dx \\ &\leq \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty} \|\partial_{x_i} u\|_{L^2} \|\partial_{x_j} v\|_{L^2} \\ &\quad + \sum_{i=1}^n \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_{x_i} v\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2} \\ &\leq C \left(\sum_{i,j=1}^n \|a_{ij}\|_{L^\infty} + \sum_{i=1}^n \|b_i\|_{L^\infty} + \|c\|_{L^\infty} \right) \|u\|_{H_0^1} \|v\|_{H_0^1}, \end{aligned}$$

which yields (2.9).

Step 2. Using the uniform ellipticity condition (2.4), we have

$$\begin{aligned} \theta \|Du\|_{L^2}^2 &= \theta \int_{\Omega} |Du|^2 \, dx \\ &\leq \sum_{i,j=1}^n \int_{\Omega} a_{ij} \partial_{x_i} u \partial_{x_j} u \, dx \\ &\leq a(u, u) + \sum_{i=1}^n \int_{\Omega} b_i u \partial_{x_i} u \, dx - \int_{\Omega} cu^2 \, dx \\ &\leq a(u, u) + \sum_{i=1}^n \int_{\Omega} |b_i u \partial_{x_i} u| \, dx - c_0 \int_{\Omega} u^2 \, dx \\ &\leq a(u, u) + \sum_{i=1}^n \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_{x_i} u\|_{L^2} - c_0 \|u\|_{L^2}^2 \\ &\leq a(u, u) + \beta \|u\|_{L^2} \|Du\|_{L^2} - c_0 \|u\|_{L^2}^2, \end{aligned}$$

¹ Named after Lars Gårding [Gär53]; cf. also [Hör18].

where $c(x) \geq c_0$ a.e. in Ω , and $\beta = \left(\sum_{i=1}^n \|b_i\|_{L^\infty}^2 \right)^{1/2}$.

If $\beta = 0$, we get (2.8) with

$$\gamma := \theta - c_0, \quad C_1 := \theta.$$

On the other hand, if $\beta > 0$, by Young's inequality² with ϵ , we have, for any $\epsilon > 0$,

$$\|u\|_{L^2} \|Du\|_{L^2} \leq \epsilon \|Du\|_{L^2}^2 + \frac{1}{4\epsilon} \|u\|_{L^2}^2.$$

Hence, choosing $\epsilon = \theta/2\beta$, we get

$$\frac{\theta}{2} \|Du\|_{L^2}^2 \leq a(u, u) + \left(\frac{\beta^2}{2\theta} - c_0 \right) \|u\|_{L^2}^2,$$

and (2.8) follows with

$$\gamma := \frac{\beta^2}{2\theta} + \frac{\theta}{2} - c_0, \quad C_1 := \frac{\theta}{2}.$$

□

THEOREM 5.6 (Application to the Dirichlet problem). *Suppose that Ω is an open set in \mathbb{R}^n , and $f \in H^{-1}(\Omega)$. Let L be a differential operator (2.2) (with coefficients satisfying (2.3)), and let $\gamma \in \mathbb{R}$ be a constant for which Theorem 5.5 holds. Then, for every $\mu \geq \gamma$, there exists one and only one weak solution of the Dirichlet problem (2.1).*

PROOF. For $\mu \in \mathbb{R}$, let us define $a_\mu : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$a_\mu(u, v) = a(u, v) + \mu(u, v)_{L^2},$$

where a is defined in (2.5). Then $u \in H_0^1(\Omega)$ is a weak solution of $Lu + \mu u = f$ if and only if

$$a_\mu(u, \phi) = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega).$$

We want to apply Lax–Milgram's theorem (Theorem 5.4) to establish the existence of one and only one such weak solution.

We rely on Theorem 5.5. From (2.9), we get

$$|a_\mu(u, v)| \leq C_2 \|u\|_{H_0^1} \|v\|_{H_0^1} + |\mu| \|u\|_{L^2} \|v\|_{L^2} \leq (C_2 + |\mu|) \|u\|_{H_0^1} \|v\|_{H_0^1}$$

so a_μ is bounded on $H_0^1(\Omega)$. From (2.8),

$$C_1 \|u\|_{H_0^1}^2 \leq a(u, u) + \gamma \|u\|_{L^2}^2 \leq a_\mu(u, u)$$

whenever $\mu \geq \gamma$.

Thus, the assumptions of Lax–Milgram's theorem are satisfied and we can conclude that, for every $f \in H^{-1}(\Omega)$, there is a unique $u \in H_0^1(\Omega)$ such that $\langle f, \phi \rangle = a_\mu(u, \phi)$ for all $v \in H_0^1(\Omega)$. This concludes the proof. □

² Named after William Henry Young [You12].

APPENDIX A

Some background information

1. Some notation

Let \mathbb{R}^n be n -dimensional Euclidean space. We denote the Euclidean norm of a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ by

$$|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

and the inner product of vectors $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$ by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

We denote Lebesgue measure on \mathbb{R}^n by dx , and the Lebesgue measure of a set $E \subset \mathbb{R}^n$ by $|E|$.

If E is a subset of \mathbb{R}^n , we denote the complement by $E^c = \mathbb{R}^n \setminus E$, the closure by \bar{E} , the interior by E° and the boundary by $\partial E = \bar{E} \setminus E^\circ$. The characteristic function $\chi_E : \mathbb{R}^n \rightarrow \mathbb{R}$ of E is defined by

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

A set E is bounded if $\{|x| : x \in E\}$ is bounded in \mathbb{R} . A set is *connected* if it is not the disjoint union of two non-empty relatively open subsets. We sometimes refer to a connected open set as a *domain*.

We say that an open set $\Omega' \subset \mathbb{R}^n$ is *compactly contained* in an open set Ω , written $\Omega' \subset\subset \Omega$ (or $\Omega' \Subset \Omega$), if $\bar{\Omega}' \subset \Omega$ and $\bar{\Omega}'$ is compact. If $\bar{\Omega}' \subset \Omega$, then

$$\text{dist}(\Omega', \partial\Omega) = \inf\{|x - y| : x \in \Omega', y \in \partial\Omega\} > 0.$$

This distance is finite provided that $\Omega' \neq \emptyset$ and $\Omega \neq \mathbb{R}^n$.

2. Integration by part formulas

We denote by $B_r(x) \subset \mathbb{R}^n$ the open ball of radius r and center $x : B_r(x) = \{y \in \mathbb{R}^n : \|y - x\| < r\}$ with respect to the Euclidean norm, and we will often use the shorthand notation $B_r = B_r(0)$. With \mathbb{R}_+^n we denote the half-plane $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$.

DEFINITION A.1. *Given a domain $\Omega \subset \mathbb{R}^n$, we say that $\partial\Omega$ is of class C^k if, for each $y \in \partial\Omega$, there exists $r > 0$ and a C^k function $\varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ such that, upon relabeling and reorienting the coordinates axis if necessary, we have*

$$\Omega \cap B_r(y) = \{x \in B_r(y) : x_n > \varphi(x_1, \dots, x_{n-1})\}$$

i.e. locally, in a neighborhood of $y \in \partial\Omega$, the boundary can be expressed as the graph of a C^k function and the domain Ω lies only on one side of the graph. If $\partial\Omega \in C^1$, then $\forall y \in \partial\Omega$ the outward pointing normal vector is well defined and denoted $\nu(y) = (\nu_1(y), \dots, \nu_n(y))$.

We recall an equivalent characterization of a domain with C^k boundary.

PROPOSITION A.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega$ of class C^k . Then for each $x \in \partial\Omega$ there exists an open set $V_x \ni x$ and a C^k -diffeomorphism $\phi_x : V_x \rightarrow B_1 \subset \mathbb{R}^n$ satisfying:*

- (1) $\phi_x(V_x \cap \Omega) = B_1^+ = B_1 \cap \mathbb{R}_+^n$,
- (2) $\phi_x(V_x \cap \partial\Omega) = B_1 \cap \partial\mathbb{R}_+^n$,
- (3) $\phi_x \in C^k(V_x)$, $\phi_x^{-1} \in C^k(B_1)$.

We say that each mapping ϕ_x straightens locally the boundary.

With the notion of the outward normal unit vector, we can now recall the Green–Gauss and integration by parts formulas.

Gauss–Ostrogradsky’s formula: Let $\Omega \subset \mathbb{R}^n$ be a bounded set with $\partial\Omega \subset C^1$. For $u \in C^1(\bar{\Omega})$,

$$\int_{\Omega} u_{x_i}(x) \, dx = \int_{\partial\Omega} u(y)\nu_i(y) \, dS(y), \quad i = 1, \dots, n.$$

For more historical information of the *divergence theorem*, see [Kat79].

Integration by parts formula: Let $\Omega \subset \mathbb{R}^n$ be a bounded set with $\partial\Omega \subset C^1$. For $u, v \in C^1(\bar{\Omega})$,

$$\int_{\Omega} u_{x_i}(x)v(x) \, dx = - \int_{\Omega} u(x)v_{x_i}(x) \, dx + \int_{\partial\Omega} u(y)v(y)\nu_i(y) \, dS(y), \quad i = 1, \dots, n.$$

or, in vectorial form,

$$\int_{\Omega} \nabla u(x)v(x) \, dx = - \int_{\Omega} u(x)\nabla v(x) \, dx + \int_{\partial\Omega} u(y)v(y)\nu(y) \, dS(y)$$

Green’s identities: Let $\Omega \subset \mathbb{R}^n$ be a bounded set with $\partial\Omega \subset C^1$. For $u, v \in C^2(\bar{\Omega})$,

$$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \partial_{\nu} u \, dS,$$

with $\partial_{\nu} u = \nabla u \cdot \nu$ being the normal derivative of u , and

$$\int_{\Omega} v\Delta u \, dx = - \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} v\partial_{\nu} u \, dS = \int_{\Omega} u\Delta v \, dx + \int_{\partial\Omega} (v\partial_{\nu} u - u\partial_{\nu} v) \, dS.$$

All previous identities are valid also if the boundary $\partial\Omega$ is only Lipschitz continuous (since Lipschitz functions are differentiable everywhere but a set of points of Lebesgue measure zero).

Bibliography

- [AD04] G. Acosta and R. G. Durán. “An optimal Poincaré inequality in L^1 for convex domains”. In: *Proc. Am. Math. Soc.* 132.1 (2004), pp. 195–202.
- [AN16] D. E. Apushkinskaya and A. I. Nazarov. “A counterexample to the Hopf-Oleinik lemma (elliptic case)”. In: *Anal. PDE* 9.2 (2016), pp. 439–458.
- [Aub76] T. Aubin. “Problèmes isopérimétriques et espaces de Sobolev”. In: *J. Differential Geometry* 11.4 (1976), pp. 573–598.
- [Bab71] I. Babuška. “Error-bounds for finite element method”. In: *Numer. Math.* 16 (1970/71), pp. 322–333.
- [BL83] H. Berestycki and P.-L. Lions. “Nonlinear scalar field equations. I. Existence of a ground state”. In: *Arch. Rational Mech. Anal.* 82.4 (1983), pp. 313–345.
- [Bôc03] M. Bôcher. “Singular points of functions which satisfy partial differential equations of the elliptic type”. In: *Bull. Amer. Math. Soc.* 9.9 (1903), pp. 455–465.
- [Bre13] A. Bressan. *Lecture notes on functional analysis. With applications to linear partial differential equations*. Vol. 143. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2013.
- [Bre11] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. New York, NY: Springer, 2011.
- [Bun59] V. Bunyakovsky. “Sur quelques inégalités concernant les intégrales aux différences finies”. In: *Mémoires de l’Académie des Sciences de Saint-Pétersbourg* 7.1 (1859), p. 6.
- [Cau44] A. Cauchy. “Mémoire sur les fonctions complémentaires”. In: *C. R. Acad. Sci. Paris* 19 (1844), pp. 1377–1384.
- [Cau21] A.-L. Cauchy. “Sur les formules qui résultent de l’emploi du signe $>$ ou $<$, et sur les moyennes entre plusieurs quantités”. In: *Cours d’Analyse, 1^{er} Partie: Analyse Algébrique*. Vol. Ser. 2, III. OEuvres. 1821, pp. 373–377.
- [CEL84] M. G. Crandall, L. C. Evans, and P.-L. Lions. “Some properties of viscosity solutions of Hamilton-Jacobi equations”. In: *Trans. Am. Math. Soc.* 282 (1984), pp. 487–502.
- [CIL92] M. G. Crandall, H. Ishii, and P.-L. Lions. “User’s guide to viscosity solutions of second order partial differential equations”. In: *Bull. Am. Math. Soc., New Ser.* 27.1 (1992), pp. 1–67.
- [CL83] M. G. Crandall and P.-L. Lions. “Viscosity solutions of Hamilton-Jacobi equations”. In: *Trans. Am. Math. Soc.* 277 (1983), pp. 1–42.
- [Der64] G. H. Derrick. “Comments on nonlinear wave equations as models for elementary particles”. In: *J. Mathematical Phys.* 5 (1964), pp. 1252–1254.
- [Ehr54] L. Ehrenpreis. “Solution of some problems of division. I. Division by a polynomial of derivation”. In: *Amer. J. Math.* 76 (1954), pp. 883–903.
- [Eva10] L. C. Evans. *Partial differential equations*. 2nd ed. Vol. 19. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2010.
- [FR22] X. Fernández-Real and X. Ros-Oton. *Regularity theory for elliptic PDE*. Zur. Lect. Adv. Math. Berlin: European Mathematical Society (EMS), 2022.
- [Fré07] M. Fréchet. “Sur les ensembles de fonctions et les opérations linéaires.” In: *C. R. Acad. Sci., Paris* 144 (1907), pp. 1414–1416.
- [Fri27] K. Friedrichs. “Eine invariante Formulierung des Newtonschen Gravitationsgesetzes und des Grenzüberganges vom Einsteinschen zum Newtonschen Gesetz.” English. In: *Math. Ann.* 98 (1927), pp. 566–575.
- [Fri44] K. O. Friedrichs. “The identity of weak and strong extensions of differential operators”. In: *Trans. Amer. Math. Soc.* 55 (1944), pp. 132–151.

- [Gag58] E. Gagliardo. “Proprietà di alcune classi di funzioni in più variabili”. In: *Ricerche Mat.* 7 (1958), pp. 102–137.
- [Går53] L. Gårding. “Dirichlet’s problem for linear elliptic partial differential equations”. In: *Math. Scand.* 1 (1953), pp. 55–72.
- [GT01] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Reprint of the 1998 ed. Class. Math. Berlin: Springer, 2001.
- [GA98] K. Gustafson and T. Abe. “The third boundary condition—was it Robin’s?” In: *Math. Intelligencer* 20.1 (1998), pp. 63–71.
- [Had02] J. Hadamard. “Sur les problèmes aux dérivées partielles et leur signification physique”. In: *Princeton University Bulletin* 13 (1902), pp. 49–52.
- [HL11] Q. Han and F. Lin. *Elliptic partial differential equations*. 2nd ed. Vol. 1. Courant Lect. Notes Math. New York, NY: Courant Institute of Mathematical Sciences; Providence, RI: American Mathematical Society (AMS), 2011.
- [Höl89] O. Hölder. “Ueber einen Mittelwerthabsatz”. In: *Nachrichten von der Königl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen* 1889 (1889), pp. 38–47.
- [Hop27] E. Hopf. “Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus”. In: *Sitzungsberichte Preussische Akademie Wissenschaften* (1927), pp. 147–152.
- [Hör18] L. Hörmander. “Gårding’s inequality during three decades”. In: *Unpublished Manuscripts: from 1951 to 2007*. Springer International Publishing, 2018, pp. 137–146.
- [Hun14] J. K. Hunter. *Notes on partial differential equations*. 2014. URL: <http://www.math.ucdavis.edu/~hunter/pdes/pdes.html>.
- [Joh82] F. John. *Partial differential equations*. 4th ed. Vol. 1. Appl. Math. Sci. Springer, Cham, 1982.
- [Jos07] J. Jost. *Partial differential equations*. 2nd ed. Vol. 214. Grad. Texts Math. New York, NY: Springer, 2007.
- [Kat79] V. J. Katz. “The history of Stokes’ theorem”. In: *Math. Mag.* 52.3 (1979), pp. 146–156.
- [Kon45] W. Kondrachov. “Sur certaines propriétés des fonctions dans l’espace L^p ”. In: *C. R. (Dokl.) Acad. Sci. URSS, n. Ser.* 48 (1945), pp. 535–538.
- [Lan72] N. S. Landkof. *Foundations of modern potential theory*. Die Grundlehren der mathematischen Wissenschaften, Band 180. Translated from the Russian by A. P. Doohovskoy. Springer-Verlag, New York-Heidelberg, 1972, pp. x+424.
- [LM54] P. D. Lax and A. N. Milgram. “Parabolic equations”. In: *Contributions to the theory of partial differential equations*. Ann. of Math. Stud., no. 33. Princeton Univ. Press, Princeton, NJ, 1954, pp. 167–190.
- [LL23] K.-A. Lee and S.-C. Lee. *The Wiener criterion for fully nonlinear elliptic equations*. 2023. arXiv: [2301.00960](https://arxiv.org/abs/2301.00960) [math.AP].
- [Lio80] M. J. Liouville. “Leçons sur les fonctions doublement périodiques faites en 1847 par M. J. Liouville”. In: *J. Reine Angew. Math.* 88 (1880), pp. 277–310.
- [LSW63] W. Littman, G. Stampacchia, and H. F. Weinberger. “Regular points for elliptic equations with discontinuous coefficients”. In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* 17 (1963), pp. 43–77.
- [Mal56] B. Malgrange. “Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution”. In: *Ann. Inst. Fourier (Grenoble)* 6 (1955/56), pp. 271–355.
- [Mal98] L. Maligranda. “Why Hölder’s inequality should be called Rogers’ inequality”. In: *Math. Inequal. Appl.* 1.1 (1998), pp. 69–83.
- [MS64] N. G. Meyers and J. Serrin. “ $H = W$ ”. In: *Proc. Nat. Acad. Sci. U.S.A.* 51 (1964), pp. 1055–1056.
- [Mor40] C. B. j. Morrey. “Functions of several variables and absolute continuity. II.” In: *Duke Math. J.* 6 (1940), pp. 187–215.
- [Mor66] C. B. j. Morrey. *Multiple integrals in the calculus of variations*. Vol. 130. Grundlehren Math. Wiss. Springer, Cham, 1966.

- [Nau02] J. Naumann. *Remarks on the Prehistory of Sobolev Spaces*. 2002. Humboldt-Universität zu Berlin, Preprints aus dem Institut für Mathematik: <http://dx.doi.org/10.18452/2615>.
- [Naz12] A. I. Nazarov. “A centennial of the Zaremba-Hopf-Oleinik lemma”. In: *SIAM J. Math. Anal.* 44.1 (2012), pp. 437–453.
- [Nel61] E. Nelson. “A proof of Liouville’s theorem”. In: *Proc. Amer. Math. Soc.* 12 (1961), p. 995.
- [Nir59] L. Nirenberg. “On elliptic partial differential equations”. In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* 13 (1959), pp. 115–162.
- [Ole52] O. A. Oleĭnik. “Über die Eigenschaften der Lösungen gewisser Randwertaufgaben für Gleichungen vom elliptischen Typus”. Russian. In: *Mat. Sb., Nov. Ser.* 30 (1952), pp. 695–702.
- [OW96] N. Ortner and P. Wagner. “A short proof of the Malgrange-Ehrenpreis theorem”. In: *Functional analysis (Trier, 1994)*. de Gruyter, Berlin, 1996, pp. 343–352.
- [PW60] L. E. Payne and H. F. Weinberger. “An optimal Poincaré inequality for convex domains”. In: *Arch. Ration. Mech. Anal.* 5 (1960), pp. 286–292.
- [Per23] O. Perron. “Eine neue Behandlung der ersten Randwertaufgabe für $\Delta u = 0$ ”. In: *Math. Z.* 18.1 (1923), pp. 42–54.
- [Pic24] E. Picard. “Quelques théorèmes élémentaires sur les fonctions harmoniques”. In: *Bull. Soc. Math. France* 52 (1924), pp. 162–166.
- [Pie07] A. Pietsch. *History of Banach spaces and linear operators*. Birkhäuser Boston, Inc., Boston, MA, 2007, pp. xxiv+855.
- [Poh65] S. I. Pohožaev. “On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$ ”. In: *Dokl. Akad. Nauk SSSR* 165 (1965), pp. 36–39.
- [Poi90] H. Poincaré. “Sur les Equations aux Derivees Partielles de la Physique Mathematique”. In: *Amer. J. Math.* 12.3 (1890), pp. 211–294.
- [PS04a] P. Pucci and J. Serrin. “Erratum: “The strong maximum principle revisited” [J. Differential Equations **196** (2004), no. 1, 1–66; MR2025185]”. In: *J. Differential Equations* 207.1 (2004), pp. 226–227.
- [PS04b] P. Pucci and J. Serrin. “The strong maximum principle revisited”. In: *J. Differential Equations* 196.1 (2004), pp. 1–66.
- [Püs32] W. Püschel. “Die erste Randwertaufgabe der allgemeinen selbstadjungierten elliptischen Differentialgleichung zweiter Ordnung im Raum für beliebige Gebiete”. In: *Math. Z.* 34.1 (1932), pp. 535–553.
- [Rel30] F. Rellich. “Ein Satz über mittlere Konvergenz.” In: *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.* 1930 (1930), pp. 30–35.
- [Rie07] F. Riesz. “Sur une espèce de géométrie analytique des systèmes de fonctions sommables.” In: *C. R. Acad. Sci., Paris* 144 (1907), pp. 1409–1411.
- [Rie10] F. Riesz. “Untersuchungen über Systeme integrierbarer Funktionen”. In: *Math. Ann.* 69.4 (1910), pp. 449–497.
- [Rog88] L. J. Roge-rs. “An extension of a certain theorem in inequalities”. In: *Mess. Math.* 17 (1888), pp. 145–150.
- [Ros91] J.-P. Rosay. “A very elementary proof of the Malgrange-Ehrenpreis theorem”. In: *Amer. Math. Monthly* 98.6 (1991), pp. 518–523.
- [Sch50] L. Schwartz. *Théorie des distributions. Tome I*. Vol. 9. Publications de l’Institut de Mathématiques de l’Université de Strasbourg [Publications of the Mathematical Institute of the University of Strasbourg]. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1091. Hermann & Cie, Paris, 1950, p. 148.
- [Sch66] L. Schwartz. *Théorie des distributions*. Publications de l’Institut de Mathématique de l’Université de Strasbourg, IX-X. Nouvelle édition, entièrement corrigée, refondue et augmentée. Hermann, Paris, 1966, pp. xiii+420.
- [Sch51] L. Schwartz. *Théorie des distributions. Tome II*. Vol. 10. Publications de l’Institut de Mathématiques de l’Université de Strasbourg [Publications of the Mathematical Institute of the University of Strasbourg]. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1122. Hermann & Cie, Paris, 1951, p. 169.

- [Sch88] H. A. Schwarz. “Über ein Flächen kleinsten Flächeninhalts betreffendes Problem der Variationsrechnung”. In: *Acta Societatis Scientiarum Fennicae* XV (1888), p. 318.
- [Sob91] S. L. Sobolev. *Some applications of functional analysis in mathematical physics*. Vol. 90. Translations of Mathematical Monographs. Translated from the third Russian edition by Harold H. McFaden, With comments by V. P. Palamodov. American Mathematical Society, Providence, RI, 1991, pp. viii+286.
- [Sob38] S. L. Sobolev. “Sur un théorème de l’analyse fonctionnelle”. In: *C. R. (Dokl.) Acad. Sci. URSS, n. Ser.* 20 (1938), pp. 5–9.
- [Str08] D. W. Stroock. “Weyl’s lemma, one of many”. In: *Groups and analysis*. Vol. 354. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2008, pp. 164–173.
- [Tal76] G. Talenti. “Best constant in Sobolev inequality”. In: *Ann. Mat. Pura Appl. (4)* 110 (1976), pp. 353–372.
- [Wag09] P. Wagner. “A new constructive proof of the Malgrange-Ehrenpreis theorem”. In: *Amer. Math. Monthly* 116.5 (2009), pp. 457–462.
- [Wey40] H. Weyl. “The method of orthogonal projection in potential theory”. In: *Duke Math. J.* 7 (1940), pp. 411–444.
- [Wie24] N. Wiener. “Certain notions in potential theory.” In: *J. Math. Phys., Mass. Inst. Techn.* 3 (1924), pp. 24–51.
- [You12] W. H. Young. “On classes of summable functions and their *Fourier* series.” In: *Proc. R. Soc. Lond., Ser. A* 87 (1912), pp. 225–229.
- [Zar10] S. Zaremba. *Sur un problème mixte relatif à l’équation de Laplace*. Krakau Anz. (A), 313-344 (1910). 1910.